II Southern-Summer School on Mathematical Biology

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Lecture VI

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Outline

1. Semi-arid and arid regions

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2. Model
3. Hysteresis

Outline

Vegetation in semi-arid regions

Eremology: science of arid regions.

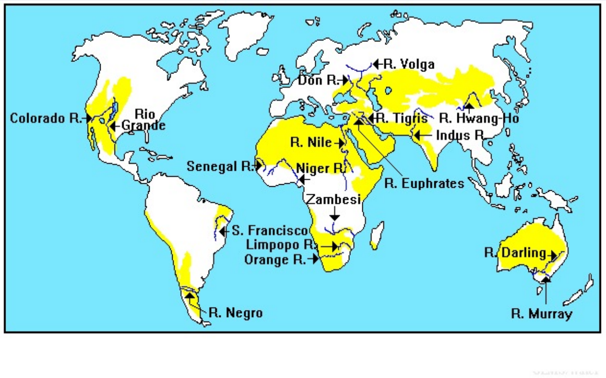

Figura : Arid and semi-arid regions of the world

Vegetation in semi-arid regions

Eremology: science of arid regions.

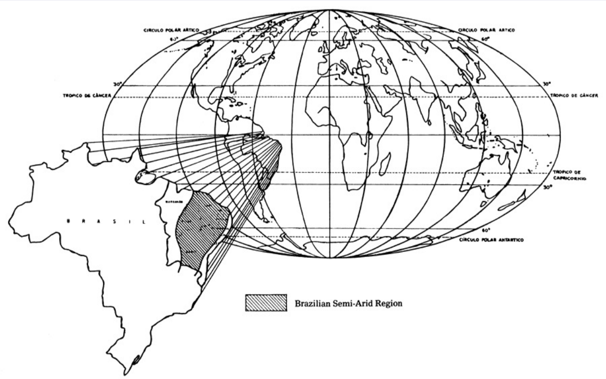

Figura : Arid and semi-arid regions of the world

Vegetation in Semi-Arid Regions

Figura : Bahia

- Consider the vegetation cover in water-poor regions.
- In this case, water is a ***limiting factor.***
- quite different from tropical regions, where competition for water is irrelevant. One of the main limiting factor is light.
- We want to build a mathematical model — (**simple, please**) — to describe the mutual relation between water in soil and biomass in semi-arid regions.
- **Let us do it**

Klausmeier Model

Figura : Colorado, USA

Figura : Kalahari, Namibia

- Water and vegetation, in a first approximation, entertain a relation similar to *predador-prey dynamics*.
- The presence of water is incremental for vegetation;
- Vegetation consumes water.
- But note that water does not originate from water.. It is an abiotic variable.
- The usual predator-prey dynamics does not apply.
- Consider two variables:
  - w,*the amount of water in soil.*
  - u, *the vegetation biomass (proportional to the area with vegetation cover).*

Klausmeier Model

Equation for the amount of water in soil

$$\frac{dw}{dt} = \underbrace{a}_{\text{precipitation}} - \underbrace{bw}_{\text{evaporation}} - \underbrace{cu^2w}_{\text{absorption by vegetation}}$$

Water in soil increases due to precipitation (a) , evaporates at a constant *per volume* rate (b), and is absorbed by vegetation in a per volume rate that depends on u^2 (c). This is phenomenological law coming from lab fittings

Equation for biomass

$$\frac{du}{dt} = \underbrace{-du}_{\text{natural death}} + \underbrace{eu^2w}_{\text{absorption of water}}$$

Vegetation has a natural death rate, (d) and absorbs water at a per volume rate (e) proportional to uw.

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Analysis

$$\frac{dw}{dt} = a - bw - cu^2 w \qquad \frac{du}{dt} = -du + eu^2 w$$

- Let us begin by defining two new variables, rescaled ones:

$$W = w\left[\frac{e}{\sqrt{b^3 c}}\right]$$

$$U = u\sqrt{bc}$$

$$T = tb$$

- They are dimensionless.
- Plug them into the equations and you will get....

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Analysis of the model

$$\frac{dW}{dT} = A - W - WU^2$$

$$\frac{dU}{dT} = WU^2 - BU$$

where

$$A = \frac{ae}{\sqrt{b^3c}}$$

and

$$B = d/b$$

$\Rightarrow$ the equations depend only on **two** parameters, instead of five

What do these equations tell us?

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Analysis of the model

$\frac{dW}{dT} = A - W - WU^2$ $\frac{dU}{dT} = WU^2 - BU$

- Let us look for fixed points:
- The points U^* e W^* such that
  - $$\frac{dW^*}{dT} = 0$$
  - $$\frac{dU^*}{dT} = 0$$
- or
  - $$A - W^* - W^*(U^*)^2 = 0$$
  - $$W^*(U^*)^2 - BU^* = 0$$

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Analysis of the model

$$\frac{dW}{dT} = A - W - WU^2 \qquad \frac{dU}{dT} = WU^2 - BU$$

- The algebraic equations has three roots:

$$U^* = 0$$
$$W^* = A$$

If $A > 2B$

$$U^* = \frac{2B}{A - \sqrt{A^2 - 4B^2}}$$

$$W^* = \frac{A - \sqrt{A^2 - 4B^2}}{2}$$

If $A > 2B$

$$U^* = \frac{2B}{A + \sqrt{A^2 - 4B^2}}$$

$$W^* = \frac{A + \sqrt{A^2 - 4B^2}}{2}$$

Analysis of the Model

Interpretation

Our first conclusion:

- If $A < 2B$ the only solution is $U^* = 0$ e $W^* = A$.
- This represents a bare state. **A desert**.
- The condition $A > 2B \Rightarrow a > \frac{2d\sqrt{bc}}{e}$ shows that there must be a minimum amount of precipitation to sustain vegetation.
- Moreover, the higher e the easier to have a state with vegetation.Recall that e represents the absorption rate. The higher, the better.
- On the other, the higher (b) and the death rate of the population, (d) easier it is to have a vegetationless solution.
- **Seems reasonable!**

Analysis of the model

So, let $A > 2B$

- If $A > 2B$, we can have two fixed points.
- What about their stability?.
- The linear stability analysis results in:
  - The fixed point $U^* = 0$ and $W^* = A$ is always **stable**.
  - The fixed point

$$U^* = \frac{2B}{A + \sqrt{A^2 - 4B^2}}, \quad W^* = \frac{A + \sqrt{A^2 - 4B^2}}{2}$$

    is always **unstable**.
  - The fixed point

$$U^* = \frac{2B}{A - \sqrt{A^2 - 4B^2}}, W^* = \frac{A - \sqrt{A^2 - 4B^2}}{2}$$

    is always **stable**.

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Analysis of the Model

- So, if $A > 2B$, and $B > 2$, we have **two** stable fixed points, each of them with its basin of attraction.
- A pictorial view is as follows:

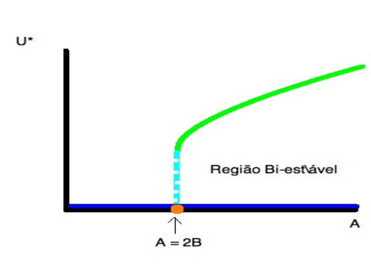


Figura : *B* is fixed, and we plot U^* (the biomass) in terms of *A*. The solution representing a desert ($U^* = 0$) and the solution corresponding to vegetation cover are both stable

Hysteresis

- The existence of a region of bi-stability ($A > 2B$, $B > 2$), can take us to the following situation.
- Take a fixed B. Consider that A can change slowly (*think of a more formal definition of "slowly"*).
- Let us begin in the bi-stability. And let A decrease. At a certain moment, A will cross the critical value $A = \mathbf{2}B$.
- At this time a sudden transition occurs, a jump, in which $U^* \to \mathbf{0}$. **Desertification!!!**.
- Suppose now that A begins again to increase - slowly. As $U^* = \mathbf{0}$ is stable, even with $A > \mathbf{2}B$ we will continue in the "desertic"region, as at the moment of crossing back the critical point we were in its the basin of attraction..
- **In summary**: if we begin with a certain A, decrease it $A < \mathbf{2}B$ and then come back to our initial value of A, the state of the system can transit from $U^* \neq \mathbf{0}$ to $U^* = \mathbf{0}$.
- This is called **Hysteresis**.

Hysteresis

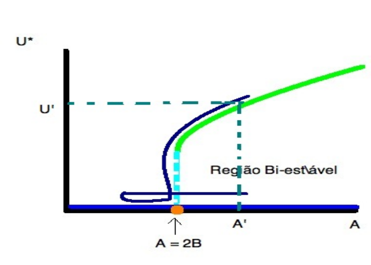


Figura : B is fixed. A begins at a value $A^{'}$ with $U^{*} = U^{'}$, decreases, crosses a critical point at $A = 2B$. It goes to , $U^{*} \to 0$. When we increase again A, even with$A > 2B$, we have $U^{*} = 0$.

Once the "desertic"state is attained it is <u>not</u> sufficient to change the external conditions back (in our model, this is the rainfall) in order to get back a vegetation-cover state . Terrible!

Glory and Misery of the Model

Glory

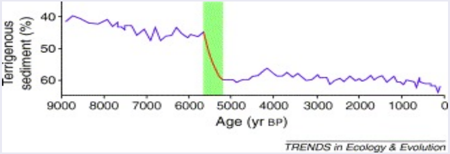


Figura : Estimated vegetation cover in the region of Sahara, over a long time span. We see a sudden change around 5500 BP.

- Existence of sudden transitions can be understood rather simply. The same kind of phenomenon appears in other systems as well.
- The model is **Simple**.

Glory and Misery of the Model

Misery

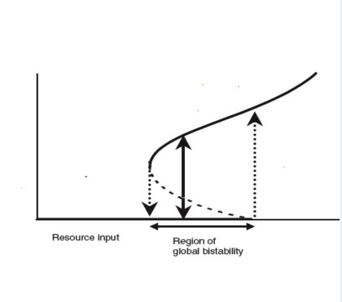
Figura : Desertification Region in China

Figura : Senegal, at the Sahel region, south to Sahara.

- The model is **very simple**
- The transition is towards a completely vegetationless state. Actual desertification processes allow for a remnants of vegetation.
- The model predicts an infinite bi-stability region... We could think that enough rain could reverse desertification.
- There are indeed better models.

More realistic models

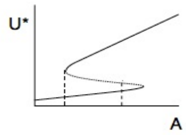
- More realistic models give bifurcation diagrams like the one below.



Biomass in terms of the rainfall, in a static case.. The blue curve represents **two transition regions.** The one to the left implies a vegetation $\rightarrow$ desert transition. To the right , a reversed transition.

More realist models

- Still another curve.



This diagram is similar to the preceding one, but U^* does not tend to zero.

Bibliography

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