Semiclassical Quantum States of Fields on Black Hole Space-times

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Outline

- QFT in Curved Space
- States in Schwarzschild
- States in Kerr: bosons vs fermions
- Conclusions

QFT in Curved Space

- Search for a theory of Quantum Gravity: unify GR and Quantum Physics

- QFT in GR: keep gravity classical but quantize 'matter' fields

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \left\langle \hat{T}_{\mu\nu} \right\rangle_{ren}^{\psi}$$

$|\psi\rangle$: quantum state of the field

- Any theory of Quantum Gravity should reproduce QFT in GR for time&length >> Planck time&length

- Calculation of $\left\langle \hat{T}_{\mu\nu} \right\rangle_{ren}^{\psi}$ is important, e.g., for backreaction (Ashtekar's School talk)

QFT in Curved Space

- **Ultraviolet divergences require renormalization:**

$$\left\langle \hat{T}_{\mu\nu} \right\rangle_{ren}^{\psi} = \left\langle \hat{T}_{\mu\nu} \right\rangle^{\psi} - T_{\mu\nu}^{div}$$

- $T_{\mu\nu}^{div}$ **is purely geometric - independent of the quantum state.**

Therefore, differences between states do not require renormalization:

$$\left\langle \hat{T}_{\mu\nu} \right\rangle_{ren}^{\psi_2} - \left\langle \hat{T}_{\mu\nu} \right\rangle_{ren}^{\psi_1} = \left\langle \hat{T}_{\mu\nu} \right\rangle^{\psi_2} - \left\langle \hat{T}_{\mu\nu} \right\rangle^{\psi_1}$$

Kerr spacetime

- **Rotating black hole with angular momentum per unit mass a and mass M**

$$ds^2 = -\frac{\Delta}{\Sigma}\left[dt^2 - a\sin^2\theta d\varphi\right]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[\left(r^2+a^2\right)d\varphi - adt\right]^2$$

$$\Delta \equiv r^2 - 2Mr + a^2 \qquad \Sigma \equiv r^2 + a^2\cos^2\theta$$

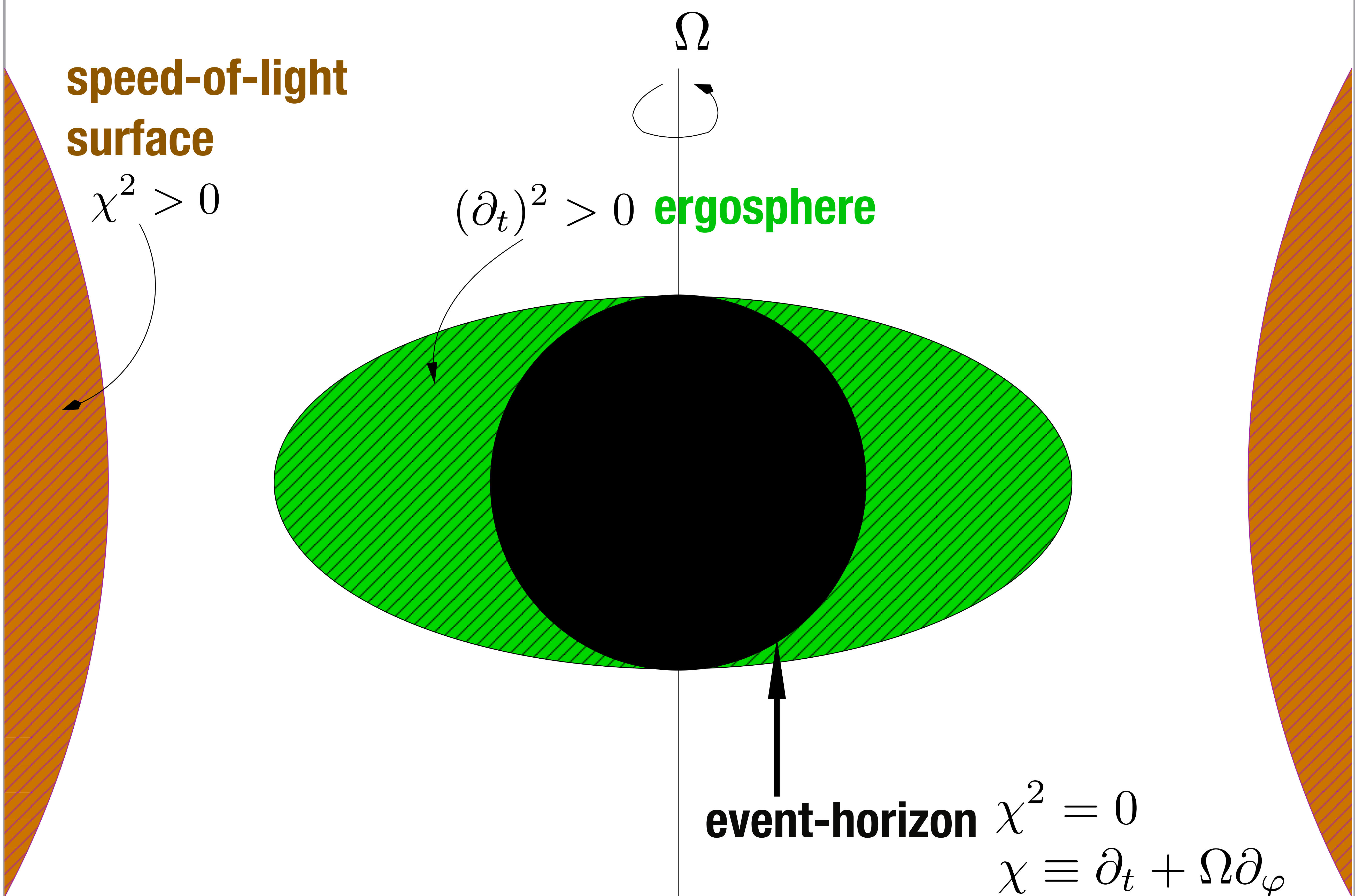
- **Event-horizon:** $r_h = M + \sqrt{M^2 - a^2}$

- **Angular velocity:** $\Omega = \dfrac{a}{r_h^2 + a^2}$

- **Axisymmetric & stationary:** ∂_t **&** ∂_φ **Killing vectors**

- **Killing vector** $\chi \equiv \partial_t + \Omega\partial_\varphi$ **is null generator of the horizon**

Kerr surfaces



QFT in Kerr: Bosons

- **(massless) Bosons. Mode decomposition of field operator:**

$$\hat{\phi} = \sum_{\Lambda} \hat{a}_{\Lambda}\phi_{\Lambda} + \hat{a}^{\dagger}_{\Lambda}\phi_{-\Lambda}$$

Field eq: $``\Box" \phi_{\Lambda} = 0$ **separates by variables** $\Lambda \equiv \{\ell, m, \omega\}$

$-\Lambda \equiv \{\ell, -m, -\omega\}$

Basis: $\phi_{\Lambda} = R_{\Lambda}(r)S_{\Lambda}(\theta)e^{im\varphi - i\omega t}$

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- **If** $(\phi_{\pm\Lambda}, \phi_{\pm\Lambda'}) = \pm\delta_{\Lambda,\Lambda'}$ **positive norm wrt an 'inner product'**
negative

then commutation rlns. $\left[\hat{a}_\Lambda, \hat{a}^\dagger_{\Lambda'}\right] = \delta_{\Lambda,\Lambda'}$

QFT in Kerr: Fermions

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Dirac eq. $\gamma^\mu \left(\partial_\mu - \Gamma_\mu\right) \phi_{\pm\Lambda} = 0$

curved st-t Dirac matrices

4-spinors

flat s-t Dirac matrix

Spinor connection matrices

- **Conserved current** $J^\mu = \left(\phi_1^\dagger \tilde{\gamma}^0\right) \gamma^\mu \phi_2 \qquad \nabla_\mu J^\mu = 0$

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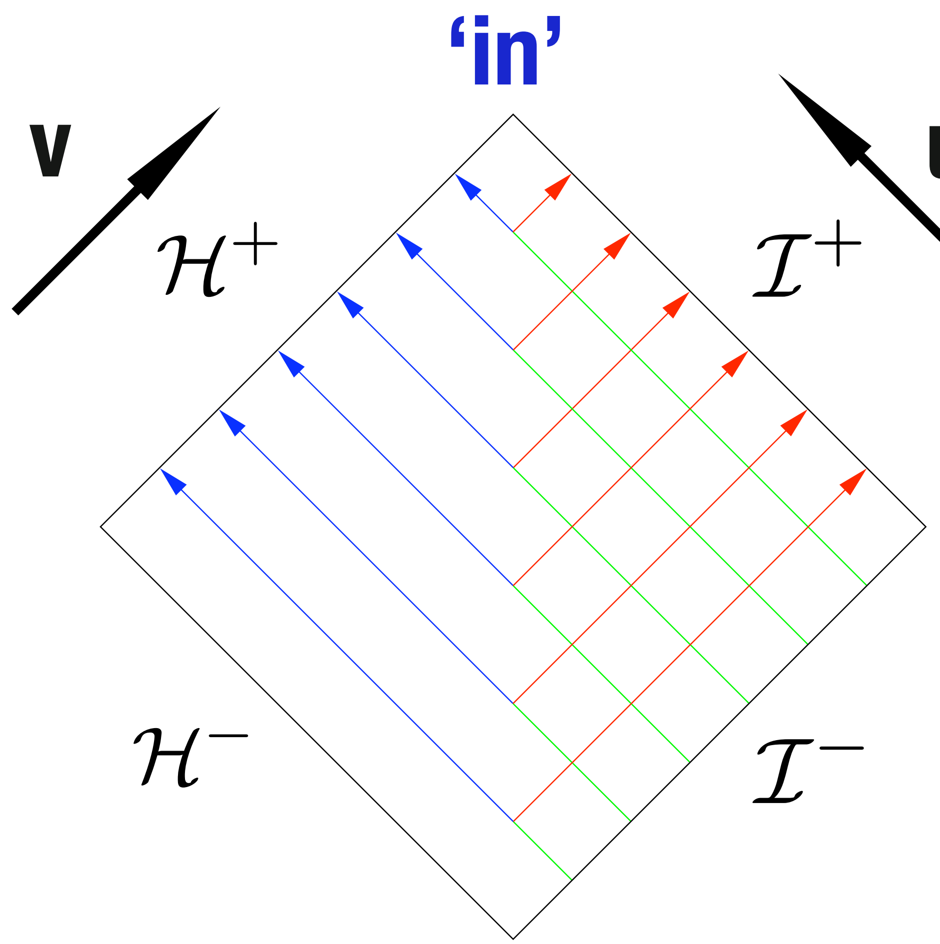
If $(\phi_{\pm\Lambda}, \phi_{\pm\Lambda'}) = \delta_{\Lambda,\Lambda'}$ **all positive norm wrt inner product**

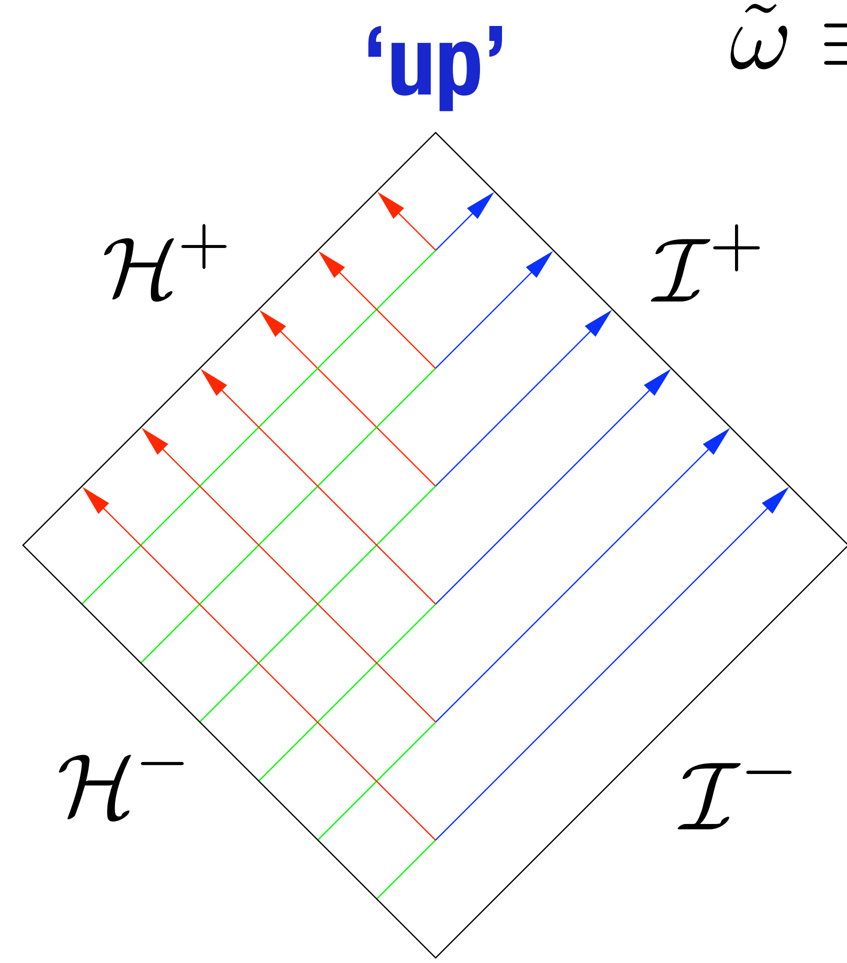
$$(\phi_1, \phi_2) = \int_{t=const} J^t dS$$

then anticommutation rlns. $\left\{\hat{a}_\Lambda, \hat{a}^\dagger_{\Lambda'}\right\} = \left\{\hat{b}_\Lambda, \hat{b}^\dagger_{\Lambda'}\right\} = \delta_{\Lambda,\Lambda'}$

Positive frequency solutions: when Fourier-decomposed, they only have positive-frequency modes

$$\tilde{\omega} \equiv \omega - m\Omega$$





'in': p.f. at $\mathcal{I}^-$ wrt null coord. v $\quad \partial_v \phi_\Lambda^{in} \sim -i\omega \phi_\Lambda^{in} \quad \forall \omega > 0$

'up': p.f. at $\mathcal{H}^-$ wrt null coord. u $\quad \partial_u \phi_\Lambda^{up} \sim -i\tilde{\omega} \phi_\Lambda^{up} \quad \forall \tilde{\omega} > 0$

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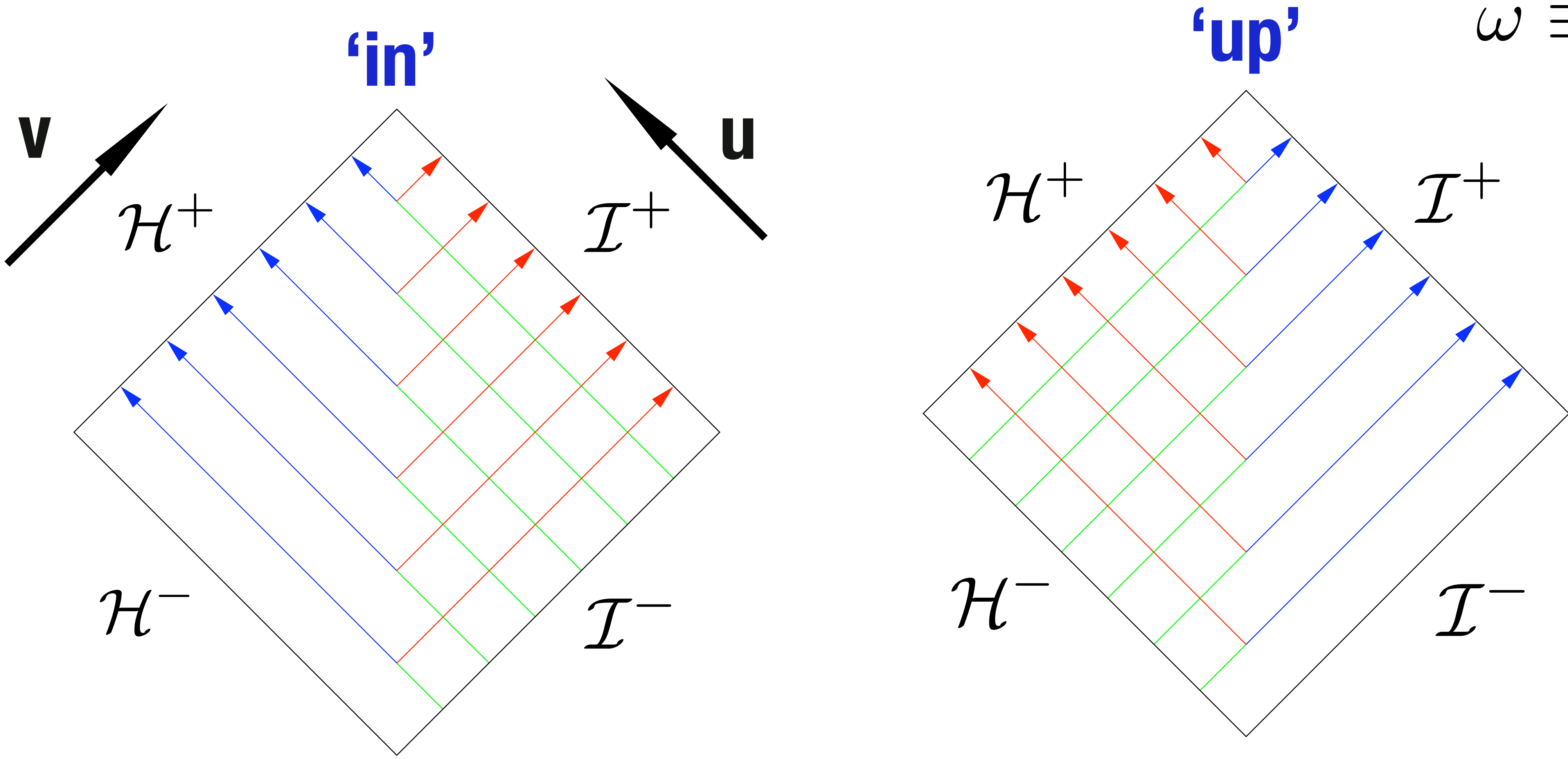

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'IN': p.f. at $\mathcal{H}^+$ wrt Kruskal coord. V $\quad \forall \omega \in \mathbb{R}$

'UP': p.f. at $\mathcal{H}^-$ wrt Kruskal coord. U $\quad \forall \tilde{\omega} \in \mathbb{R}$

Boson modes

- **Positive/negative norm:**

$$\left(\phi_{\Lambda}^{in},\phi_{\Lambda}^{in}\right)\gtrless 0 \qquad \forall\omega\gtrless 0 \qquad\qquad (\phi_{\Lambda}^{up},\phi_{\Lambda}^{up})\gtrless 0 \qquad \forall\tilde{\omega}\gtrless 0$$

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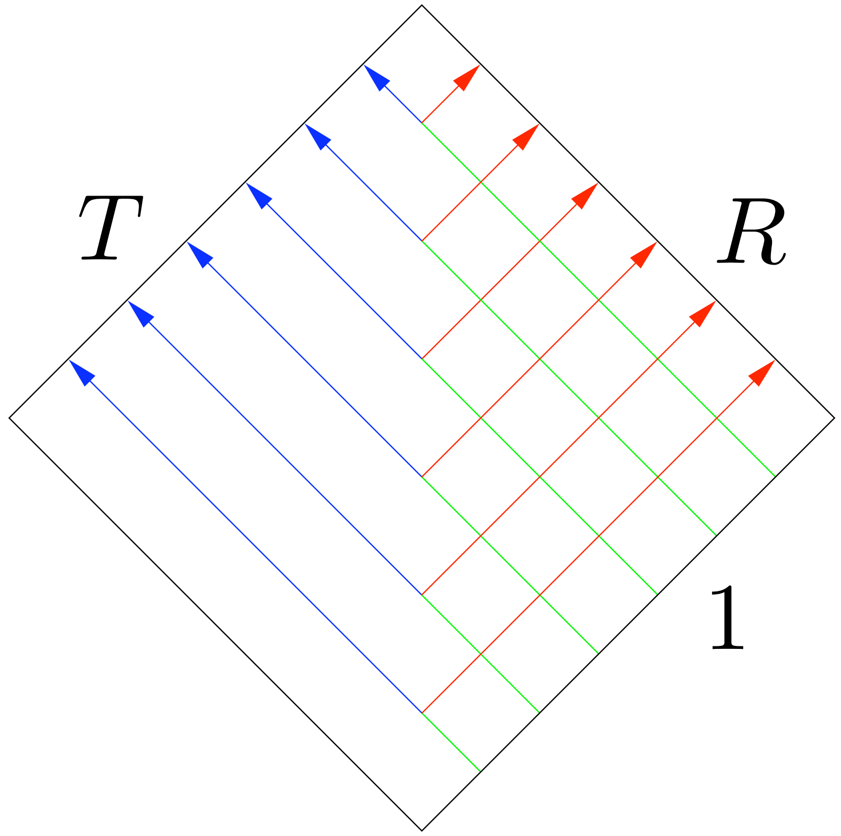
- **Classical superradiance for** $\omega\tilde{\omega} < 0$

$$1 - |R|^2 = \frac{\tilde{\omega}}{\omega}|T|^2$$



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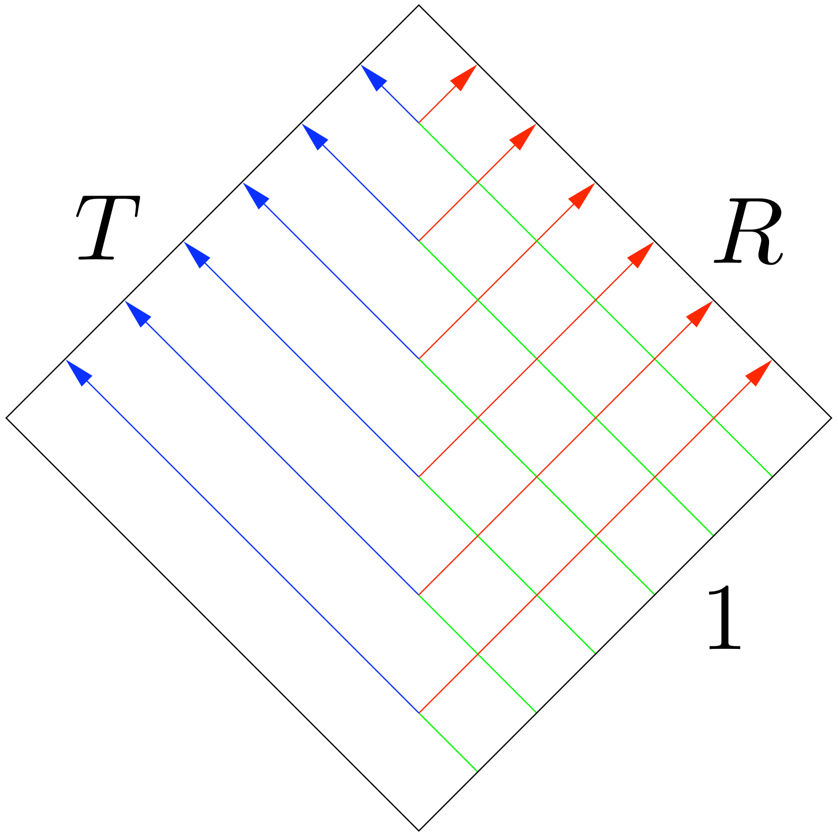

- **1st law of black holes thermodynamics:**

$$dM = T_H d\left(\frac{A}{4}\right) + \Omega dJ$$

Send wavepacket in with $\frac{dM}{dJ} = \frac{\omega}{m} \quad \longrightarrow \quad \frac{\tilde{\omega}}{\omega} dM = T_H d\left(\frac{A}{4}\right)$

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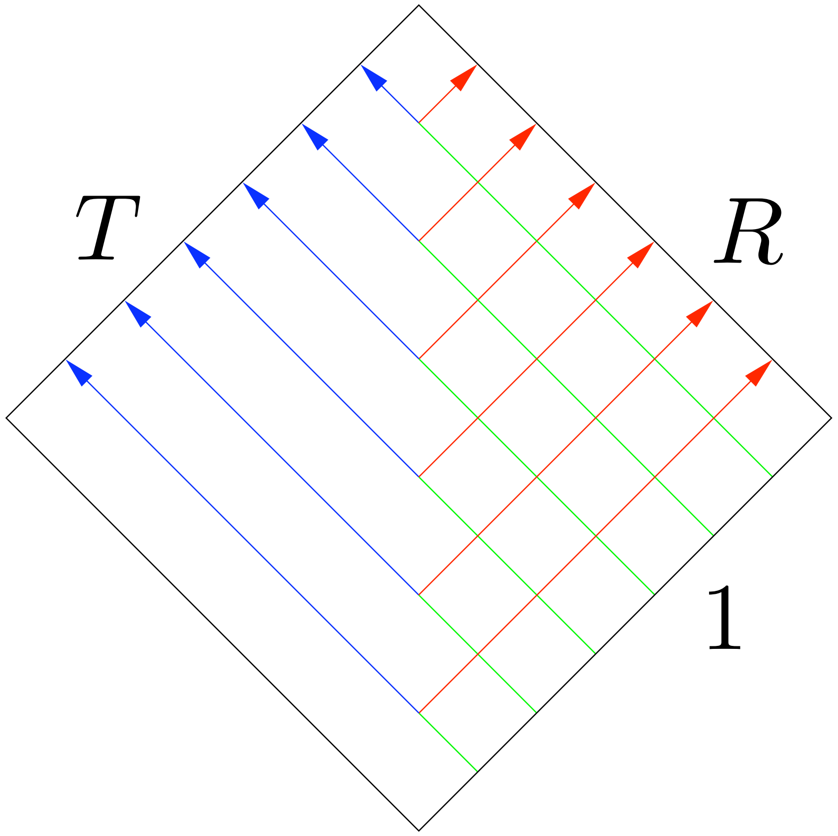

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Area theorem: $dA > 0$ **if weak-energy condition** $T_{\mu\nu}\xi^\mu\xi^\nu \geq 0 \quad \forall\xi^\mu \text{ tlike}$

is satisfied $\quad\Rightarrow\quad dM < 0$ **for** $\omega\tilde\omega < 0$

Fermion modes

- All modes are **positive norm**:

$$\left(\phi_{\Lambda}^{in}, \phi_{\Lambda}^{in}\right) > 0 \quad \left(\phi_{\Lambda}^{up}, \phi_{\Lambda}^{up}\right) > 0 \quad \forall \omega \in \mathbb{R}$$

-> freedom in choice of p.f. modes

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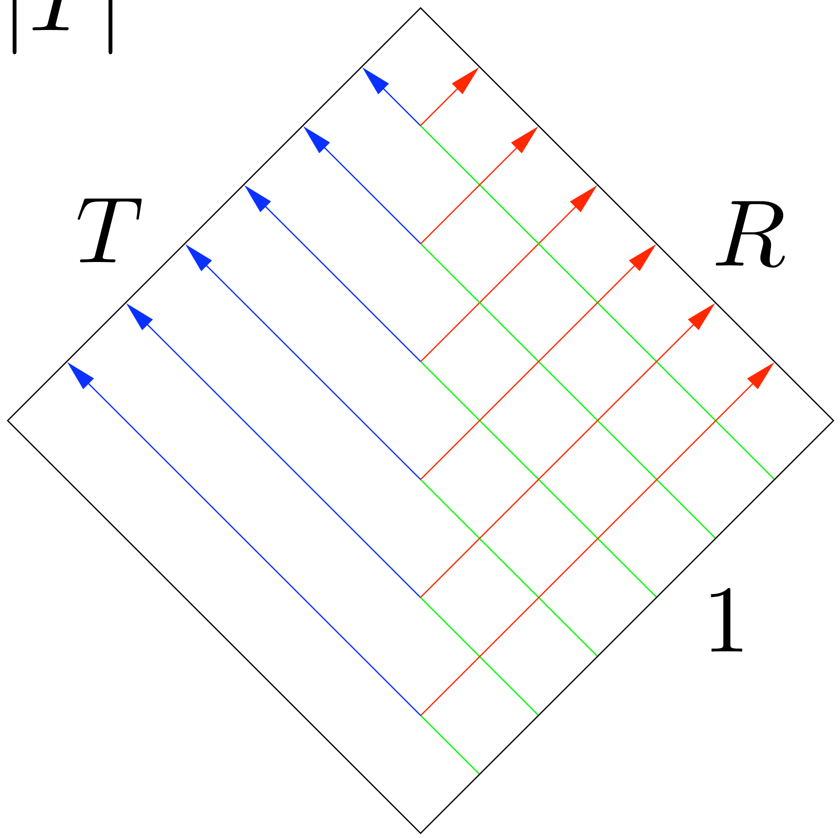

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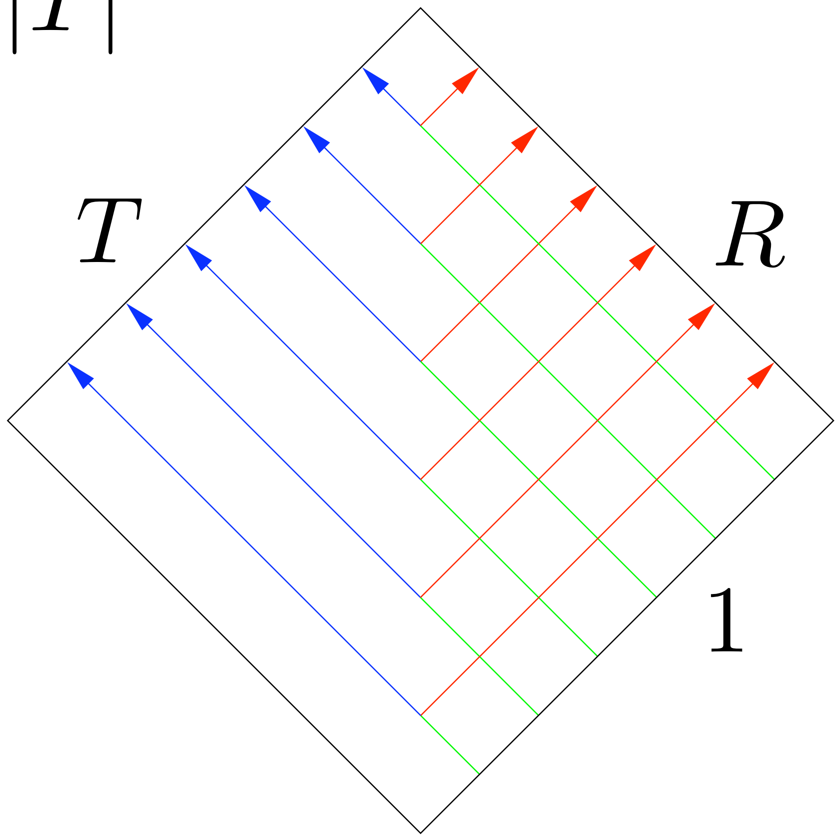

- **Fermions do not satisfy the weak-energy condition -> Area th. does not apply**

States in Schwarzschild

$\tilde{\omega} = \omega$

- **Boulware** $\hat{\phi} = \sum_{\ell,m} \int_0^\infty d\omega \left(\phi_\Lambda^{in} \hat{a}_\Lambda^{in} + \phi_\Lambda^{up} \hat{a}_\Lambda^{up} + \text{n.f.} \right)$

$\hat{a}_\Lambda^{in/up} |B\rangle = 0 \quad \forall \omega > 0$

(1) empty at $\mathcal{I}^\pm$ **(2) diverges at $\mathcal{H}^\pm$** **-> 'cold star'**

States in Schwarzschild

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$$\hat{a}_\Lambda^{in/up} |B\rangle = 0 \quad \forall \omega > 0$$

(1) empty at $\mathcal{I}^\pm$ (2) diverges at $\mathcal{H}^\pm$ -> 'cold star'

- **Unruh** $\hat{\phi} = \sum_{\ell,m} \int_0^\infty d\omega \left(\phi_\Lambda^{in} \hat{a}_\Lambda^{in} + \text{n.f.} \right) + \int_{-\infty}^\infty d\omega \left(\phi_\Lambda^{UP} \hat{a}_\Lambda^{UP} + \text{n.f.} \right)$

$$\hat{a}_\Lambda^{in} |U\rangle = 0 \quad \forall \omega > 0 \qquad \hat{a}_\Lambda^{UP} |U\rangle = 0 \quad \forall \omega \in \mathbb{R}$$

(1) Hawking radiation at $\mathcal{I}^+$ (2) diverges at $\mathcal{H}^-$ ->

-> evaporating black hole

States in Schwarzschild

- **Hartle-Hawking** $\hat{\phi} = \sum_{\ell,m} \int_{-\infty}^{\infty} d\omega \left(\phi_{\Lambda}^{IN} \hat{a}_{\Lambda}^{IN} + \text{n.f.} + \phi_{\Lambda}^{UP} \hat{a}_{\Lambda}^{UP} + \text{n.f.} \right)$

$$\hat{\ }_{\Lambda}^{IN/UP} | \quad \rangle = 0 \qquad \forall \omega \in \mathbb{R}$$

(1) has symmetries of spacetime (2) regular everywhere ->

-> black hole in (unstable) thermal equilibrium at Hawking temperature

$$T_H = \frac{1}{8\pi M}$$

[State used in ‘gauge-gravity duality’]

States in Kerr - bosons

Defined as in Schwarzschild but $\rightarrow$ ~ for 'up' (because of positive norm)

- **'past Boulware'** $\hat{a}^{in}_{\Lambda} \left| B^- \right\rangle = 0 \quad \forall \omega > 0 \qquad \hat{a}^{up}_{\Lambda} \left| B^- \right\rangle = 0 \quad \forall \tilde{\omega} > 0$

(1) empty at $\mathcal{I}^-$ and $\mathcal{H}^-$

(2) has **Unruh-Starobinsky radiation** at $\mathcal{I}^+$ from superradiant modes

-> there is **no state empty** at $\mathcal{I}^{\pm}$

States in Kerr - bosons

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- **'past Boulware'** $\hat{a}_{\Lambda}^{in} \left| B^{-} \right\rangle = 0 \quad \forall \omega > 0 \qquad \hat{a}_{\Lambda}^{up} \left| B^{-} \right\rangle = 0 \quad \forall \tilde{\omega} > 0$

(1) empty at $\mathcal{I}^{-}$ and $\mathcal{H}^{-}$

(2) has **Unruh-Starobinsky radiation** at $\mathcal{I}^{+}$ from superradiant modes

-> there is **no state empty** at $\mathcal{I}^{\pm}$

- **'past Unruh'**

As in Schwarzschild: (1) empty at $\mathcal{I}^{-}$, (2) Hawking radiation at $\mathcal{I}^{+}$

at Hawking temperature: $T_H = \dfrac{r_h^2 - a^2}{4\pi r_h (r_h^2 + a^2)}$

States in Kerr - bosons

- **'Hartle-Hawking'**: constructed by Frolov&Thorne'89.

But Ottewill&Winstanley'00 show it is **ill-defined** everywhere (except on axis of symmetry) due to a pole at $\tilde{\omega} = 0$:

$$\left\langle U^- \right| \hat{\phi}^2 \left| U^- \right\rangle - \left\langle FT \right| \hat{\phi}^2 \left| FT \right\rangle \sim \sum_{\ell,m} \int_0^\infty d\omega \frac{|T|^2}{\omega \left(e^{\tilde{\omega}/T_H} - 1 \right)}, \quad r \to r_h$$

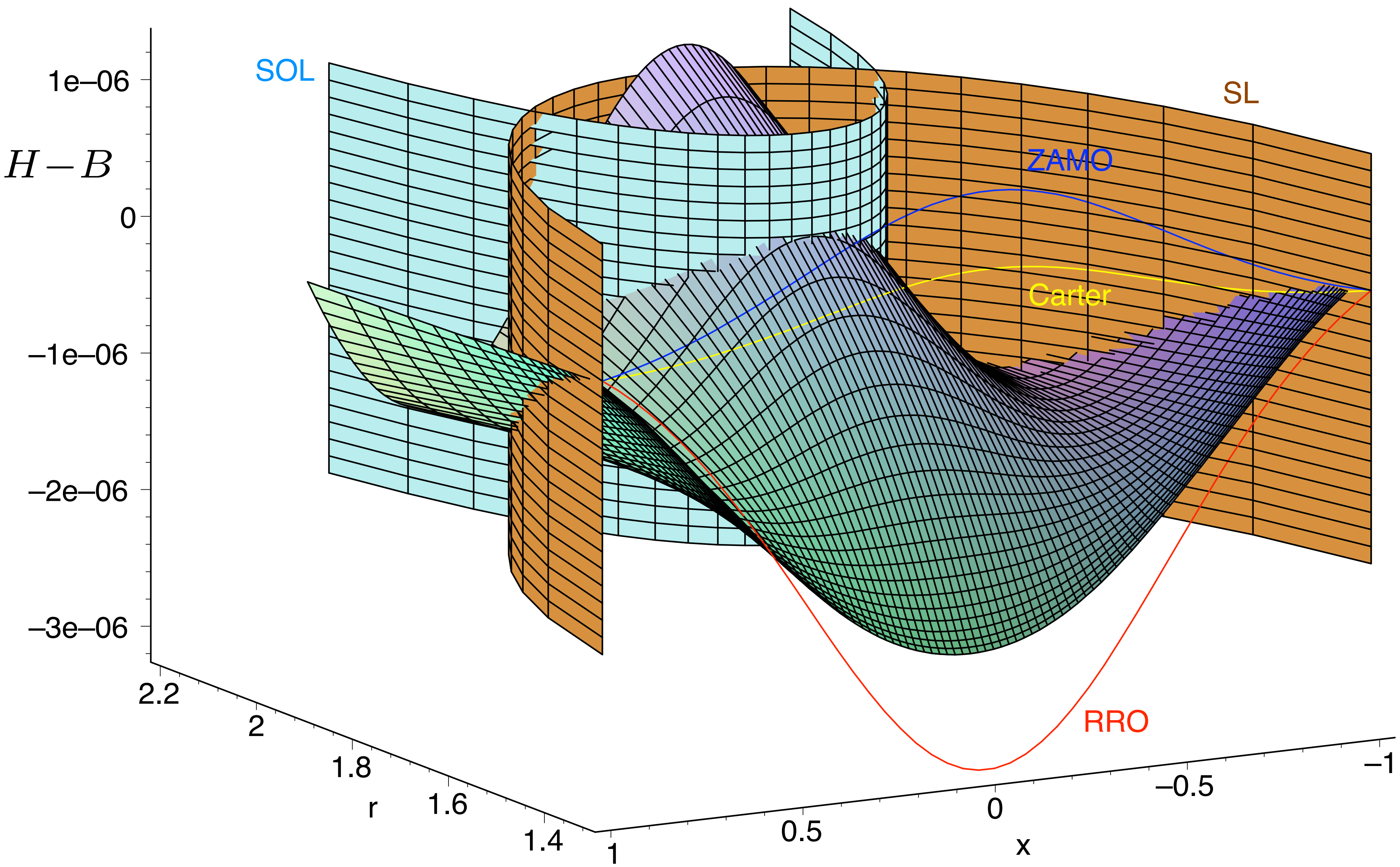
[so its use in Kerr-CFT is not well justified]

States in Kerr - bosons

- **'CCH'** : constructed by Candelas, Chrzanowsky & Howard'81. Modes thermalized wrt their 'natural energy'.

Casals&Ottewill'05: CCH is regular everywhere but does not satisfy symmetries of space-time

$$\Delta \left\langle \hat{T}_{t_+\varphi_+} \right\rangle^{CCH-B}$$



States in Kerr - bosons

- Kay&Wald'91: if there exists a (Hadamard) state which is regular everywhere and has symmetries of spacetime then it is thermal (ie, Hartle-Hawking).

They prove that such a state does not exist in Kerr....but what about fermions?

States in Kerr - fermions

Remember: 'in' & 'up' have positive norm $\forall \omega \in \mathbb{R}$

- **'past Boulware': Similar as for bosons. It has Unruh-Starobinsky radiation (even if fermions have no classical superradiance) at** $\mathcal{I}^+$
- **Boulware:** $\hat{a}^{in}_{\Lambda} |B\rangle = 0 \quad \forall \omega > 0$

  $\hat{a}^{up}_{\Lambda} |B\rangle = 0 \quad \forall \omega > 0$ **(for bosons:** $\forall \tilde{\omega} > 0$ **)**

  Expected to be empty at $\mathcal{I}^{\pm}$ **!**
- **'past Unruh' similar as for bosons (Hawking radiation at** $\mathcal{I}^+$**)**
- **Hartle-Hawking is well-defined!**

  but where is it regular?

Results: Fermions in Kerr

[Casals,Dolan,Nolan,Ottewill,Winstanley'13]

$a = 0.91M$

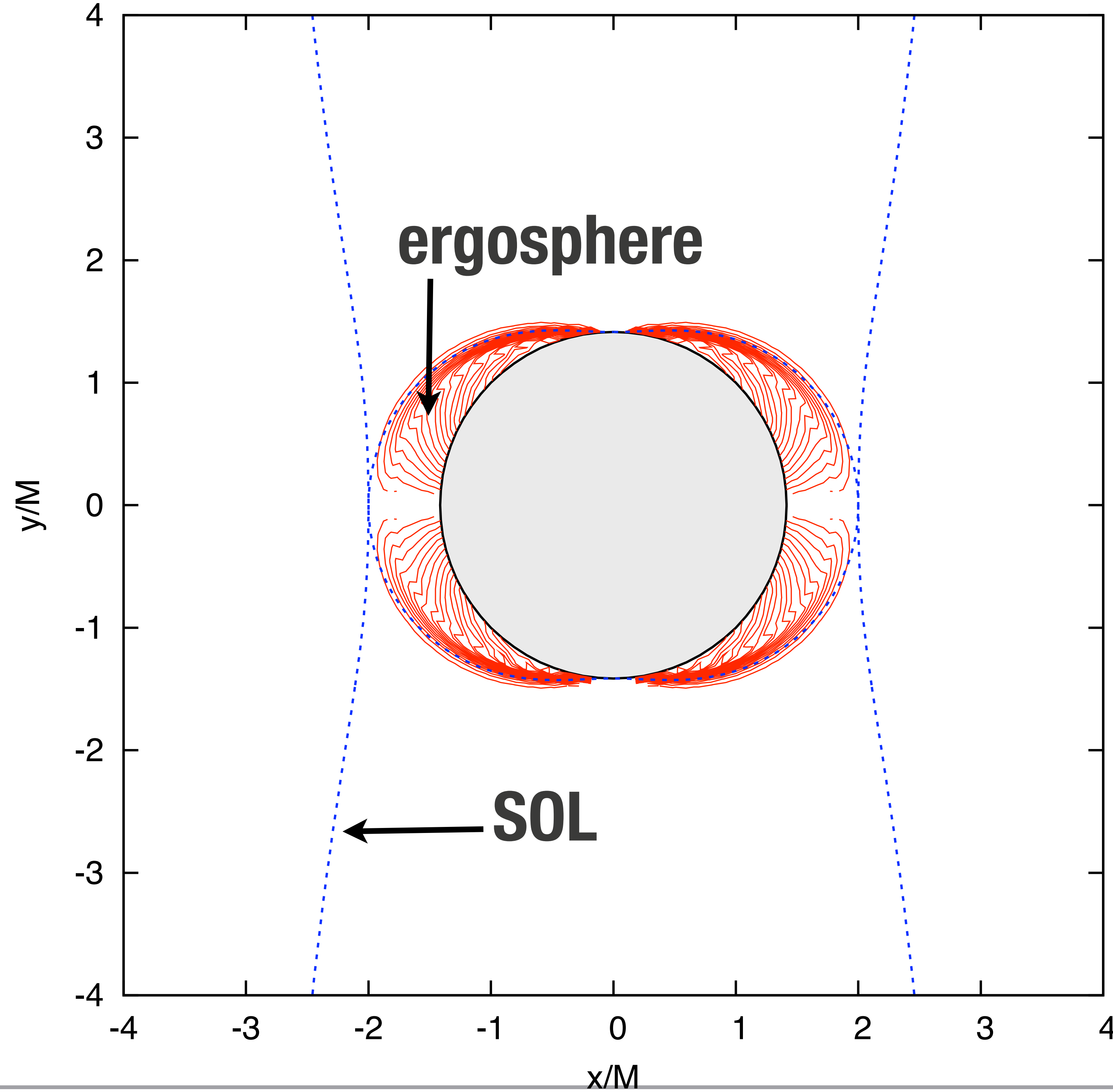
- Ratio test for terms in ℓ-sum for

particle number current

$$\left\langle \hat{J}^r \right\rangle^{U^- - B}$$

- Regular on and outside SOL

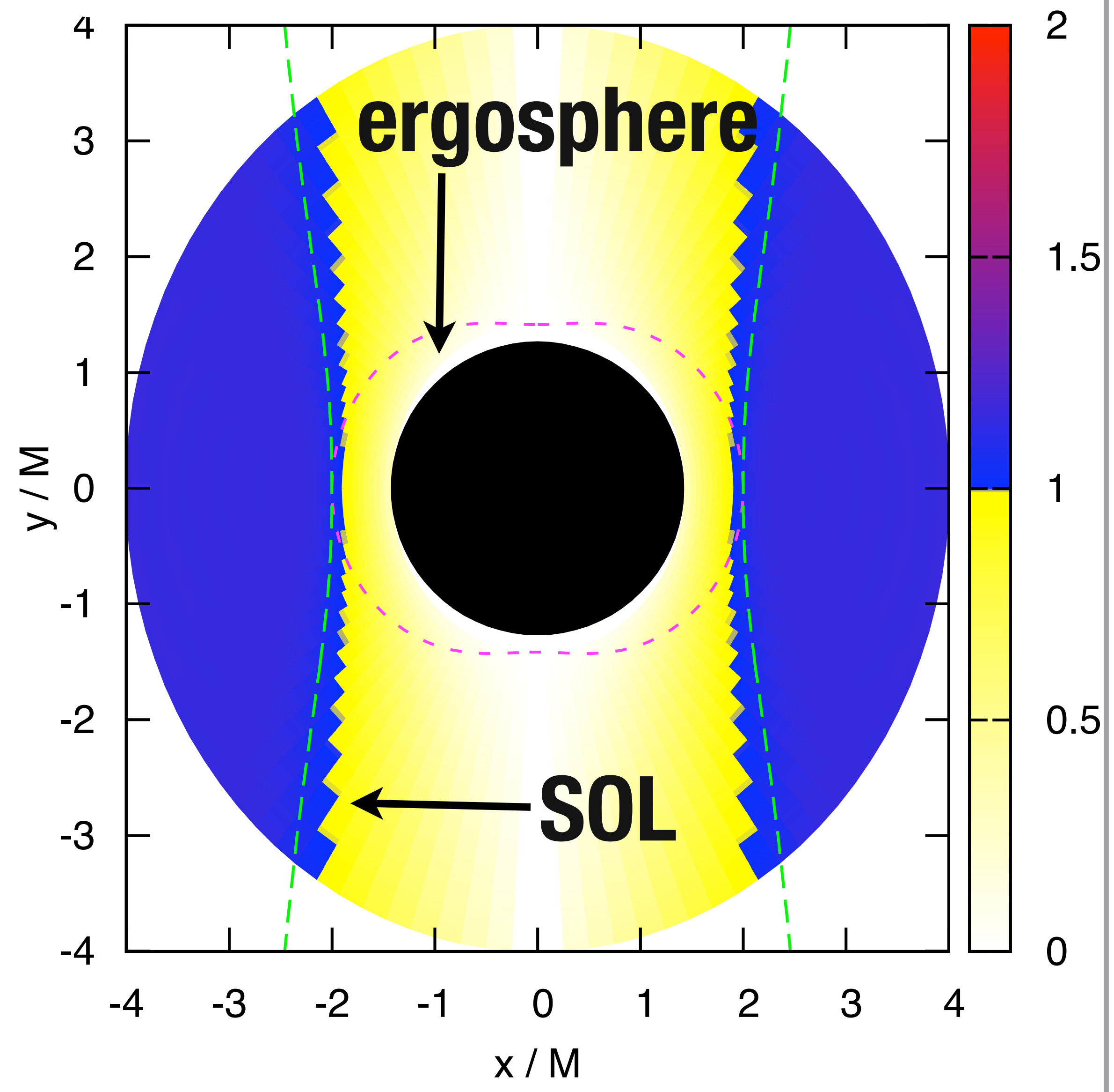
- Divergence in ergosphere due to B-state



Results: Fermions in Kerr

- Ratio test for terms in ℓ -sum for $\left\langle \hat{T}_{\theta\theta} \right\rangle^{H-U}$

- Divergence on SOL due to large- ℓ modes: thermal bath rotating with horizon?

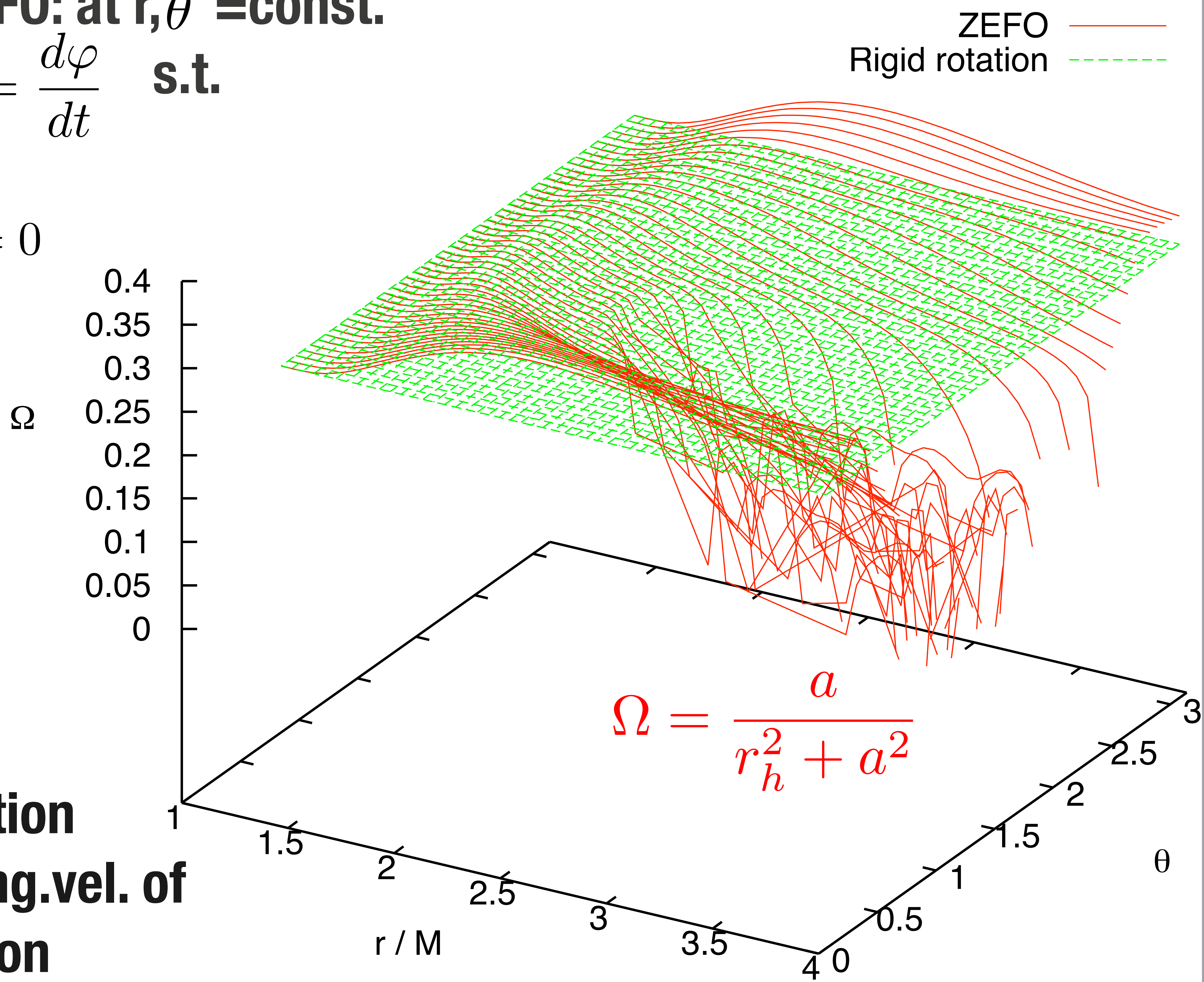


Results: Fermions in Kerr

- **Observer ZEFO: at r, θ =const.**

and $\Omega_{ZEFO} = \frac{d\varphi}{dt}$ **s.t.**

$$\left\langle \hat{T}_{(t)(\varphi)} \right\rangle^{H-\tilde{B}} = 0$$



- **Rate of rotation approaches ang.vel. of b-h near horizon**

Conclusions

- **Bosons** in Kerr:

  - There exists no 'Boulware' state, empty at $\mathcal{I}^{\pm}$

  - 'Hartle-Hawking' state is ill-defined

- **Fermion** modes have positive norm for all frequencies. Therefore:

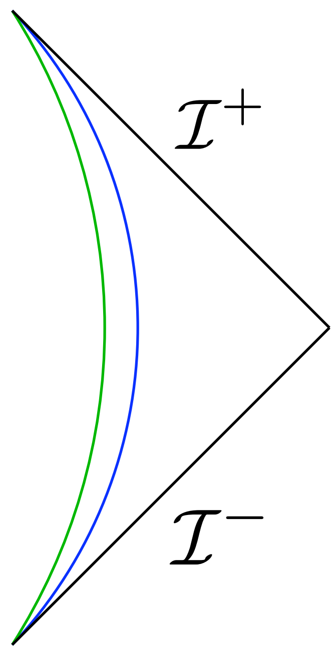
  - First ever construction of state ('Boulware') empty at $\mathcal{I}^{\pm}$
  Diverges in ergosphere

  - First ever construction of thermal state ('Hartle-Hawking')
  'Physical' divergence on SOL surface

- Future project: **QFT on Kerr-AdS**. For some values of cosmological constant and angular velocity there is no SOL

Classical stability for bosons

Mirror outside ergosphere



stable

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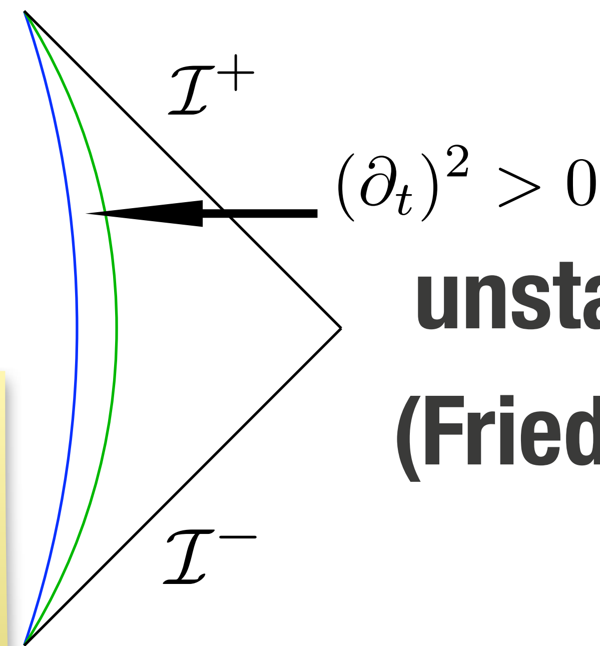

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Also Cardoso,Dias,Lemos \& Yoshida'04, Card \&Dias'04+H\&R'99? aquests miren a classical stability o thermodynamic stability?

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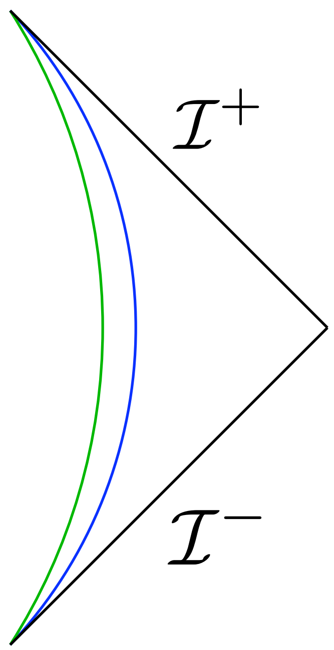

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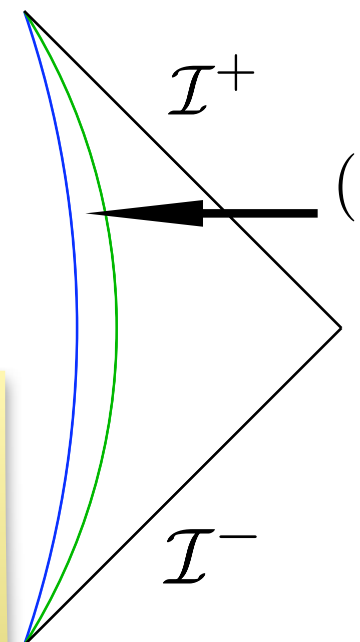

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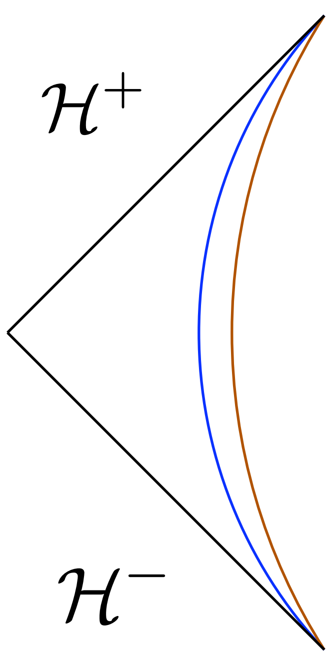

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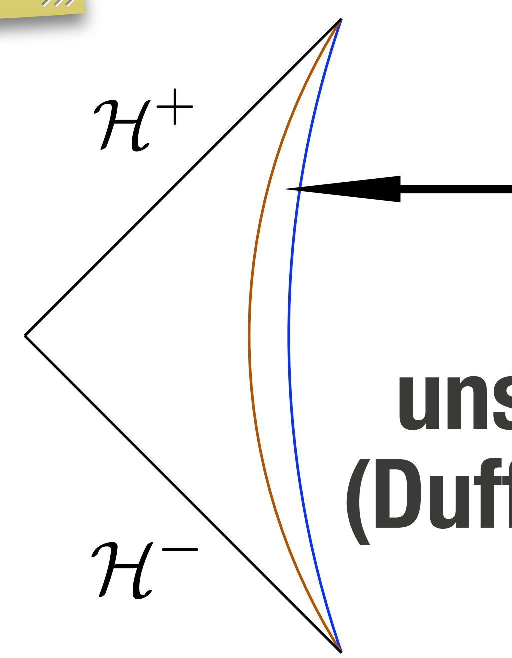

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Solving Dirac eq.

Dirac eq. $\gamma^{\mu}\left(\partial_{\mu}-\Gamma_{\mu}\right)\phi_{\Lambda}=0$

4-spinor: $$\phi_{\Lambda}=\frac{e^{im\varphi-i\omega t}}{4\pi\left[\Delta(r-ia\cos\theta)^{2}\right]^{1/4}}\begin{pmatrix} {}_{-1/2}R_{\Lambda}\cdot{}_{-1/2}S_{\Lambda} \\ \Delta^{1/2}{}_{1/2}R_{\Lambda}\cdot{}_{1/2}S_{\Lambda} \\ {}_{-1/2}R_{\Lambda}\cdot{}_{-1/2}S_{\Lambda} \\ \Delta^{1/2}{}_{1/2}R_{\Lambda}\cdot{}_{1/2}S_{\Lambda} \end{pmatrix}$$

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Decouple and separate by variables into 2nd order linear ODEs for radial and angular parts:

$$\left[\frac{d}{dx}\left((1-x^2)\frac{d}{dx}\right)+(a\omega)^2(x^2-1)-\frac{(m\pm x/2)^2}{1-x^2}+\lambda+a\omega(2m\mp x)\pm\frac{1}{2}\right]{}_{\pm1/2}S_{\Lambda}(x)=0$$

Eigenvalue-λ problem solved with 'shooting method' (5th order Runge-Kutta)

$x\equiv\cos\theta$

Solving Dirac eq.

Dirac eq. $\gamma^\mu \left(\partial_\mu - \Gamma_\mu\right) \phi_\Lambda = 0$

4-spinor: $$\phi_\Lambda = \frac{e^{im\varphi - i\omega t}}{4\pi \left[\Delta (r - ia\cos\theta)^2\right]^{1/4}} \begin{pmatrix} {}_{-1/2}R_\Lambda \cdot {}_{-1/2}S_\Lambda \\ \Delta^{1/2}{}_{1/2}R_\Lambda \cdot {}_{1/2}S_\Lambda \\ {}_{-1/2}R_\Lambda \cdot {}_{-1/2}S_\Lambda \\ \Delta^{1/2}{}_{1/2}R_\Lambda \cdot {}_{1/2}S_\Lambda \end{pmatrix}$$

Decouple and separate by variables into 2nd order linear ODEs for radial and angular parts:

$$\left[\frac{d}{dx}\left((1-x^2)\frac{d}{dx}\right) + (a\omega)^2(x^2-1) - \frac{(m \pm x/2)^2}{1-x^2} + \lambda + a\omega(2m \mp x) \pm \frac{1}{2}\right] {}_{\pm 1/2}S_\Lambda(x) = 0$$

Eigenvalue-λ problem solved with 'shooting method' (5th order Runge-Kutta)

$$x \equiv \cos\theta$$

$$\left[\Delta^{\mp 1/2}\frac{d}{dr}\left(\Delta^{\pm 1/2 + 1}\frac{d}{dr}\right) + \frac{\mp i(r-M)K + K^2}{\Delta} \pm 2i\omega r - \lambda\right] {}_{\pm 1/2}R_\Lambda(r) = 0$$

$$K \equiv (r^2 + a^2)\omega - am$$

Expand about $r = r_h, \infty$ **and integrate using Runge-Kutta**

Classical stability for fermions

stable in all cases!   (CDNOW)