Recent development in random planar maps: exercises for lecture II

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Abstract

The purpose of this exercise session is to introduce the so-called slice decomposition of planar maps, which was introduced in [1] as a reformulation of the BDG bijection bypassing the use of trees. It differs from Tutte's original recursive decomposition in the sense that it allows to write down equations characterizing map generating functions without recourse to a catalytic variable.

1 Quadrangulations

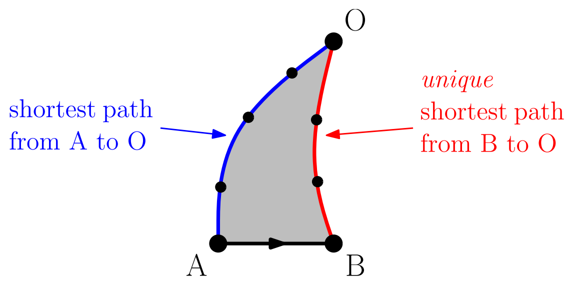

Figure 1: Schematic representation of a slice.

We first concentrate on the simpler case of quadrangulations. We define a *slice* as a quadrangulation with a simple boundary (i.e. a rooted planar map such that every inner face has degree 4, and such that the outer face is simple), satisfying the following further constraint: denoting by A and B respectively the origin and the endpoint of the root edge, there is another boundary vertex O (the *apex*) such that:

- the part of the boundary from A to O not containing B is a geodesic (i.e. a shortest path) between its endpoints,
- the part of the boundary from B to O not containing A is the *unique* geodesic between its endpoints.

See Figure 1 for a schematic representation.

Question 1. Denoting by $d(\cdot, \cdot)$ the graph distance, what is the relation between $d(\mathrm{A}, \mathrm{O})$ and $d(\mathrm{B}, \mathrm{O})$?

The distance $d(\mathrm{A},\mathrm{O})$ is called the *length* of the slice. By convention, the link-map (the map reduced to two vertices connected by a single edge) is considered as a slice of length 1 (with $\mathrm{O}=\mathrm{B}$) and 0 inner faces. For $i \geq 1$, we denote by $r_i \equiv r_i(g)$ the generating function of slices of length i, counted with a weight g per inner face. We furthermore introduce the series

$$R_i = \sum_{j=1}^{i} r_i, \qquad R = \lim_{i\to\infty} R_i = \sum_{j=1}^{\infty} r_i, \tag{1}$$

namely R_i is the generating function of slices of length at most i, and R is the generating function of all slices (note that there is only a finite number of slices with a given number n of faces, hence this series is well-defined).

Question 2. Explain why R is equal to the generating function of pointed rooted quadrangulations, i.e. of rooted quadrangulations (without a boundary) with a distinguished vertex. What is then the interpretation of r_i? (Hint: think about gluing some boundary edges together.)

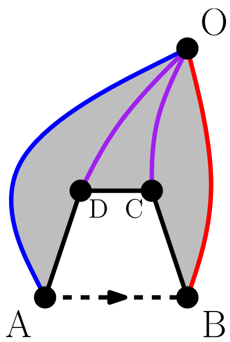

Figure 2: Recursive decomposition of a slice: let ABCD be the face on the left of the root edge, we cut along the leftmost shortest path from C to O, and along that from D to O.

Given a slice with at least one inner face, we decompose it as follows. Consider the face on the left of the root edge, and let C and D be its other incident vertices besides A and B. We then consider the *leftmost shortest path* from C to O, and that from D to O, and cut along them, see Figure 3. The root edge is finally removed.

Question 3. Show that this decomposition splits the slice into, generically, two (and not three !) pieces which are themselves slices (hint: consider the possible distances from C and D to O). Deduce that

$$R = 1 + 3gR^2. \tag{2}$$

Question 4. In this decomposition, how are the length of the subslices related to that of the original one? Deduce that, for $i \geq 1$,

$$R_i = 1 + gR_i(R_{i-1} + R_i + R_{i+1}) \tag{3}$$

with $R_0 = 0$.

Question 5. Let $F_p^\bullet$ denote the generating function of quadrangulations with a (not necessarily simple) boundary of length $2p$ and with a distinguished vertex (not necessarily on the boundary). By adapting the above slice decomposition, show that

$$F_p^\bullet = \binom{2p}{p} R^p. \tag{4}$$

Using Lagrange inversion, deduce an explicit expression for $[g^n]F_p^\bullet$.

2 $2m$-angulations

Fix an integer $m \geq 2$: we define a slice as before, except that inner faces are now assumed to have degree $2m$ instead of 4.

Question 6. By a slight adaptation of the slice decomposition for quadrangulations, show that R satisfies

$$R = 1 + g\binom{2m-1}{m} R^m. \tag{5}$$

Write down the equation satisfied by R_i in the case $m = 3$. Show that (4) holds unchanged.

3 Triangulations

We want to adapt the slice decomposition to the case of triangulations, which are generically not bipartite. Still we keep the same definition for a slice, except that inner faces are now assumed to have degree 3.

Question 7. How is the answer to Question 1 modified in the case of triangulations?

We denote by r_i (resp. s_i) the generating function of slices with $i = d(\mathrm{A}, \mathrm{O}) = d(\mathrm{B}, \mathrm{O}) + 1$ (resp. $i = d(\mathrm{A}, \mathrm{O}) = d(\mathrm{B}, \mathrm{O})$). We also define R_i and R as in (1), and let

$$S_i = \sum_{j=0}^{i} s_i, \qquad S = \lim_{i\to\infty} S_i = \sum_{j=0}^{\infty} s_i. \tag{6}$$

Question 8. By adapting the slice decomposition, show that

$$R = 1 + 2gRS, \qquad S = 2R + S^2 \tag{7}$$

and that

$$R_i = 1 + gR_i(S_{i-1} + S_i), \qquad S_i = R_{i+1} + R_i + S_i^2. \tag{8}$$

4 Further reading

We may of course adapt the slice decomposition to maps containing inner faces of arbitrary (and varying) degrees. Interestingly, it is also possible to adapt it to handle maps with connectivity (girth, etc.) constraints [2].

References

[1] J. Bouttier and E. Guitter, *Planar maps and continued fractions*, Comm. Math. Phys. 309 (2012) 623–662, arXiv:1007.0419 [math.CO].

[2] J. Bouttier and E. Guitter, *On irreducible maps and slices*, to appear in Combinatorics, Probability and Computing (2014), arXiv:1303.3728 [math.CO].