School on Gravitational Waves: from data to theory and back

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Effective field theory methods to model astrophysical binaries

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Plan of the course

- L1: EFT and compact binaries
  - hierarchy of scales
  - “integrating out” degrees of freedom
  - other mehods (overview)
- L2: conservative dynamics I
  - Feynman rules
  - systematic Post-Newtonian expansion
  - momentum integration and examples
- L3: conservative dynamics II
  - Energy in the center of mass frame
  - Spin
  - 3PN and beyond
- Exercise Session: automated computation
- L4: radiative effects
  - Power emission, backreaction, and waveform
  - Matching
  - Tail and memory effects

Effective field theory

Describes physics at a given scale, giving up UV details

It works well in presence of a hierarchy of scales

all operators compatible with the symmetries of the system

$$\mathcal{L} = \sum c_i \mathcal{O}_i$$

organized in powers of $\frac{l_{UV}}{L}$

coefficients depending on UV physics:

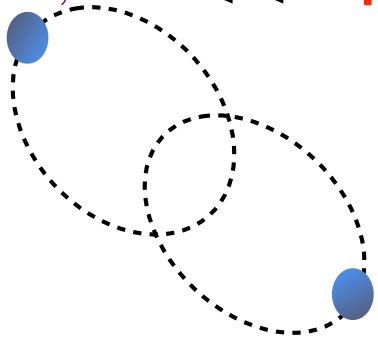
they can be derived from a microscopic theory by integrating out the UV degrees of freedom

$\mathcal{L}$ itself can be used to derive another effective lagrangian at a larger scale

Length scales in a binary system: a double hierarchy

r <<* $\lambda_{GW} \simeq \frac{r}{v}$

$l_{BH,NS}$ <<* r



in both cases
a multipole expansion is appropriate
to describe the interaction of the
source with the gravitational field:

$$S_m[x,h] = -\int \mathrm{d}\tau \; m + \frac{1}{2}\mathcal{S}_{\mu\nu}\Omega^{\mu\nu} - \frac{1}{2}I_{ij}E^{ij} + \frac{2}{3}J_{ij}B^{ij} + \dots$$

c_i $\mathcal{O}_i$

*during the inspiral phase

$$S_m[x,h] = -\int \mathrm{d}\tau \left[m + \frac{1}{2}\mathcal{S}_{\mu\nu}\Omega^{\mu\nu} + I_{ij}E^{ij} + J_{ij}B^{ij} + c_E E^{ij}E_{ij} \dots \right]$$

- Symmetries: general covariance, parity and rotations
- No Ricci tensor (vanishes outside the source)
- Hierarchy of terms:

$$I_{ij} \simeq m r_i r_j$$
$$E_{ij} = C_{\mu i \nu j} u^\mu u^\nu \quad \Longrightarrow \quad I_{ij}E^{ij} \sim m \frac{|_s^2}{\llcorner_h^2}$$

- Also terms quadratic in curvature tensors allowed:

$$c_E E_{ij}E^{ij} \qquad \text{tidal effect}$$

Integrating out
the Newtonian gravitational field in the near zone

$$S_{tot}[x,h] = S_g[h] + S_m[x,h] \qquad e^{iS_{eff}[x]} = \mathcal{D}[h]e^{iS_{tot}[x,h]}$$

Warning: the EFT approach borrows many techniques and the language of **Quantum Field Theory**, but everything is **classical** here. The integral above is **not** a Path Integral, it is rather a symbolic way to represent the solution of the classical equation of motion for h.

Dictionary:

integrate out	solve classical equations of motion
propagator	Green's function
graviton (on shell, off shell)	classical gravity field (GW, Newtonian)
renormalization	renormalization

EFT is a good tool to learn about field theory in general

For example:

$$S_g \simeq -4 \int \mathrm{d}^4 y \phi! \ \phi \qquad S_m \simeq \int \mathrm{d}^4 y \delta^3(y - x(t)) m \left[\frac{1}{2} v(t)^2 - \sqrt{32\pi G}\phi \right]$$

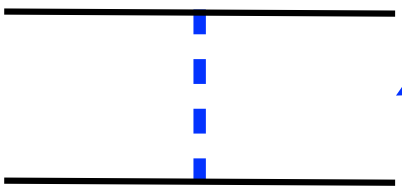
$$! \ \rightarrow ! \ - \frac{1}{8} \Delta^{! \ 1} \left(\sqrt{32" \ G} m_1 \#_1^3 \right) - \Delta^{! \ 1} \left(\frac{1}{8} \sqrt{32" \ G} m_2 \#_2^3 \right)$$

$$S_{tot} \simeq -4 \int d^4 y \phi \Delta \phi - \sum \delta_i^3 \frac{m_i}{2} v_i(t)^2 - 2\pi G \left(\sum \delta_i^3 m_i \right) \Delta^{-1} \left(\sum \delta_i^3 m_i \right)$$

$$\Delta^{-1} f(y) = \int \mathrm{d}^3 z \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathrm{e}^{ip.(y-z)} \frac{1}{-p^2} f(z)$$

Green's function

$$S_{eff} \simeq \sum \delta_i^3 \frac{m_i}{2} v_i(t)^2 + 4\pi G m_1 m_2 \int \mathrm{d}t \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathrm{e}^{ip.(x_1 - x_2)} \frac{1}{p^2} + \sum 2\pi G m_i^2 \int \mathrm{d}t \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{p^2}$$



UV divergent!

Newton's potential:

$$-\frac{G m_1 m_2}{r}$$

What about this one?

The self-interaction terms are UV divergent because the sources have been modeled as pointlike.
However these divergences are non-physical because they can be reabsorbed into redefinition of the masses. Just like in classical electrodynamics.

Dimensional regularization is an efficient way to discard this kind of effects by setting to zero all power-law divergencies:

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{p^2} = 0$$

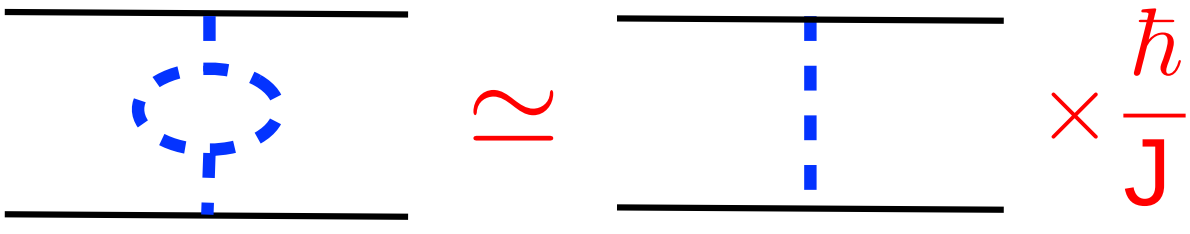
(Logarithmic divergencies can carry physical informations,
and, as we will se later on, they are accounted for in dim.reg.)

To summarize, so far:

$$\int \mathcal{D}[\phi] \mathrm{e}^{iS_{tot}[x,\phi]} \to \mathrm{e}^{iS_{Newton}[x]} \int \mathcal{D}[\phi] \mathrm{e}^{iS_g[\phi]}$$

No “quantum” integration have been done in the gravitational sector:

no loops



So far it was easy. But let's consider what happens when a term mimicking gravitational nonlinearities is added:

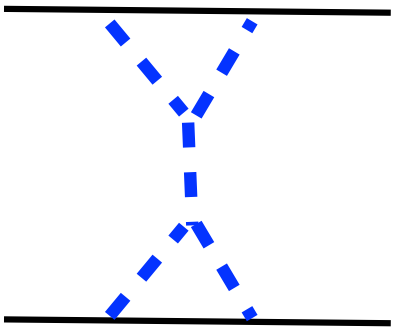
$$S_g \simeq \int \mathrm{d}^4 y \; -4\phi\Delta\phi - \sqrt{G}\mu^2\phi^3$$

$$\sqrt{G}\mu^2\phi^3 \to \mu^2 \left\{ \sqrt{G}\phi^3 + G\phi^2\Delta^{-1}(\delta m) + \sqrt{G^3}\phi[\Delta^{-1}(\delta m)]^2 + G^2[\Delta^{-1}(\delta m)]^3 \right\}.$$

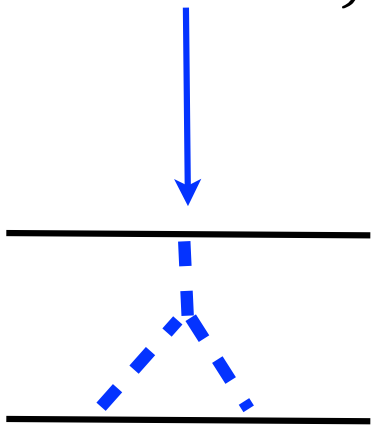
The linear term is removed by:

$$\phi \to \phi - \mu^2\sqrt{G^3}\Delta^{-1}\left[\Delta^{-1}(\delta m)\right]^2$$

and such process generates also:



and similar ones, and so on.....



a new effective interaction:

$$\frac{G^2\mu^2 m_1^2 m_2}{r^3}$$

+ other terms which vanish in dim. reg.

Looks like a complicated procedure, but it has already been met and solved in Quantum Field Theory, by means of

Wick's contractions
and systematic perturbative expansion in terms of Feynman diagrams

Having established that we are well within **classical** physics, we can nevertheless use the power of QFT formalism and compute S_eff as:

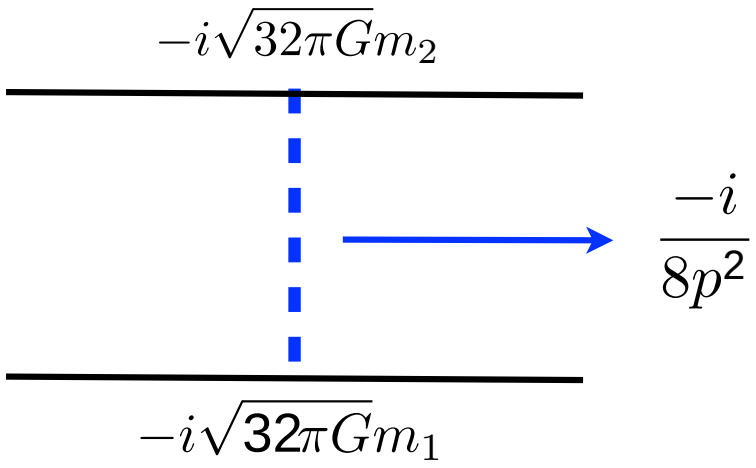
$$iS_{eff}[x] = \log \int \mathcal{D}[h] e^{iS_{tot}[x,h]} \simeq \langle e^{iS_{int}[x,h]} \rangle_{connected\ graphs}$$

The various contributions are organized in a post-Minkoskian perturbative series with expansion parameter $\sqrt{G}$ or, better: $\frac{Gm_{1,2}}{r} \simeq \frac{R_s}{r}$

G G^2 G^3

each vertex gets a $\sqrt{G}$ factor

To summarize, Newton's potential again:

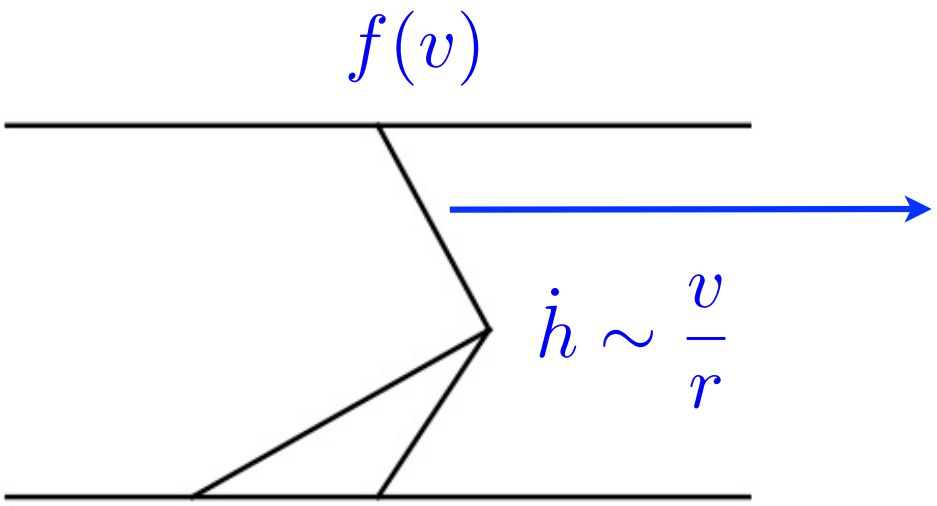


$$V = i\int \frac{\mathrm{d}^3p}{(2\pi)^3}(-i\sqrt{32\pi G}m_1)(-i\sqrt{32\pi G}m_2)\frac{-i}{8p^2}\mathrm{e}^{ip.(x_1-x_2)}$$

maybe too much formalism so far for such a simple result,
but it will be very useful when things will become more complicated

Because of the virial theorem $v^2 \simeq \dfrac{Gm}{r}$

velocity factors must be considered consistently at any given post-Minkoskian order,



$$\Box \sim \Delta + \mathcal{O}(v^2)$$

$$h(t-r) \sim h(t) - r\dot{h}$$

thus turning the post-Minkoskian expansion into the Post Newtonian one

nPN: $\mathcal{O}(G^n, v^{2n})$ wrt Newtonian

Other ways of doing the same physics

(good to have more than one, because at high PN calculations become very involved)

Direct perturbations of (Einstein + geodesic) equations

relaxed Einstein equations: $\mathbf{h}^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$

$$\Box \mathbf{h}_{\mu\nu} = -16\pi G \tau_{\mu\nu}[\mathbf{h}] \quad \text{with} \quad \tau_{\mu!}[\mathbf{h}] = (-g)T^{\mu!} + \Lambda^{\mu!}[\mathbf{h}]$$

$$T^{\mu\nu}(t,y) = \frac{1}{\sqrt{-g}} \gamma_1 m_1 \dot{x}_1^\mu \dot{x}_1^\mu \delta^3(y - x_1(t))$$

once the metric generated by particle 1 is known at a given PN order,
it is inserted into the geodesic equations for particle 2:

$$\ddot{x}_2^i = -\Gamma^i_{\mu\nu} \dot{x}_2^\mu \dot{x}_2^\nu$$

ADM formalism: Hamiltonian formulation of GR

3+1 splitting

$$g_{\mu\nu} \to \{\gamma_{ij}, N, N^i\}$$

$$\pi^{ij} \equiv \frac{\delta L_g}{\delta \dot{\gamma}_{ij}}$$

Hamiltonian constraints

$$\frac{1}{\sqrt{\gamma}}\left[\gamma\ {}^{(3)}R + \frac{1}{2}\left(\gamma_{ij}\pi^{ij}\right)^2 - \gamma_{ij}\gamma_{kl}\pi^{ik}\pi^{jl}\right] = \sum \left(m_a^2 + \gamma_{ij}p_a^i p_a^j\right)\delta^3(x - x_a)$$

$$-2\pi^{ik}_{;k} = \sum \gamma_{ik} p_k^a \delta^3(x - x_a)$$

Gauge fixing and variables decomposition

$$\gamma_{ij} \,!\, \frac{1}{3}\gamma_{kk}\delta_{ij}{}_{,j} = 0 \qquad !_{ij} \equiv (1 + \frac{"}{8})^4 \#_{ij} + h^{TT}_{ij}$$

$$\pi^{ii} = 0 \qquad \pi^{ij} \equiv \pi^{ij}_{TT} + \ldots$$

solved **implicitly**

and perturbatively

in the PN expansion

$$H_{ADM} = -\int \mathrm{d}^3x \Delta\phi\left[\vec{x}_a, \vec{p}_a, h^{TT}_{ij}, \pi^{ij}_{TT}\right] + \sum_a \vec{p}_a \cdot \dot{\vec{x}}_a$$

$$\dot{h}^{TT}_{ij}[\vec{x}_a, \vec{p}_a] = \frac{\partial H_{ADM}}{\pi^{ij}_{TT}}$$

$$\dot{\pi}^{ij}_{TT}[\vec{x}_a, \vec{p}_a] = -\frac{\partial H_{ADM}}{h^{TT}_{ij}}$$

matter-only Hamiltonian

$$H[\vec{x}_a, \vec{p}_a]$$

(can replace solutions directly in the Hamiltonian, center-of-mass frame easy to implement)

Self-force: perturbation around Schwarzschild metric

expansion parameter $\dfrac{m_1}{m_2}$

Lowest order solution of Einstein equations:

$$G_{\mu\nu}[g^{(0)}_{\mu\nu}] = 8\pi G T_{\mu\nu}[m_1]$$

$g^{(0)}_{\mu\nu}$ metric of a static BH of mass m_1

geodesic equation for m_2 $\longrightarrow$ $x^\mu_2(t)$

First order perturbation of Einstein equations:

$$G_{\mu\nu}[g^{(0)}_{\mu\nu} + h_{\mu\nu}] = 8\pi T_{\mu\nu}[m_1, m_2, x_2]$$

the metric perturbation can be decomposed into $h_{\mu\nu} = h^S_{\mu\nu} + h^R_{\mu\nu}$

in such a way that $G_{\mu\nu}[g^{(0)}_{\mu\nu} + h^R_{\mu\nu}] = 8\pi T_{\mu\nu}[m_1]$

now a new geodesic equation for m_2 in the metric $g^{(0)}_{\mu\nu} + h^R_{\mu\nu}$ is computed
and one can move to the next perturbative step

S-matrix formalism

massless spin 2
+
unitary S-matrix
+
Lorentz invariance

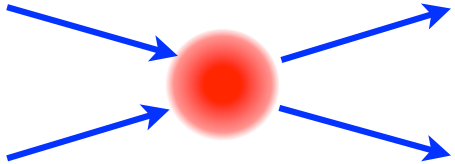
coupling to
conserved
energy momentum
+
equivalence
principle

Einstein
equations

compute effective potentials without passing through a Lagrangian

(off shell S-matrix elements derived from on-shell ones using unitarity)

compute S-matrix
for a given
scattering process



match the result to
an appropriate
effective action:

gravity $V_{p,q}\ \phi^\dagger(p+q)\phi(p-q)\phi^\dagger(-p-q)\phi(-p+q)$

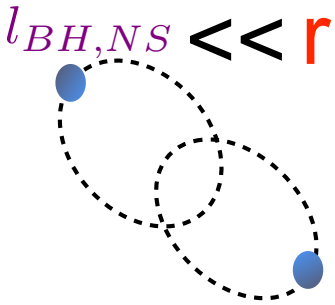
matter

use only gauge invariant quantities
work in the center of mass frame

simplified calculations

Detailed calculation of conservative dynamics

We are in the near zone:



(quasi-)Newtonian field $\nabla h \sim \frac{1}{r}$

(as opposed to GW: ! h " $\frac{v}{r}$)

$$S_m[x,h] = -\int \mathrm{d}\tau \left[m + \frac{1}{2}\mathcal{S}_{\mu\nu}\Omega^{\mu\nu} - \frac{1}{2} I_{ij} E^{ij} + \frac{2}{3} J_{ij} B^{ij} + c_E E^{ij} E_{ij} + \dots \right]$$

geometrically suppressed

5PN tidal effect

$E_{ij} \sim \frac{m}{r^3}$

$c_E \sim G l_{NS}^5$

Q ! $\epsilon m l_{NS}^2$ $\qquad \epsilon_{intrinsic} \simeq 10^{-8}$ << ! $_{tidal} \sim \left(\frac{l_{ns}}{r}\right)^3$

No spin (for the moment)

up to 4PN:

$$S_{m_a} \simeq S^a_{pp} = -m_a \int \sqrt{-g_{\mu\nu} \mathrm{d}x^\mu_a \mathrm{d}x^\nu_a} = -m_a \int \mathrm{d}t \sqrt{-g_{\mu\nu} \dot{x}^\mu_a \dot{x}^\nu_a} \qquad \mathsf{a} = 1, 2$$

$$\mathsf{S}_g = \frac{1}{32\pi \mathsf{G}} \int \mathrm{d}^{(d+1)}\mathsf{x} \sqrt{-\mathsf{g}} \left[\mathsf{R} - \mathsf{g}_{\mu\nu} \mathsf{g}^{\alpha\beta} \mathsf{g}^{\gamma\delta} \Gamma^\mu_{\alpha\beta} \Gamma^\nu_{\gamma\delta} \right]$$

gauge fixing term

KK variables

$$g_{\mu\nu} = e^{2\phi} \begin{pmatrix} -1 & A_j \\ A_i & e^{-c_d \phi}(\delta_{ij} + \sigma_{ij}) - A_i A_j \end{pmatrix} \qquad c_d = \frac{2(d-1)}{d-2}$$

$$\phi \;\; A_i \;\; \sigma_{ij} \longrightarrow \frac{\phi \;\; A_i \;\; \sigma_{ij}}{m_p} \qquad m_p \equiv \frac{1}{\sqrt{32\pi G}}$$

$$S^{\mathsf{a}}_{pp} = -m_{\mathsf{a}} \int \mathrm{d}\tau_{\mathsf{a}} = -m_{\mathsf{a}} \int \mathrm{d}t \; e^{\phi/m_{Pl}} \sqrt{\left(1 - \frac{A_i}{m_{Pl}} v^i_{\mathsf{a}}\right)^2 - e^{-c_d \phi/m_{Pl}} \left(v^2 + \frac{\sigma_{ij}}{m_{Pl}} v^i_{\mathsf{a}} v^j_{\mathsf{a}}\right)}$$

$$S_g \simeq \int \mathrm{d}^{d+1}x\sqrt{-\gamma}\left\{\frac{1}{4}\left[(\vec{\nabla}\sigma)^2 - 2(\vec{\nabla}\sigma_{ij})^2 - \left(\dot{\sigma}^2 - 2(\dot{\sigma}_{ij})^2\right)\mathrm{e}^{\frac{-c_d\phi}{m_{Pl}}}\right] - c_d\left[(\vec{\nabla}\phi)^2 - \dot{\phi}^2\mathrm{e}^{-\frac{c_d\phi}{m_{Pl}}}\right]\right.$$
$$+\left[\frac{F_{ij}^2}{2} + \left(\vec{\nabla}\cdot\vec{A}\right)^2 - \dot{\vec{A}}^2\mathrm{e}^{-\frac{c_d\phi}{m_{Pl}}}\right]\mathrm{e}^{\frac{c_d\phi}{m_{Pl}}} + 2\frac{\left[F_{ij}A^i\dot{A}^j + \vec{A}\cdot\dot{\vec{A}}(\vec{\nabla}\cdot\vec{A})\right]\mathrm{e}^{\frac{c_d\phi}{m_{Pl}}} - c_d\dot{\phi}\vec{A}\cdot\vec{\nabla}\phi}{m_{Pl}}$$
$$+2c_d\left(\dot{\phi}\vec{\nabla}\cdot\vec{A} - \dot{\vec{A}}\cdot\vec{\nabla}\phi\right) + \frac{1}{m_{Pl}}\left[-\dot{\sigma}A_i\hat{\Gamma}^i_{jj} + 2\dot{\sigma}_{ij}\left(A_k\hat{\Gamma}^k_{ij} - A_i\hat{\Gamma}^j_{kk}\right)\right] - c_d\frac{\dot{\phi}^2\vec{A}^2}{m_{Pl}^2}$$
$$+\frac{1}{m_{Pl}}\sigma^{ij}\left(\frac{1}{2}\sigma_{kl,i}\sigma^{kl}_{,j} + \sigma_{ik,l}\sigma^{k,l}_{j} + \sigma_{ik,l}\sigma^{l,k}_{j} - \sigma^{k}_{i,k}\sigma^{l}_{j,l} + \sigma_{,i}\sigma^{k}_{j,k} - \frac{1}{2}\sigma_{ij,k}\sigma^{,k} - \sigma_{ik,j}\sigma^{,k} - \frac{1}{4}\sigma_{,i}\sigma_{,j}\right)$$
$$\left.+\frac{1}{2m_{Pl}}\sigma\left(\frac{1}{4}\sigma_k\sigma k + \sigma^{ki}_{,i}\sigma^{,j}_{kj} - \sigma_{ki,j}\sigma^{kj,i} - \frac{1}{2}\sigma_{ki,j}\sigma^{ki,j}\right)\right\} + \mathcal{O}(5PN)$$

It will be useful to work in momentum space:

$$!_k(t) \equiv \int \mathrm{d}^d x ! (t,x)\mathrm{e}^{-ik.x}$$

Propagators:

$$\langle \phi_k(t)\phi'_k(t')\rangle = -\frac{1}{2c_d}(2\pi)^d\delta^d(k+k')\frac{i}{k^2-\partial_t\partial_{t'}}\delta(t-t')$$

$\partial_t\partial_{t'} \to v^2$

$$\langle A_i A_j\rangle \qquad -\delta_{ij}$$

$$\langle \sigma_{ij}\sigma_{wl}\rangle \quad \delta_{iw}\delta_{jl}+\delta_{il}\delta_{jw}+(2-c_d)\delta_{ij}\delta_{wl}$$

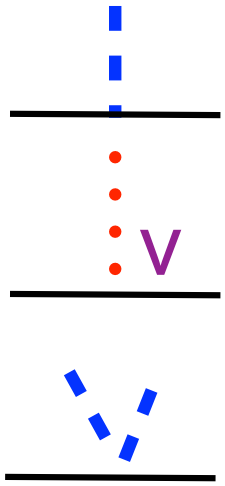
From retarded to instantaneous potentials

$$\frac{1}{k^2-\partial_t\partial'_t} \simeq \frac{1}{k^2}\left(1+\frac{\partial_t\partial'_t}{k^2}+\dots\right)$$

$$h(t-r) \sim h(t) - r\dot{h}$$

Other Feynman rules

matter interaction

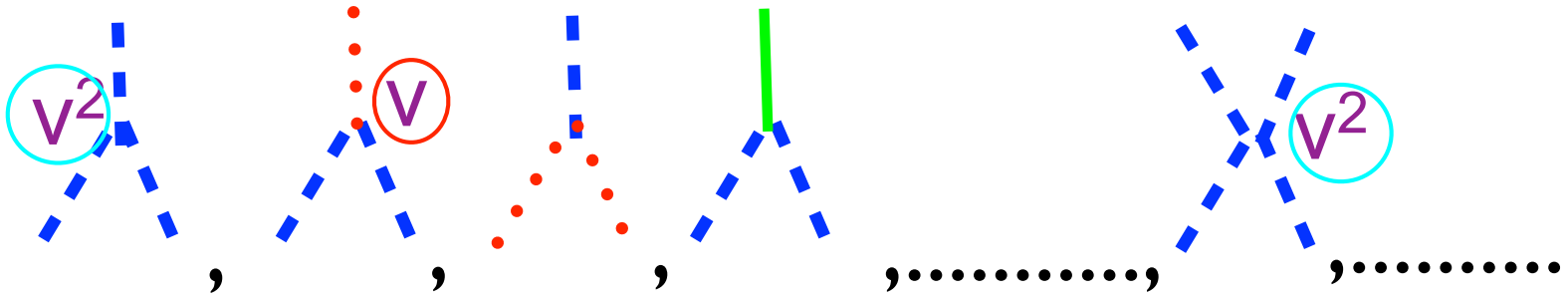


$$m_a\sqrt{G}\varphi\frac{2+(c_d!\ \ 2)v^2}{2\quad 1!\ \ v^2}$$

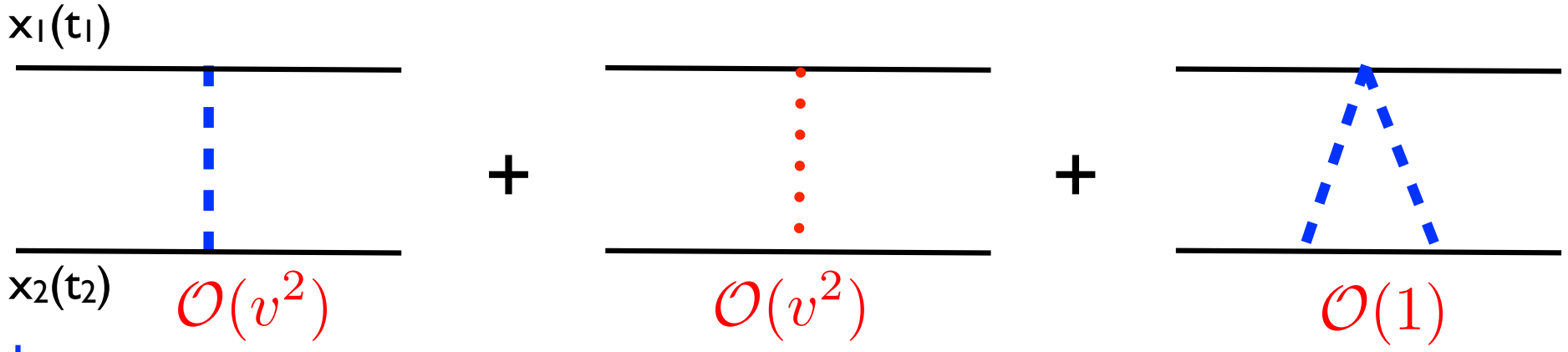
$$-m_a\sqrt{G}A_i\frac{v^i}{\sqrt{1-v^2}}$$

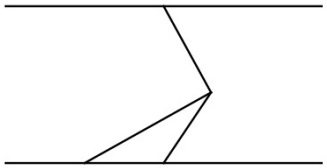
$$m_a G\,\phi^2\,\frac{4-2(c_d^2-2c_d-2)v^2+(c_d-2)^2v^4}{\sqrt{1-v^2}^3}$$

gravity self-interactions ($\Phi^3,\Phi^2A,\Phi A^2,\Phi^2\sigma,\ldots\ldots,\Phi^4,\ldots\ldots$):



1PN:



while all graphs of this kind  are suppressed by v factors so they can be neglected

$$S_{eff}[x_1,x_2] = i\int_{t_1,t_2}\left(\frac{im_1V_\phi(t_1)}{m_p}\right)\left(\frac{im_2V_\phi(t_2)}{m_p}\right)\langle\phi(t_1,x_1)\phi(t_2,x_2)\rangle$$

$$= \frac{32\pi G m_1 m_2}{8}\int_{t_1,t_2,k}\left[\frac{1}{k^2-\partial_{t_1}\partial_{t_2}}\delta(t_1-t_2)\right]V_\phi(t_1)V_\phi(t_2)e^{ik.(x_1(t_1)-x_2(t_2))}$$

Expanding at order v^2

$$V_\phi = \frac{1+v^2}{\sqrt{1-v^2}}$$

$$\simeq 4\pi G m_1 m_2 \int_{t_1,t_2,k} \left[\left(\frac{1}{k^2}(1+\frac{3}{2}v_1^2)(1+\frac{3}{2}v_2^2) + \frac{\partial_{t_1}\partial_{t_2}}{k^4} \right) \delta(t_1-t_2) \right] e^{ik.(x_1(t_1)-x_2(t_2))}$$

IR divergent:

OK in dimensional regularization

$$\int_{\mathbf{p}} \frac{e^{-i\mathbf{p}\mathbf{r}}}{p^{2\alpha}} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)} \left(\frac{2}{r}\right)^{d-2\alpha}$$

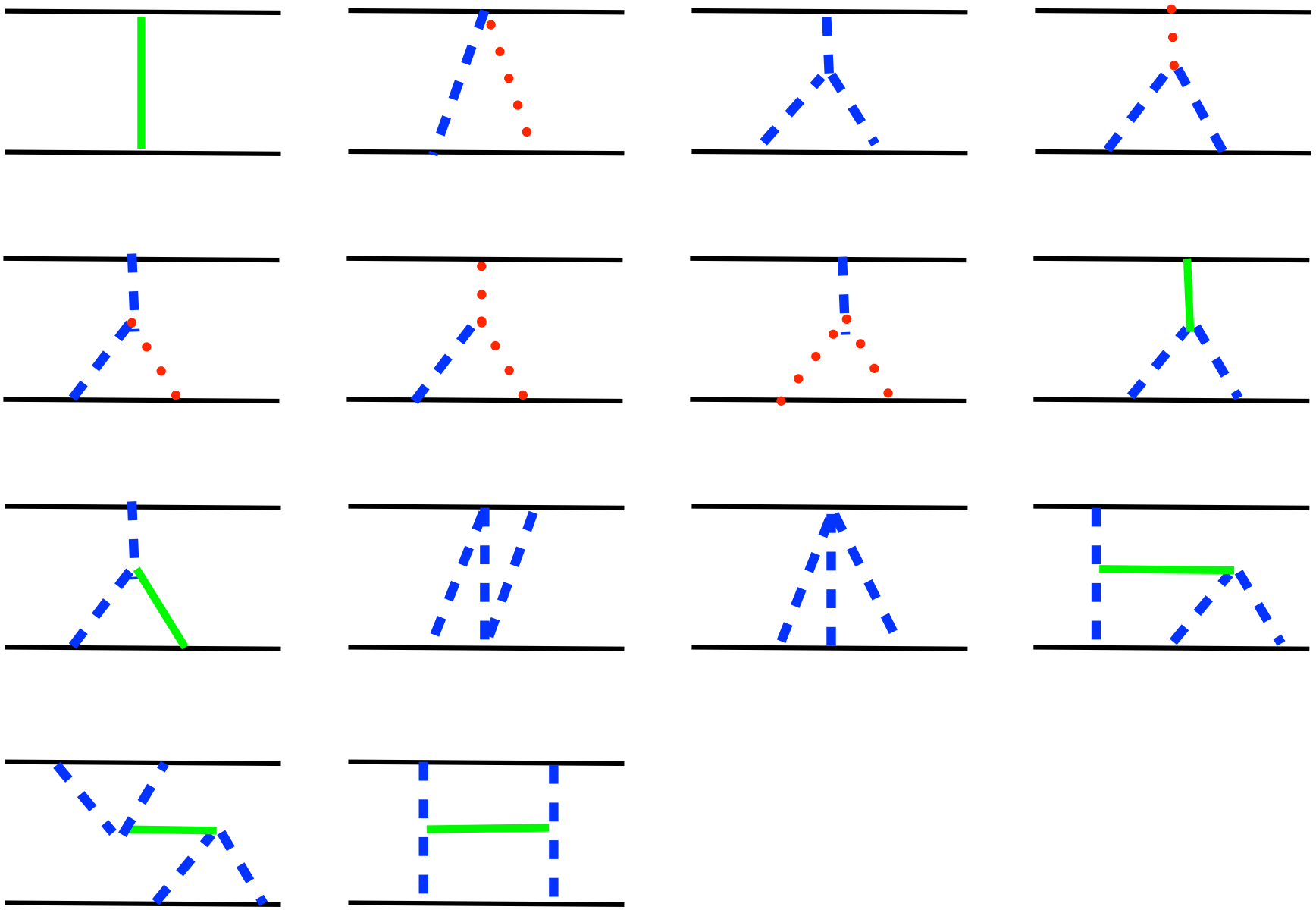
Alternatively, one can do time derivatives first,
and then integrate in momentum space using:

$$\int_{\mathbf{p}} \frac{p^i p^j}{p^{2\alpha}} e^{-ip.r} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)} \frac{1}{r} \left[\delta^{ij} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{2}{r}\right)^{d-2\alpha} + r^i r^j \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{2}{r}\right)^{d-2\alpha} \right) \right]$$

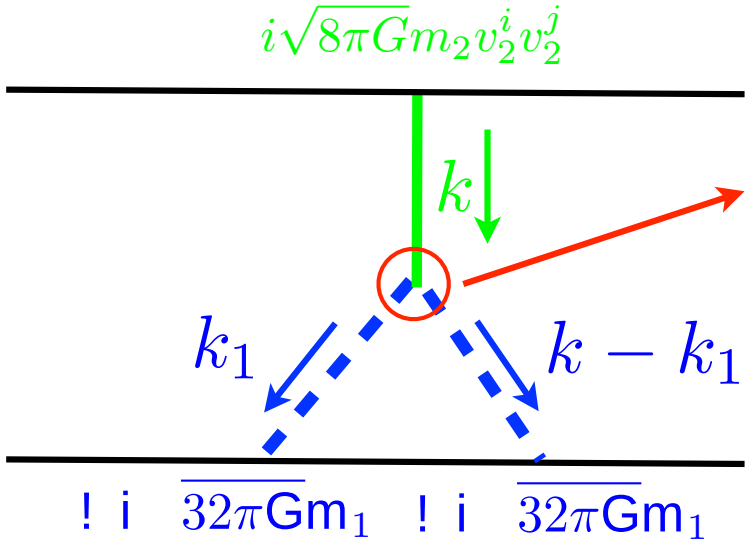
Total 1PN:

$$\mathcal{L}_{EIH} = ! \frac{G^2 M m_1 m_2}{r^2} + \frac{G m_1 m_2}{2r} [! \; v_{1r} v_{2r} + 3(v_1^2 + v_2^2) ! \; 7\vec{v_1} \acute{a} \vec{v_2}] \quad + (1 \leftrightarrow 2)$$

2PN: 3 + 14 = 17 graphs



Some nontrivial examples at $\mathcal{O}(G^2)$



$$-c_d\sqrt{\gamma}\left[\left(\vec{\nabla}\phi\right)^2 - \dot{\phi}^2\right] \simeq -c_d\left(1-\frac{\sigma}{2m_{Pl}}\right)\left(\delta_{ij}+\frac{\sigma_{ij}}{m_{Pl}}\right)\phi^{,i}\phi^{,j}$$

$$S_{eff}[x_1,x_2] \supset -i\int_x\int_{t,t_{1a},t_{1b},t_2}\left(\frac{-im_1V_\phi(t_{1a})}{m_p}\right)\left(\frac{-im_2V_\phi(t_{1b})}{m_p}\right)\left(\frac{-im_2V_\sigma^{ij}(t_2)}{m_p}\right)V_{ij}^{\phi^2\sigma}\times\{propagators\}$$

$$= 64\pi^2G^2m_1^2m_2\int_{t,k,k_1}\frac{(k_1.v_2)^2-(k.v_2)(k_1.v_2)}{k^2(k-k_1)^2k_1^2}e^{ik.r(t)}$$

$$I_1^i = \int\frac{d^dk_1}{(2\pi)^d}\frac{k_1^i}{k_1^2(k-k_1)^2} \qquad I_2^{ij} = \int\frac{d^dk_1}{(2\pi)^d}\frac{k_1^ik_1^j}{k_1^2(k-k_1)^2} \qquad ?$$

$$I_0 = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \frac{1}{k_1^{2\alpha}(k-k_1)^{2\beta}} = \frac{\Gamma(\alpha+\beta-d/2)\Gamma(d/2-\alpha)\Gamma(d/2-\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(d-\alpha-\beta)(4\pi)^{d/2}} (k^2)^{d/2-\alpha-\beta}$$

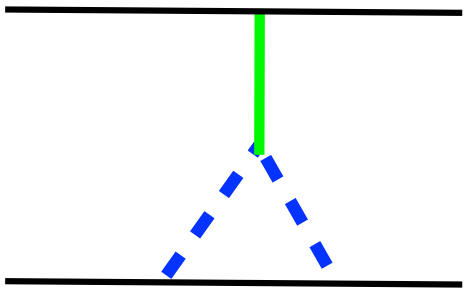
$$I_1^i = C_1 k^i$$

$$k.I_1(\alpha,\beta) = \frac{1}{2}\left[k^2 I_0(\alpha,\beta) + I_0(\alpha\,!\;\;1,\beta)\,!\;\;I_0(\alpha,\beta\,!\;\;1)\right]$$

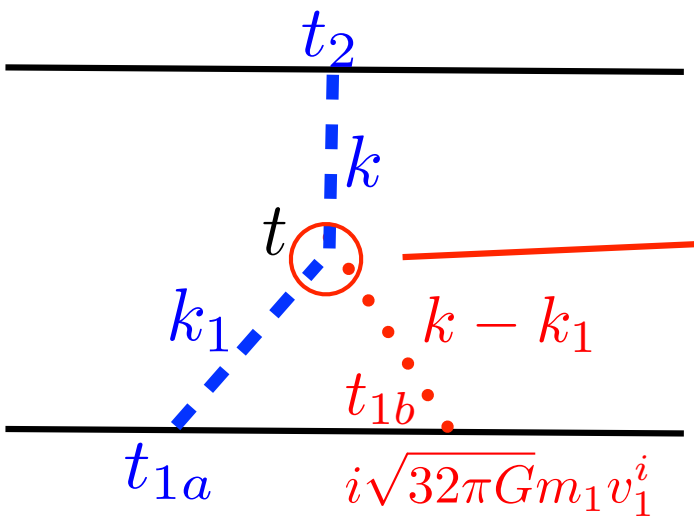
$$I_2^{ij} = C_0\delta^{ij} + C_2 k^i k^j$$

$$I_2^{ij}(\alpha,\beta)\delta_{ij} = I_0(\alpha-1,\beta)$$

$$k_i k_j I_2^{ij}(\alpha,\beta) = \frac{1}{4}\left[k^4 I_0(\alpha,\beta) + 2k^2 I_0(\alpha\,!\;\;1,\beta)\,!\;\;2k^2 I_0(\alpha,\beta\,!\;\;1)\right.$$
$$\left. + I_0(\alpha-2,\beta) - 2I_0(\alpha-1,\beta-1) + I_0(\alpha,\beta-2)\right]$$



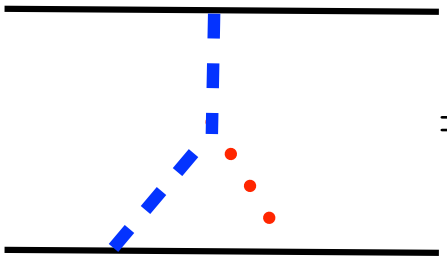
$$= \frac{G^2 m_1^2 m_2}{2r^2}\left[v_2^2 - \frac{(r.v_2)^2}{r^2}\right]$$



$$-\frac{2c_d}{m_{Pl}}\dot{\phi}\vec{A}\cdot\vec{\nabla}\phi$$

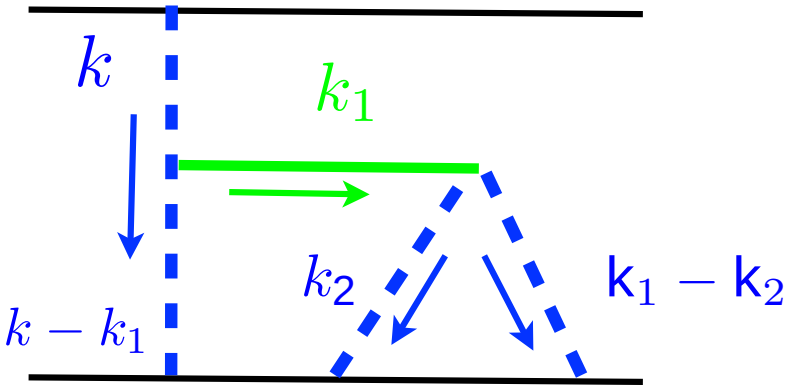
$$S_{eff}[x_1,x_2] \supset 256i\pi^2G^2m_1^2m_2\int_{t,t_{1a},t_{1b},t_2}\int_{k,k_1}\left[\frac{k.v_{1b}\delta'_{t_{1a}}-k_1.v_{1b}\delta'_{t_2}}{k^2k_1^2(k-k_1)^2}\times\delta^3_{t_i}\right]e^{-ik.x_2+ik_1.x_{1a}+i(k-k_1).x_{1b}}$$

$$=!\ 256\pi^2G^2m_1^2m_2\int_{t,k,k_1}\frac{(k.v_2)(k_1.v_1)+(k.v_1)(k_1.v_1)}{k^2(k\ !\ \ k_1)^2k_1^2}e^{ik.r(t)}$$



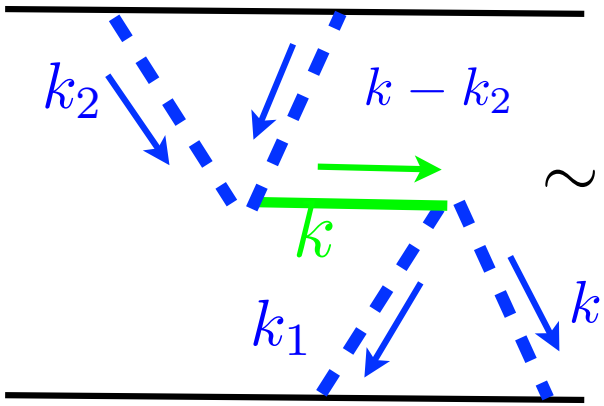
$$=\frac{2G^2m_1^2m_2}{r^2}\left[v_1^2+v_1.v_2-2\frac{(r.v_1)^2}{r^2}-2\frac{(r.v_1)(r.v_2)}{r^2}\right]$$

Nontrivial 2PN graphs at $O(G^3)$



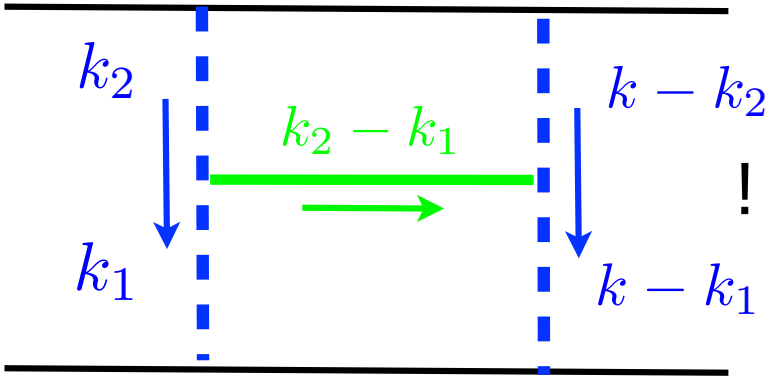
$$\sim \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \mathrm{d}^d k \; \{k^4\} \mathrm{e}^{\mathrm{ik.r}}}{k^2 k_1^2 (k-k_1)^2 k_2^2 (k_1-k_2)^2}$$

$$\sim \frac{\mathrm{d}^d k \mathrm{d}^d k_1 \{k^4\} \mathrm{e}^{\mathrm{ik.r}}}{k^2 (k-k_1)^2 k_1^{2(d/2-3)}} \sim \frac{\Gamma(d-3) \mathrm{d}^d k \{k^4\} \mathrm{e}^{\mathrm{ik.r}}}{k^{2(5-d)}} \sim \frac{\Gamma(d-3)}{\Gamma(3-d)} \longrightarrow -1$$



$$\sim \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \mathrm{d}^d k \; \{k^4\} \mathrm{e}^{\mathrm{ik.r}}}{k^2 k_1^2 (k-k_1)^2 k_2^2 (k-k_2)^2} \sim \frac{1}{\Gamma(3-d)} \longrightarrow 0$$

be careful with the d $\longrightarrow$ 3 limit



$$\frac{d^d k d^d k_1 d^d k_2 \{k^4\}\, e^{ik.r}}{k_1^2 (k - k_1)^2 k_2^2 (k - k_2)^2 (k_2 - k_1)^2}$$

Integration by parts:

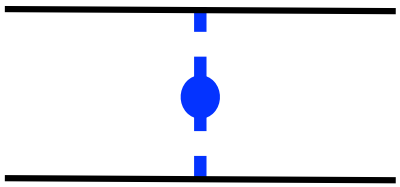
$$\begin{aligned} I(\alpha,\beta,\gamma,\delta,\epsilon) &\equiv \int_{\mathbf{p_1},\mathbf{p_2}} \frac{1}{p_1^{2\alpha}(\mathbf{p}-\mathbf{p_1})^{2\beta} p_2^{2\gamma}(\mathbf{p}-\mathbf{p_2})^{2\delta}(\mathbf{p_1}-\mathbf{p_2})^{2\epsilon}} \\ &= [\gamma\,(I(\alpha-1,\beta,\gamma+1,\delta,\epsilon) - I(\alpha,\beta,\gamma+1,\delta,\epsilon-1)) + \\ &\quad \delta\,(I(\alpha,\beta-1,\gamma,\delta+1,\epsilon) - I(\alpha,\beta,\gamma,\delta+1,\epsilon-1))] \,/\, (2\epsilon+\gamma+\delta-d) \end{aligned}$$

$$\int_{k_1,k_2} \frac{1}{k_1^2 (k-k_1)^2 k_2^2 (k-k_2)^2 (k_2-k_1)^2} = \int_{k_1,k_2} \frac{\frac{k_1^2}{k_2^2} - \frac{(k_2-k_1)^2}{k_2^2} + \frac{(k-k_1)^2}{(k-k_2)^2} - \frac{(k_2-k_1)^2}{(k-k_2)^2}}{k_1^2 (k-k_1)^2 k_2 (k-k_2)^2 (k_2-k_1)^2}$$

From the Lagrangian to the Energy in the center of mass

Several Lagrangian terms contain accelerations or higher order derivatives

some can be eliminated via ordinary integration by parts



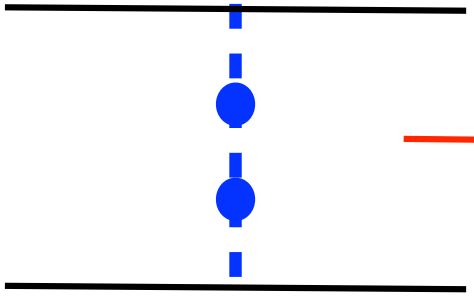
$$\int_t \frac{Gm_1m_2}{r} r.a_1$$

$$\int_t \frac{Gm_1m_2}{r}\left[v_2.v_1 - v_1^2 + \frac{(r.v_2)(r.v_1) - (r.v_1)^2}{r^2}\right]$$

in other cases a term linear in the accelerations is left:

$$\int_t \frac{Gm_1m_2}{r} v_2^2 r.a_1$$

yet in other cases one has to deal with terms quadratic in the accelerations



$$\int_t \frac{G m_1 m_2}{16 r} \left[15 r^2 a_1.a_2 - \frac{(r.a_1)(r.a_2)}{r^2} \right]$$

which can however be transformed by means of the double zero trick:

$$a_1.a_2 = (a_1 - eq_1 + eq_1).(a_2 - eq_2 + eq_2) = (a_1 - eq_1).(a_2 - eq_2) + eq_1.a_2 + a_1.eq_2 - eq_1.eq_2$$

$$\mathcal{L}^{a^2}_{G_N} \longrightarrow \mathcal{L}_{G_N^2} + \mathcal{O}(G_N^3)$$

(of course the eq1, eq2 must be taken at the appropriate PN order)

By reiterated use of the above double zero trick and integrations by parts,
one can always transform any Lagrangian into an equivalent one
(i.e. one giving the same equations of motion at a given PN order)
which is linear in the accelerations.

Energy:

$$\vec{p}_a \equiv \frac{\partial \mathcal{L}}{\partial \vec{v}_a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \vec{a}_a}\right) \qquad \vec{q}_a \; ! \; \frac{\partial \mathcal{L}}{\partial \vec{a}_a}$$

$$E \equiv \sum_{a=1,2} \vec{v}_a \cdot \vec{p}_a + \vec{a}_a \cdot \vec{q}_a - \mathcal{L}$$

$$E = \frac{m_1 v_1^2}{2} - \frac{G m_1 m_2}{2r} + \frac{3 m_1 v_1^4}{8} + \frac{G m_1 m_2}{r}\left[\frac{v_1^2}{2} - \frac{7 v_1 . v_2}{4} - \frac{v_1^r v_2^2}{4}\right] + \frac{G^2 m_1^2 m_2}{2r^2}$$

$$+\frac{5 m_1 v_1^6}{16} + \frac{G m_1 m_2}{r}\left[\frac{21 v_1^4}{8} + \frac{17 (v_1 . v_2)^2}{8} + \frac{31 v_1^2 v_2^2}{16} - \frac{55 v_1^2 v_1 . v_2}{8} - \frac{13 v_1^2 v_2^{r\,2}}{8} + \frac{13 v_1 . v_2 v_1^{r\,2}}{8} - \frac{9 v_1^2 v_1^r v_2^r}{8} + \frac{3 v_1^{r\,2} v_2^{r\,2}}{16} + \frac{3 v_1^{r\,3} v_2^r}{8}\right]$$

$$+\frac{G^2 m_1^2 m_2}{r^2}\left[\frac{7 v_2^2}{4} - \frac{3 v_1^2}{2} + \frac{29 v_1^{r\,2}}{4} - \frac{13 v_1^r v_2^r}{4} + \frac{v_2^{r\,2}}{2}\right] - \frac{19 G^3 m_1^2 m_2^2}{8 r^3} - \frac{G^3 m_1^3 m_2}{2 r^3}$$

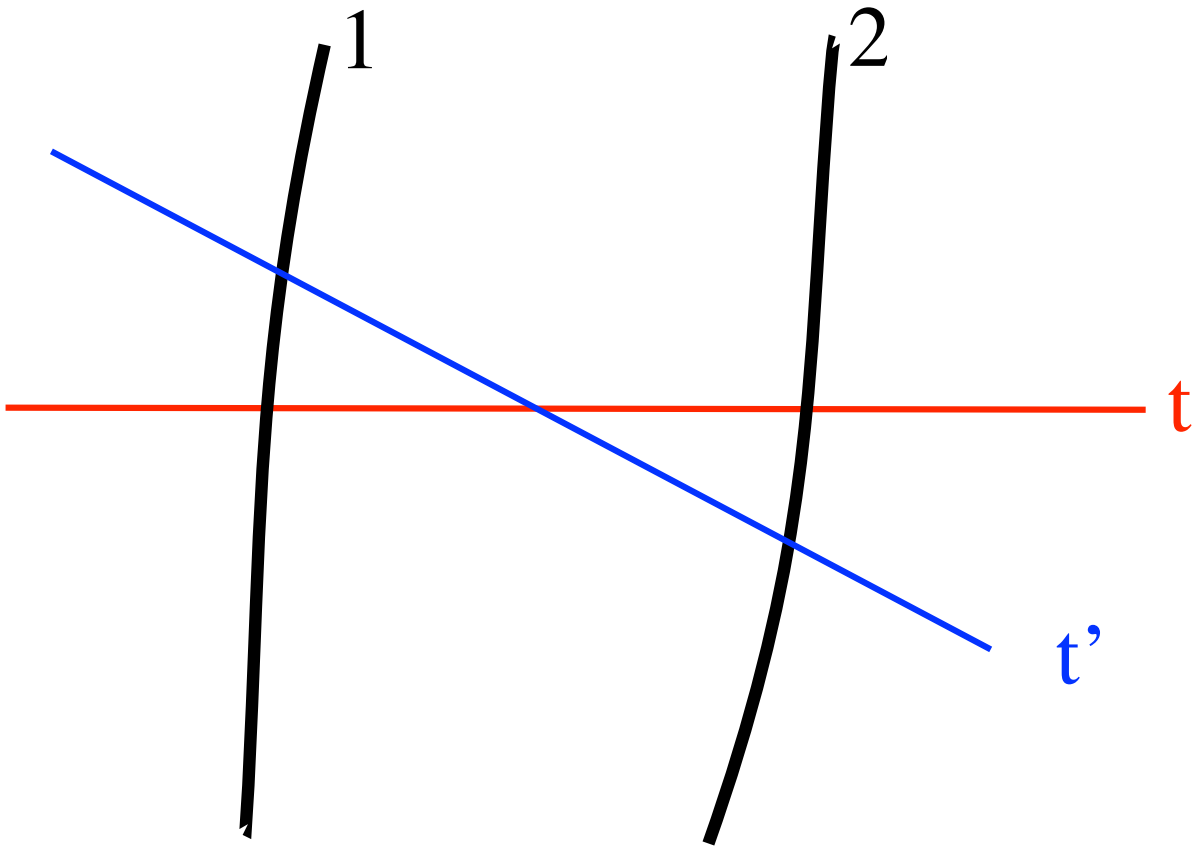
$$+(1 \leftrightarrow 2)$$

$$v_a^r \equiv \frac{\vec{v_a} \cdot \vec{r}}{r}$$

Lorentz invariance and center-of-mass position

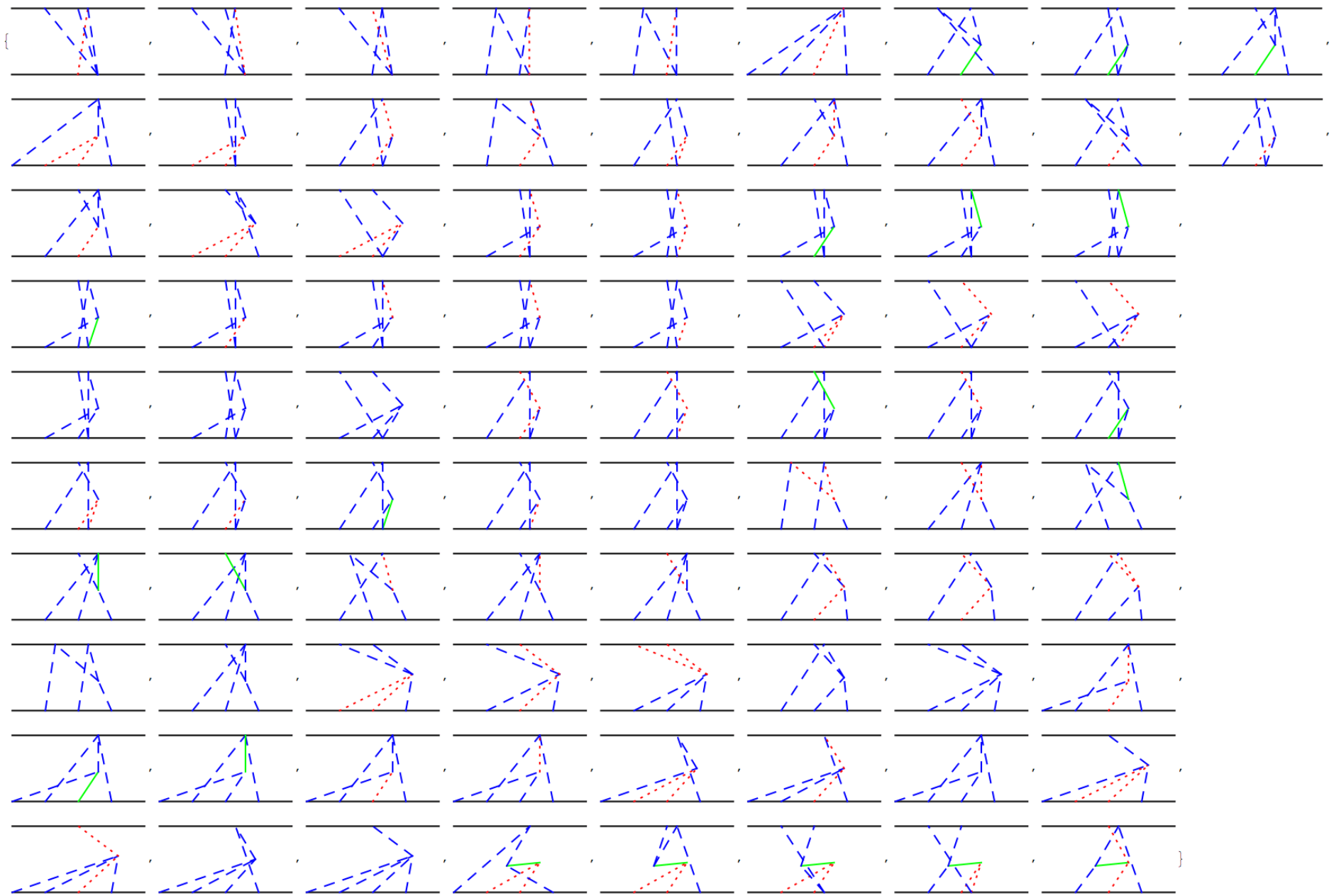
$$\mathcal{L}\left[\vec{x}_{1,2}(t)\right]$$

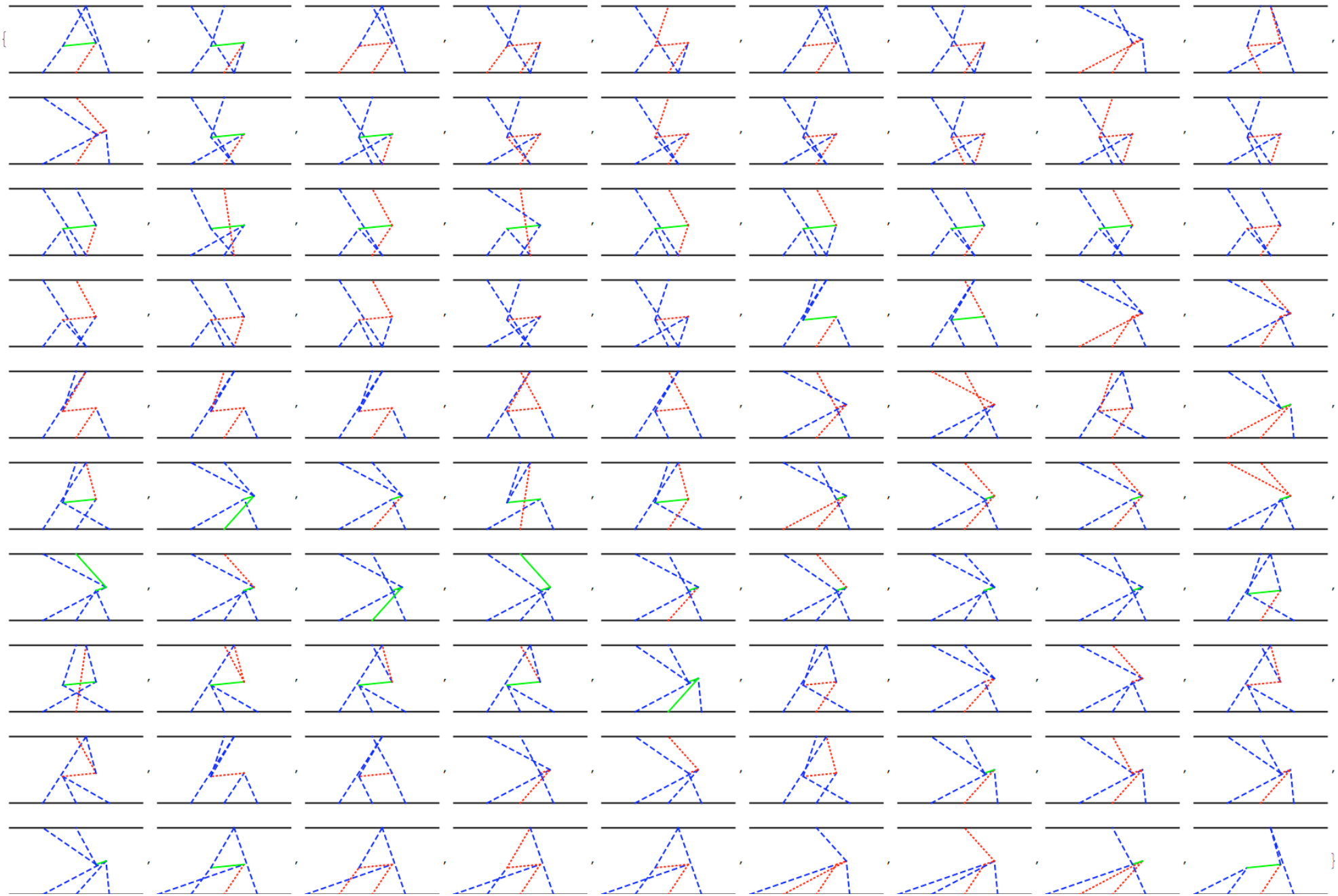
Boost:

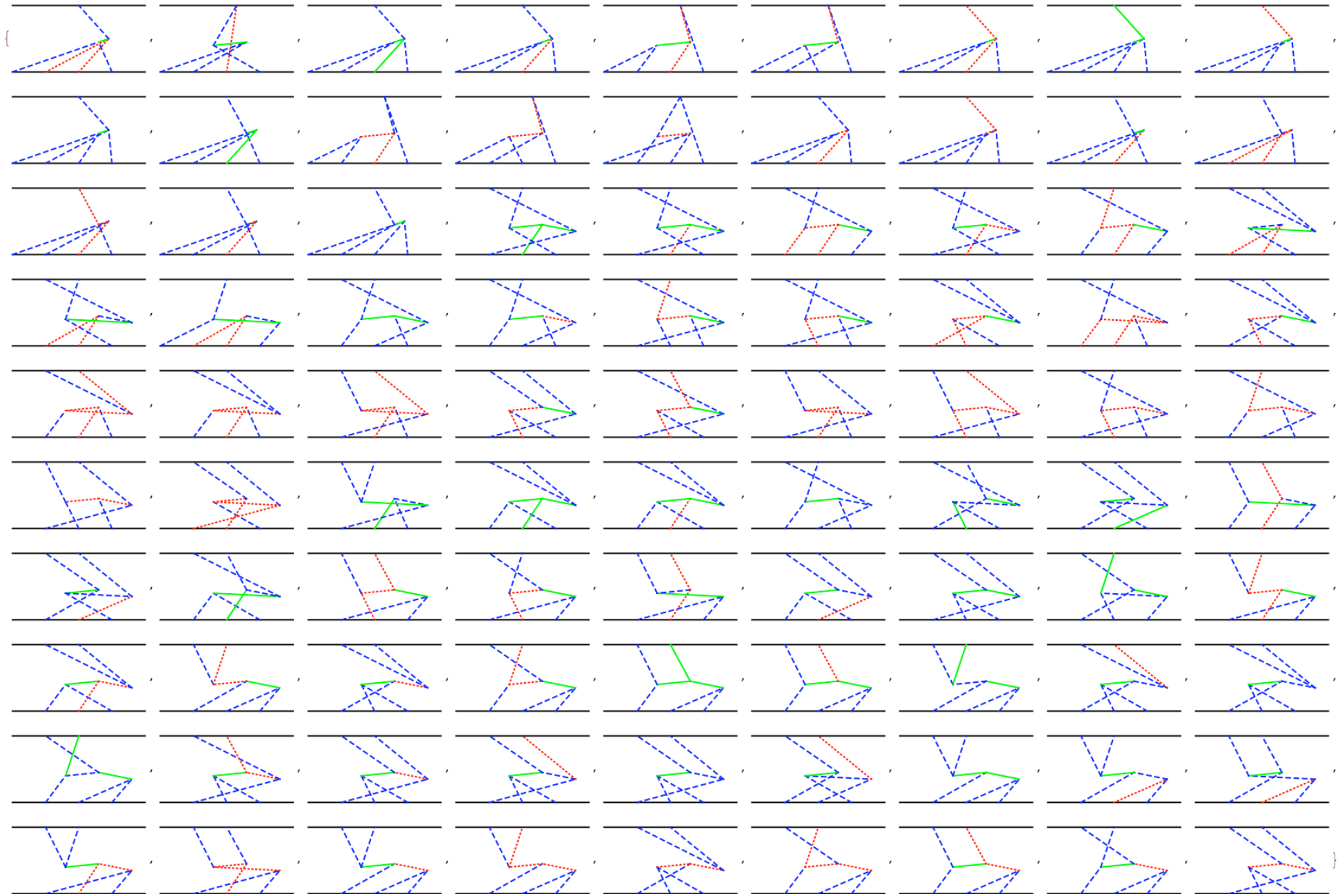


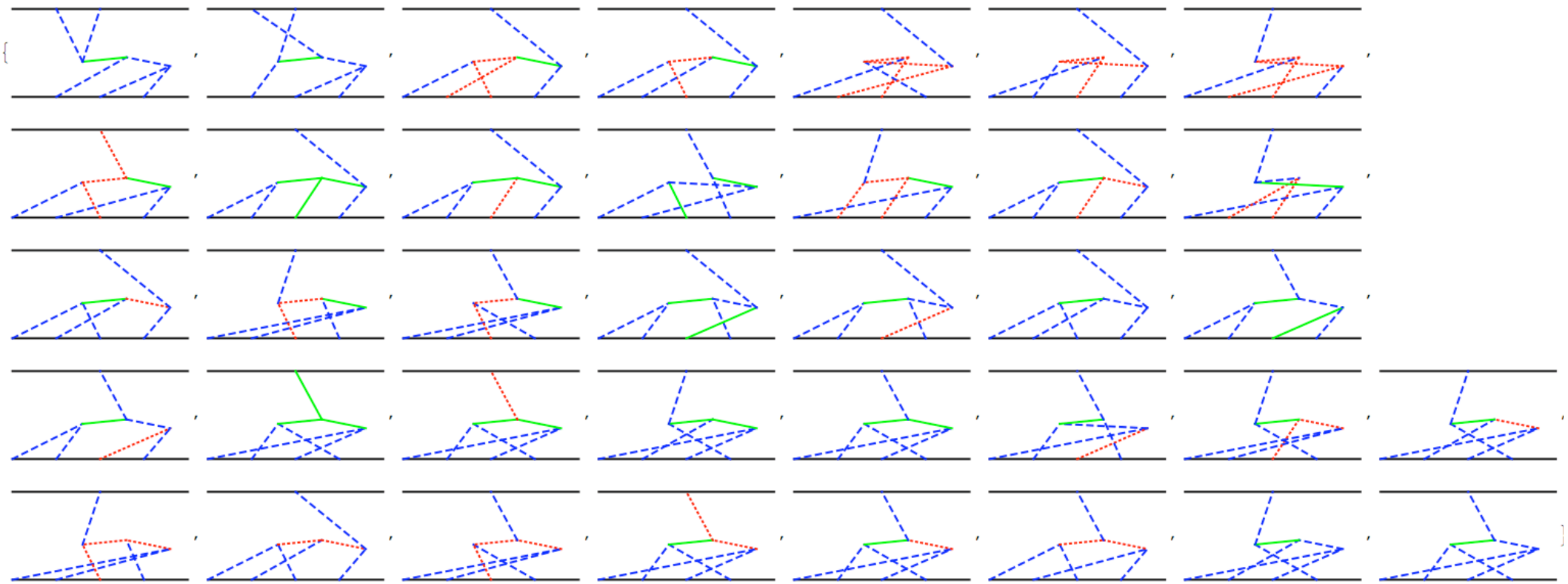
$$\delta\vec{x}_{1,2} = -\vec{V}t + (V.x_{1,2})\vec{v}_{1,2}$$

$$\delta\mathcal{L} = V.\dot{Z}\left[\vec{x}_{1,2}(t)\right] \quad + \quad \text{double zeroes}$$









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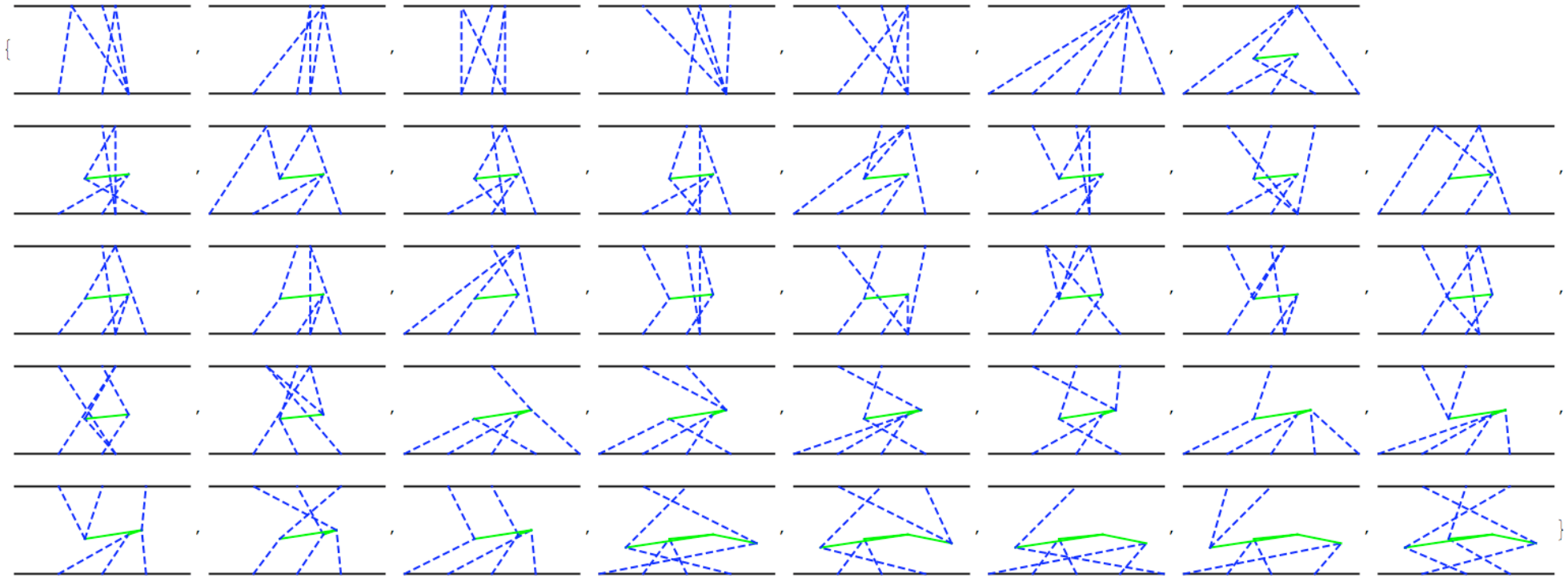

ated by *Wolfram Mathematica 6.0* (09 December 2009)

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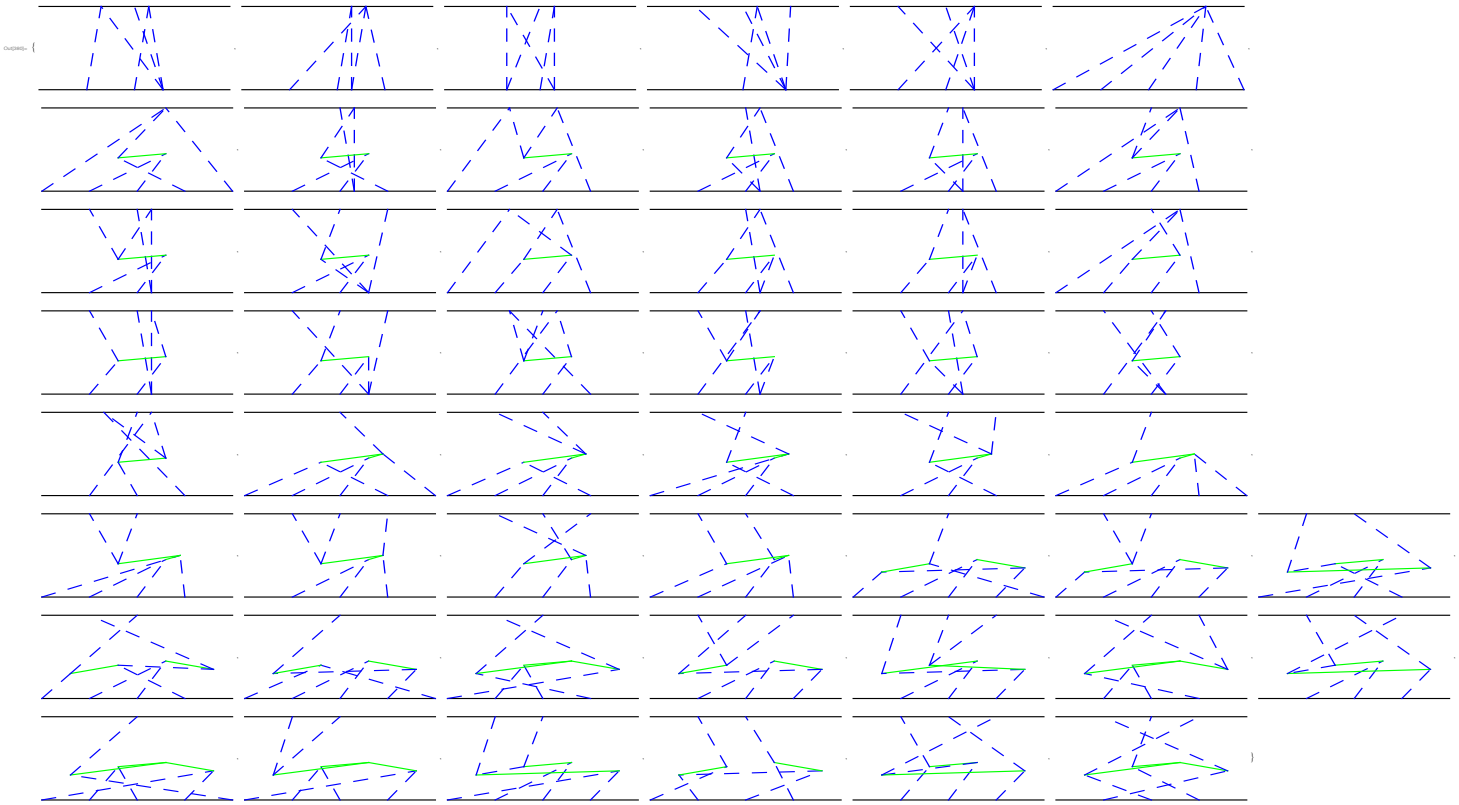
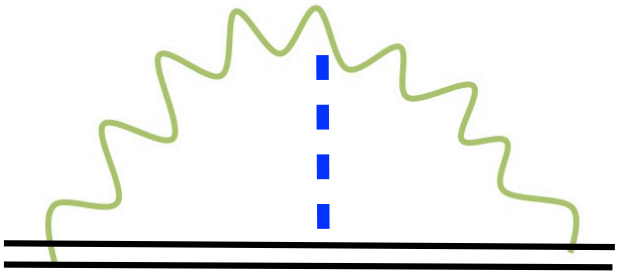


595 Feynman graphs in total

+ tail terms:

Re 

$$E = -\frac{\mu x}{2}\left[1 - \left(\frac{3}{4} + \frac{\nu}{12}\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right)x^2\right.$$

012PN

$$+\left(-\frac{675}{64} + \frac{34445}{576} - \frac{205}{96}\pi^2\right)\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\Big)x^3$$

3PN

$$+\left[-\frac{3969}{128} + \left(\frac{9037}{1536}\pi^2 - \frac{123671}{5760} + \frac{448}{15}\left[2\gamma_E + \log(16x)\right]\right)\nu + \right.$$

Tail

4PN

BH perturbation

$$\left.\left.+\left(\frac{3137}{576}\pi^2 - \frac{498449}{3456}\right)\nu^2 + \frac{301}{1728}\nu^3 + \frac{77}{31104}\nu^4\right]x^4\right\}$$

post-Minkoskian

$$\dot{E}[\omega(t)] = -P_{gw}[\omega(t)] \longrightarrow \omega(t) \longrightarrow \Phi_{gw}(t)$$

verified only up to NLO in PN expansion