1. Convert the covariant expression $\bar{\Psi}\rho^\alpha\partial_\alpha\Psi$, where

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{1}$$

and $\bar{\Psi} = \Psi^\dagger\rho^0$ are two-dimensional spinors, into the component expression

$$-2(\Psi_+\partial_-\Psi_+ + \Psi_-\partial_+\Psi_-) \tag{2}$$

with $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$.

Hint: Use the following convenient basis for the Dirac matrices

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

2. Prove the following property (valid in two dimensions)

$$\rho^\alpha\rho_\beta\rho_\alpha = 0 \tag{4}$$

3. Show that the action

$$\begin{aligned} S &= -\frac{1}{8\pi}\int d^2\sigma e\left[\frac{2}{\alpha'}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + 2i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu\right. \\ &\qquad \left. -i\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi^\mu\left(\sqrt{\frac{2}{\alpha'}}\partial_\beta X_\mu - \frac{i}{4}\bar{\chi}_\beta\psi_\mu\right)\right] \end{aligned} \tag{5}$$

is invariant under the following local world-sheet symmetries:

(a) Supersymmetry

$$\begin{aligned} \sqrt{\frac{2}{\alpha'}}\delta_\epsilon X^\mu &= i\bar{\epsilon}\psi^\mu\,, \\ \delta_\epsilon\psi^\mu &= \frac{1}{2}\rho^\alpha\left(\sqrt{\frac{2}{\alpha'}}\psi^\mu\partial_\alpha - \frac{i}{2}\bar{\chi}_\alpha\right)\epsilon\,, \\ \delta_\epsilon e_\alpha{}^a &= \frac{i}{2}\bar{\epsilon}\rho^\alpha\chi_\alpha\,, \\ \delta_\epsilon\chi_\alpha &= 2D_\alpha\epsilon \end{aligned}$$

where $\epsilon(\tau,\sigma)$ is a Majorana spinor which parametrizes susy transformations and D_α is a covariant derivative with torsion:

$$\begin{aligned} D_\alpha\epsilon &= \partial_\alpha\epsilon - \frac{1}{2}\omega_\alpha\bar{\rho}\epsilon \\ \omega_\alpha &= -\frac{1}{2}\epsilon^{ab}\omega_{\alpha ab} = \omega_\alpha(e) + \frac{i}{4}\bar{\chi}_\alpha\bar{\rho}\rho^\beta\chi_\beta \\ \omega_\alpha(e) &= -\frac{1}{e}e_{\alpha a}\epsilon^{\beta\gamma}\partial_\beta e_\gamma{}^a \end{aligned}$$

where $\omega_\alpha(e)$ is the spin connection without torsion.

(b) Weyl transformations

$$\begin{aligned}\delta_\Lambda X^\mu &= 0\\ \delta_\Lambda \psi^\mu &= -\frac{1}{2}\Lambda\psi^\mu\\ \delta_\Lambda e_\alpha{}^a &= \Lambda e_\alpha{}^a\\ \delta_\Lambda \chi_\alpha &= \frac{1}{2}\Lambda\chi_\alpha\end{aligned}$$

(c) Super-Weyl transformations

$$\begin{aligned}\delta_\eta \chi_\alpha &= \rho_\alpha \eta\\ \delta_\eta(\text{others}) &= 0\end{aligned}$$

with $\eta(\tau,\sigma)$ a Majorana spinor parameter.

(d) Two-dimensional Lorentz transformations

$$\begin{aligned}\delta_l X^\mu &= 0\\ \delta_l \psi^\mu &= -\frac{1}{2}l\bar{\rho}\psi^\mu\\ \delta_l e_\alpha{}^a &= l\epsilon^a{}_b e_\alpha{}^b\\ \delta_l \chi_\alpha &= -\frac{1}{2}l\bar{\rho}\chi_\alpha\end{aligned}$$

(e) Reparametrizations

$$\begin{aligned}\delta_\xi X^\mu &= -\xi^\beta\partial_\beta X^\mu\\ \delta_\xi \psi^\mu &= -\xi^\beta\partial_\beta \psi^\mu\\ \delta_\xi e_\alpha{}^a &= -\xi^\beta\partial_\beta e_\alpha{}^a - e_\beta{}^a\partial_\alpha\xi^\beta\\ \delta_\xi \chi_\alpha &= -\xi^\beta\partial_\beta\chi_\alpha - \chi_\beta\partial_\alpha\xi^\beta\end{aligned}$$

Note that Λ, ξ and l are infinitesimal functions of (τ,σ).

(f) Global space-time Poincaré transformations:

$$\begin{aligned}\delta X^\mu &= a^\mu{}_\nu X^\nu + b^\mu\,, \qquad a_{\mu\nu} = -a_{\nu\mu}\\ \delta h_{\alpha\beta} &= 0\\ \delta\psi^\mu &= a^\mu{}_\nu\psi^\nu\\ \delta\chi_\alpha &= 0\end{aligned}$$

4. Show that the commutator of two supersymmetry transformations on the fields X^μ and ψ^μ, i.e. $[\delta_1,\delta_2]X^\mu$ and $[\delta_1,\delta_2]\psi^\mu$, gives a translation.

5. Find the equations of motion for the fields derived from the action (5). Show the tracelessness of the energy momentum tensor $T_{\alpha\beta}$ and the ρ-tracelessness of its superpartner T_F.

6. Write the non-vanishing components of the energy-momentum tensor and the supercurrent in light-cone coordinates and show that they are conserved.

7. Show that the NS and R boundary conditions on the fermions allow to cancel the boundary terms in the variation of the action.

8. Obtain the mode expansions for the fermionic fields in the Ramond and Neveu-Schwarz sectors of the open string.

9. Obtain the superVirasoro generators L_m and G_r in terms of oscillators.

10. Obtain the classical superVirasoro algebra:

$$\begin{aligned} [L_m, L_n] &= -i(m-n)L_{m+n} \\ [L_m, G_r] &= -i\left(\frac{1}{2}m - r\right)G_{m+r} \\ \{G_r, G_s\} &= -2iL_{r+s} \end{aligned} \tag{6}$$

Useful identities:

1. $\bar{\lambda}_1\rho^{\alpha_1}\cdots\rho^{\alpha_n}\lambda_2 = (-1)^n\bar{\lambda}_2\rho^{\alpha_n}\cdots\rho^{\alpha_1}\lambda_1$
2. $(\bar{\psi}\lambda)(\bar{\phi}\chi) = -\frac{1}{2}\{(\bar{\psi}\chi)(\bar{\phi}\lambda) + (\bar{\psi}\bar{\rho}\chi)(\bar{\phi}\bar{\rho}\bar{\lambda}) + (\bar{\psi}\rho^{\alpha}\chi)(\bar{\phi}\rho_{\alpha}\lambda)\}$