Effective Field Theory for Cold Atoms III

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Agenda

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1. EFT for Ultracold Atoms I: Effective Field Theories & Universality
2. EFT for Ultracold Atoms II: Cold Atoms & the Unitary Limit
3. EFT for Ultracold Atoms III: Weak Coupling at Finite Density
4. EFT for Ultracold Atoms IV: Few-Body Systems in the Unitary Limit
5. Beyond Ultracold Atoms: Halo Nuclei and Hadronic Molecules

Literature

G.P. Lepage, TASI Lectures 1989, arXiv:hep-ph/0506330

D.B. Kaplan, arXiv:nucl-th/9506035

E. Braaten, HWH, Phys. Rep. **428** (2006) 259 [arXiv:cond-mat/0410417]

EFT at Finite density

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- Consider (repulsive) dilute Fermi gas $\longrightarrow$ perturbative
  - Experiments with ultracold atoms
  - Structure similar to nuclear Skyrme energy functionals
  - Toy model for theory questions
- No interactions $\Rightarrow$ free Fermi gas: $E/N = (3/5)E_F$
- Diagrammatic density expansion for E/N from EFT
- Low density $\Rightarrow$ probe at low resolution ($k_{\rm F} \ll 1/R$)
  - Details of interaction unresolved ($L = 0, 1, \ldots$)

$$k \cot \delta_0(k) = -\frac{1}{a} + \frac{1}{2} r_e k^2 + \cdots \,, \qquad k^3 \cot \delta_1(k) = -\frac{3}{a_p^3} + \cdots$$

  - Natural parameters ($a, a_p, r_e \sim R$), e.g. hard spheres
- Start with EFT in vacuum

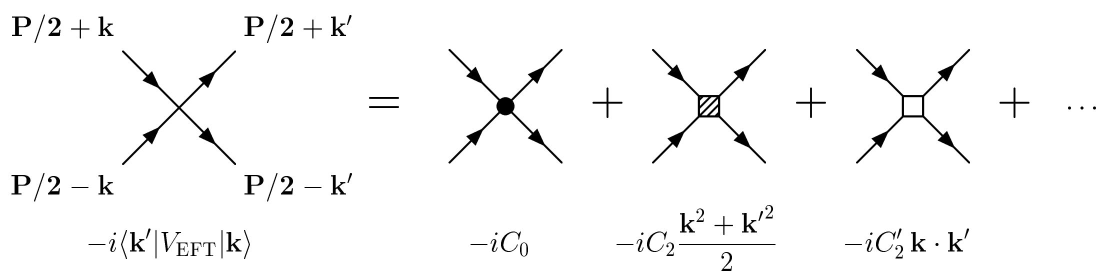
EFT in the Vacuum

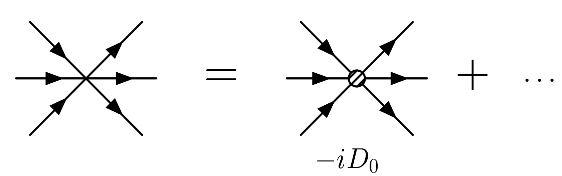
- $\mathcal{L}$ with most general local (contact) interactions:

$$\mathcal{L} = \psi^\dagger\Big(i\frac{\partial}{\partial t} + \frac{\overrightarrow{\nabla}^{\,2}}{2m}\Big)\psi - \frac{C_0}{2}(\psi^\dagger\psi)^2 + \frac{C_2}{16}\Big[(\psi\psi)^\dagger(\psi\overleftrightarrow{\nabla}^2\psi) + \text{h.c.}\Big]$$
$$+ \frac{C_2'}{8}(\psi\overleftrightarrow{\nabla}\psi)^\dagger\cdot(\psi\overleftrightarrow{\nabla}\psi) - \frac{D_0}{6}(\psi^\dagger\psi)^3 + \ldots\,, \quad \text{with } \overleftrightarrow{\nabla} = \overleftarrow{\nabla} - \overrightarrow{\nabla}$$

- Dimensional Analysis: $C_{2i} \sim \frac{4\pi}{m}R^{2i+1}\,,\quad D_{2i} \sim \frac{4\pi}{m}R^{2i+4}$

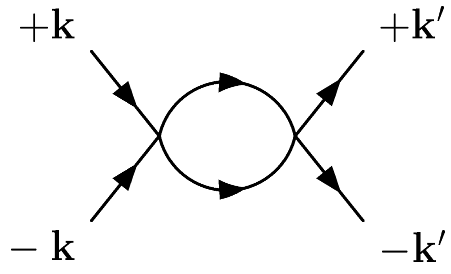
- Feynman rules:





Dimensional Regularization

- Use dimensional regularization with minimal subtraction
- Evaluate loop integral in D spatial dimensions:



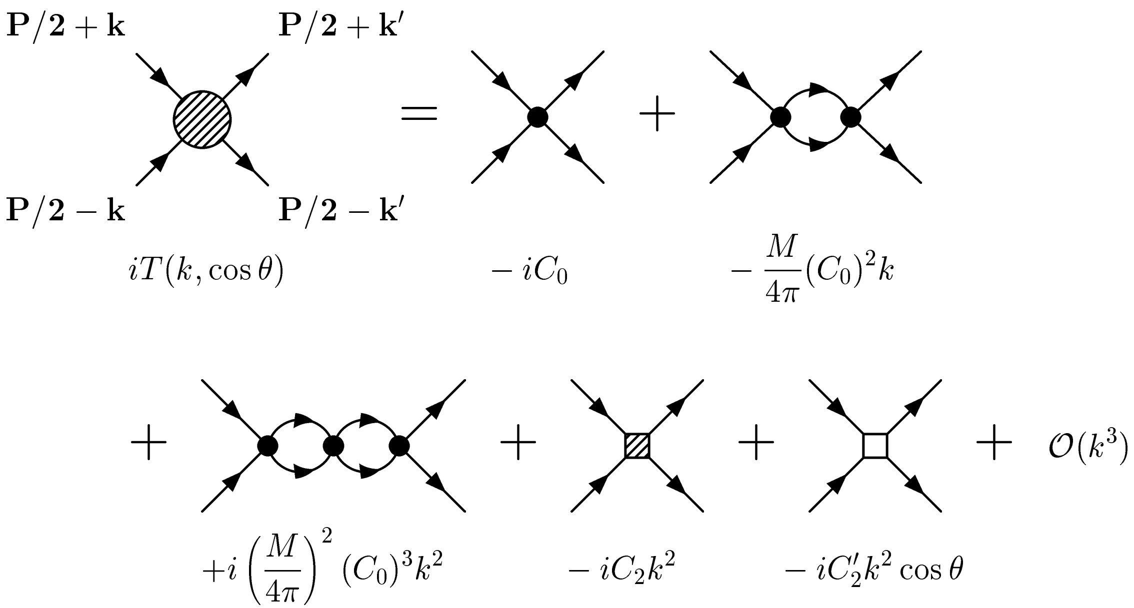
$$\Longrightarrow \int \frac{d^D q}{(2\pi)^D} \frac{q^{2n}}{k^2 - q^2 + i\epsilon}$$

$$= -k^{2n}(-k^2 - i\epsilon)^{D/2-1}\Gamma\left(1 - \frac{D}{2}\right)(4\pi)^{-D/2} \quad \longrightarrow \quad -\frac{ik^{2n+1}}{4\pi}$$

- We consider limit $D \to 3$:
  - Pure power divergence
  - Completely subtracted in DR
- No poles in $D = 3 \Rightarrow C_{2i}$ independent of renormalization scale μ

Low-Energy Scattering

- Match EFT to vacuum scattering amplitude at each order in k



- Using dimensional regularization with minimal subtraction

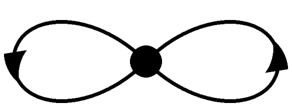
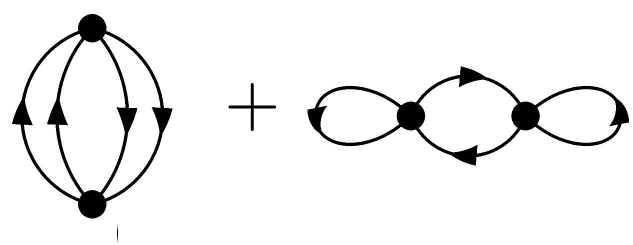
$$C_0=\frac{4\pi}{m}a,\quad C_2=\frac{4\pi}{m}\frac{a^2r_e}{2},\quad \text{and}\quad C_2'=\frac{4\pi}{m}a_p^3$$

- Calculate many-body observables

Energy Density

- Energy density $\mathcal{E} = E/V$ is sum of closed, connected Feynman diagrams
- Power counting rules at finite density
  1. for every propagator: $m/{k_\mathrm{F}}^2$
  2. for every loop integration: ${k_\mathrm{F}}^5/m$
  3. for every n–body vertex with $2i$ derivatives: ${k_\mathrm{F}}^{2i} R^{2i+3n-5}/m$
- Diagram with V_{2i}^n n–body vertices of each type scales as $(k_\mathrm{F})^\nu$:

$$\nu = 5 + \sum_{n=2}^{\infty} \sum_{i=0}^{\infty} (3n + 2i - 5) V_{2i}^n \,.$$

- E.g.: $\mathcal{O}({k_\mathrm{F}}^6)$:  $\mathcal{O}(k_\mathrm{F}^7)$: 
- Perturbation theory in $k_\mathrm{F} R$

Feynman rules

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(cf. Fetter, Walecka, Quantum Theory of Many-Particle Systems)

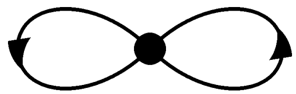
1. Write all diagrams up to given order in $k_{\rm F}$
2. Each line assigned conserved $\tilde{k} \equiv (k_0, \vec{k})$ and

$$iG_0(\tilde{k})_{\alpha\beta} = i\delta_{\alpha\beta} \left(\frac{\theta(k-k_{\rm F})}{k_0 - \omega_{\vec{k}} + i\epsilon} + \frac{\theta(k_{\rm F}-k)}{k_0 - \omega_{\vec{k}} - i\epsilon} \right) , \quad \omega_{\vec{k}} = \frac{k^2}{2m}$$

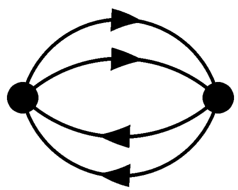
3. For each vertex $\longrightarrow (\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})$ (spin-independent interactions)
4. After spin summations, $\delta_{\alpha\alpha} \rightarrow -g$ in every closed fermion loop
5. Integrate $\int d^4k/(2\pi)^4$ with convergence factor $e^{ik_0 0^+}$ for tadpoles
6. Assign symmetry factor: $i/(S \prod_{l=2}^{l_{\max}} (l!)^k)$

   counts vertex permutations and equivalent l–tuples of lines

Energy density: examples

- $\mathcal{O}({k_{\mathrm{F}}}^6)$: 

$$= \frac{i}{1 \cdot 2} (-iC_0)\, g(g-1) \Big[\int \frac{d^4k}{(2\pi)^4}\, e^{ik_0 0^+} iG_0(\tilde{k}) \Big]^2 = \rho\, (g-1)\, \frac{{k_{\mathrm{F}}}^2}{2m}\, \frac{2}{3\pi}\, k_{\mathrm{F}} a$$

- $\mathcal{O}({k_{\mathrm{F}}}^7)$: 

$$= \frac{i(-iC_0)^2}{2 \cdot 2^2} 2g(g-1) \int \frac{d^4k_1}{(2\pi)^4} .. \frac{d^4k_3}{(2\pi)^4}\, G_0(\tilde{k}_2) G_0(\tilde{k}_2 - \tilde{k}_3) G_0(\tilde{k}_1 + \tilde{k}_3) G_0(\tilde{k}_3)$$

- Second diagram has UV divergence

  $\rightarrow$ same renormalization as in vacuum

Energy density: examples

- Typical UV divergence:

$$\mathcal{P}\int_{k_\mathrm{F}}^{\infty}\frac{d^3k}{q^2-k^2} = \mathcal{P}\int_{0}^{\infty}\frac{d^3k}{q^2-k^2} - \mathcal{P}\int_{0}^{k_\mathrm{F}}\frac{d^3k}{q^2-k^2}$$

- Medium is a IR effect $\Rightarrow$ renormalization unchanged
- Power divergences removed by dimensional regularization/MS
- Holes and particles count same

  $\Rightarrow$ no point in separate Goldstone diagrams here
- Textbook approach: (e.g. Fetter & Walecka)

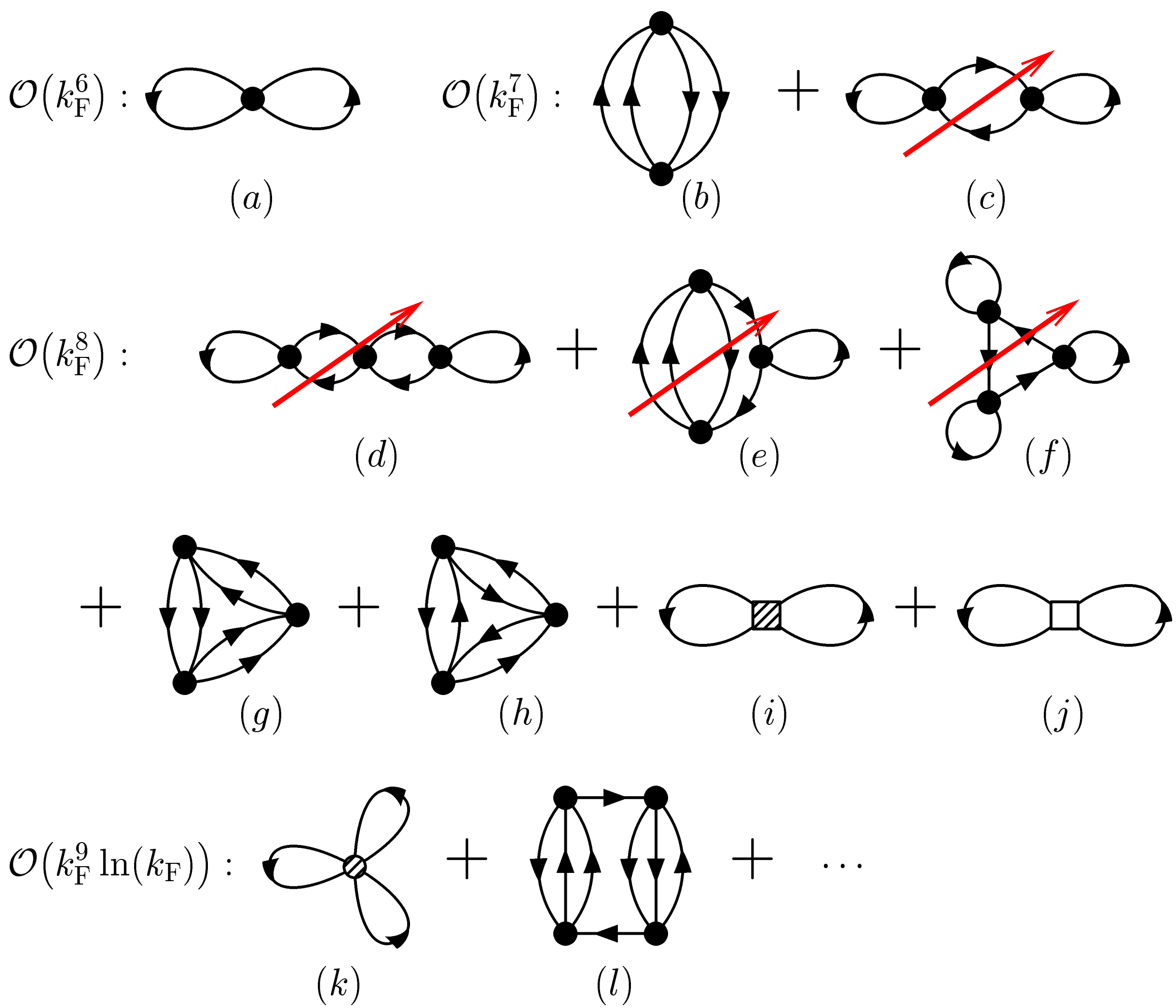
  $\Rightarrow$ nonperturbative in potential

  $\Rightarrow$ sum ladders to remove divergence

  $\Rightarrow$ many diagrams

Energy density: dilute Fermi gas

- Contributions to $\mathcal{E}$ at $T = 0$: (HWH, Furnstahl, Nucl. Phys. A **678** (2000) 277)



Results

- Energy per particle for dilute Fermi gas

$$
\begin{aligned}
\frac{E}{N} &= \frac{{k_\mathrm{F}}^2}{2m}\Big[\frac{3}{5} + (g-1)\frac{2}{3\pi}(k_\mathrm{F}a) + (g-1)\frac{4}{35\pi^2}(11-2\ln 2)(k_\mathrm{F}a)^2 \\
&+ (g-1)(0.0076+0.057(g-3))(k_\mathrm{F}a)^3 + (g-1)\frac{1}{10\pi}(k_\mathrm{F}r_e)(k_\mathrm{F}a)^2 \\
&+ (g+1)\frac{1}{5\pi}(k_\mathrm{F}a_p)^3 + (g-2)(g-1)\frac{16}{27\pi^3}(4\pi-3\sqrt{3})(k_\mathrm{F}a)^4\ln(k_\mathrm{F}a) + \ldots
\end{aligned}
$$

- Previously derived with correlation functions, Goldstone/Feynman diagrams (nonperturbative in potential!)

  (Lee, Yang, Efimov, Amusya,... 1960's)

Conventional Approaches

- Consider $(g-1)(g-3)(k_{\mathrm{F}}a)^3{k_{\mathrm{F}}}^2/(2m)$ term
- K-matrix Goldstone diagrams/T-matrix Feynman diagrams
  (Bishop, 1973)

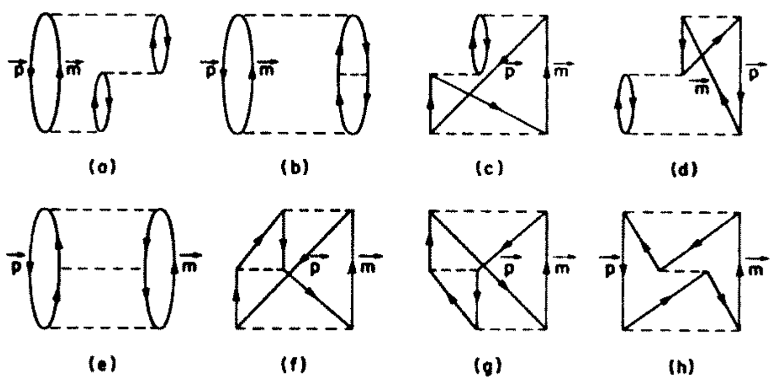


FIG. 9. The eight third-order K-matrix Goldstone diagrams contributing to ϵ_s.

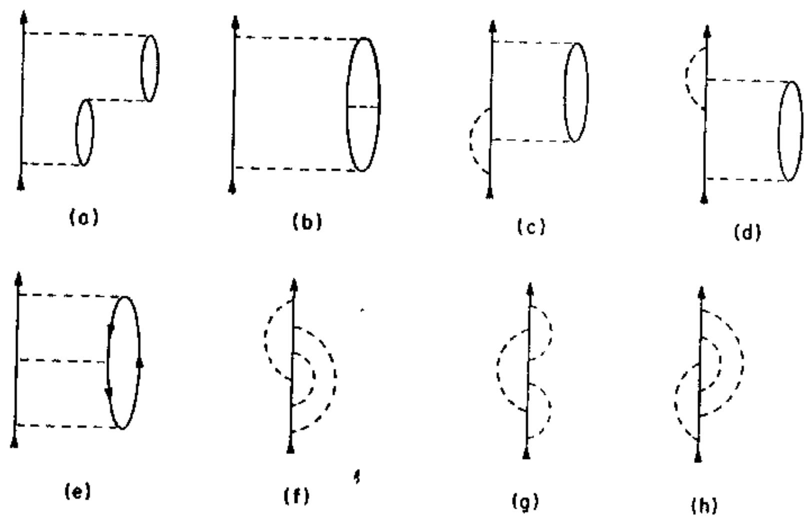
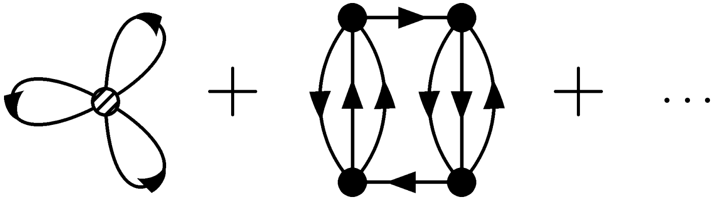


FIG. 15. The eight third-order T-matrix diagrams contributing to ϵ_s.

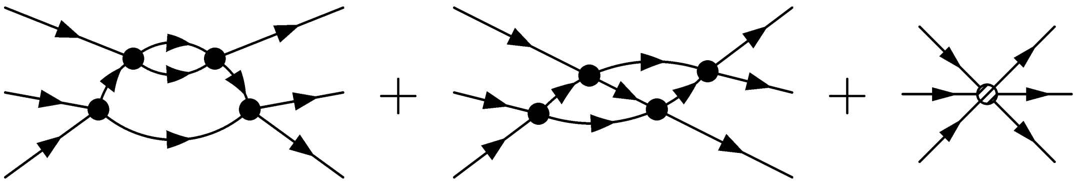
Many-Body Interactions

- Many-body interactions appear naturally and are essential
- Nonanalytic term $\rightarrow$ Three-body interactions

$\mathcal{O}\big(k_{\mathrm{F}}^9 \ln(k_{\mathrm{F}})\big)$:



- Log-divergences in 3 $\rightarrow$ 3 scattering (Braaten, Nieto, PRB **55** ('97) 8090)



$$-i\mathcal{T}_{3\rightarrow 3} = -im^3 (C_0)^4 \,\frac{4\pi - 3\sqrt{3}}{8\pi^3}\left[\frac{1}{D-3} - 2\ln\mu + \cdots\right] - iD_0(\mu)$$

Many-Body Interactions

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- $\mathcal{T}_{3\to 3}$ independent of μ: $\mu\frac{d}{d\mu}\mathcal{T}_{3\to 3} = 0$

$$\Longrightarrow \quad \mu\frac{d}{d\mu}D_0(\mu) = m^3(C_0)^4\,\frac{4\pi - 3\sqrt{3}}{4\pi^3} \qquad [D \to 3]$$

- Integrate: $D_0(\mu) = D_0(1/a) + m^3(C_0)^4\,\frac{4\pi - 3\sqrt{3}}{4\pi^3}\,\ln(a\mu)$
- Logarithm must match logarithm from remaining diagrams!

$\Longrightarrow$ for $\mathcal{O}({k_{\mathrm{F}}}^9(\ln k_{\mathrm{F}}))$ contribution to $\mathcal{E}$, evaluate D_0 term at $\mu = k_{\mathrm{F}}$
(avoids large cancellations between diagrams)

- Need renormalization of $3 \to 3$ scattering for complete $\mathcal{O}({k_{\mathrm{F}}}^9)$

Summary & Outlook

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- EFT $\rightarrow$ general approach to systematically exploit separation of scales
- Dilute & large g expansions for many-body systems
  - Dilute case: holes and particles count the same
- Many-body forces are unavoidable and essential
  - Nonanalytic term $\leftrightarrow$ three-body force
- Pairing, finite systems using DFT,...
- Next:
  - Extension to large scattering length $\Rightarrow$ unitary limit