Practical QCD at colliders

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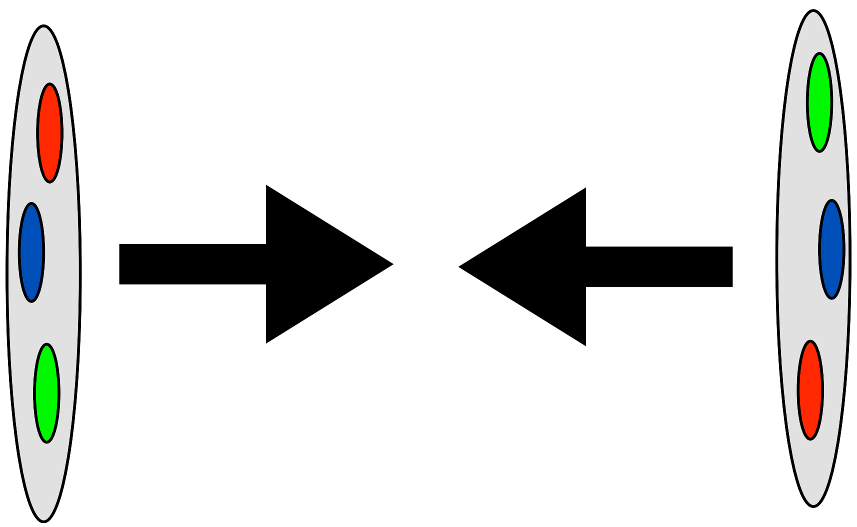
3rd Lecture

Joint ICTP-SAIFR school on Particle Physics — June 2018

Partons in the initial state

- We talked a lot about final state QCD effects
- This is the only thing to worry about at e^+e^- colliders (LEP)
- Hera/Tevatron/LHC involve protons in the initial state
- Proton are made of QCD constituents

Next we will focus mainly on aspects related to initial state effects

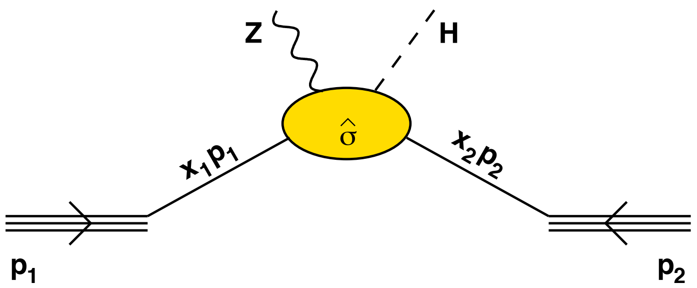


The parton model

Basic idea of the parton model: intuitive picture where in a high transverse momentum scattering partons behave as quasi free in the collision

$\Longrightarrow$ cross section is the incoherent sum of all partonic cross-sections

$$\sigma = \int dx_1 dx_2 f_1^{(P_1)}(x_1) f_2^{(P_2)}(x_2) \hat{\sigma}(x_1 x_2 s) \qquad \hat{s} = x_1 x_2 s$$



NB: This formula is wrong/incomplete (see later)

$f_i^{(P_j)}(x_i)$: parton distribution function (PDF) is the probability to find parton i in hadron j with a fraction x_i of the longitudinal momentum (transverse momentum neglected), extracted from data

$\hat{\sigma}(x_1 x_2 s)$: partonic cross-section for a given scattering process, computed in perturbative QCD

Sum rules

Momentum sum rule: conservation of incoming total momentum

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$$\int_0^1 dx \left(f_u^{(p)}(x) - f_{\bar{u}}^{(p)}(x) \right) = 2$$

$$\int_0^1 dx \left(f_d^{(p)}(x) - f_{\bar{d}}^{(p)}(x) \right) = 1$$

$$\int_0^1 dx \left(f_s^{(p)}(x) - f_{\bar{s}}^{(p)}(x) \right) = 0$$

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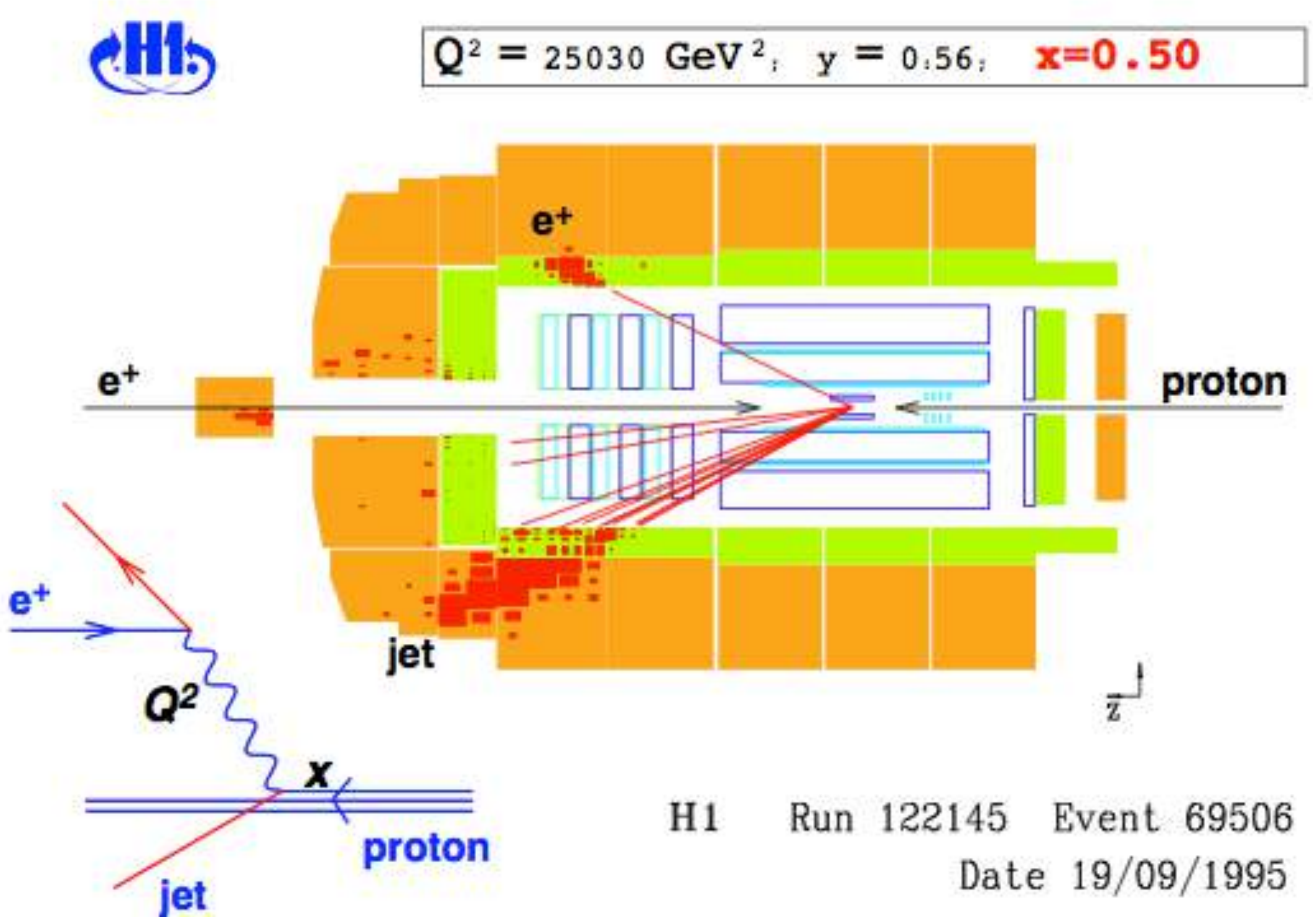
How can parton densities be extracted from data?

Deep inelastic scattering

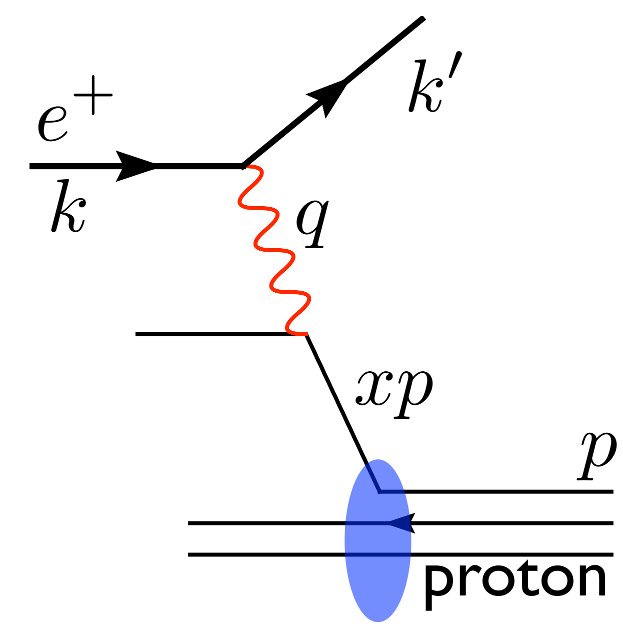
Easier than processes with two incoming hadrons is the scattering of a lepton on a (anti)-proton



Deep inelastic scattering

Protons made up of point-like quarks.
Different momentum scales involved:

- hard photon virtuality (sets the resolution scale) Q
- hard photon-quark interaction Q
- soft interaction between partons in the proton $m_p \ll Q$



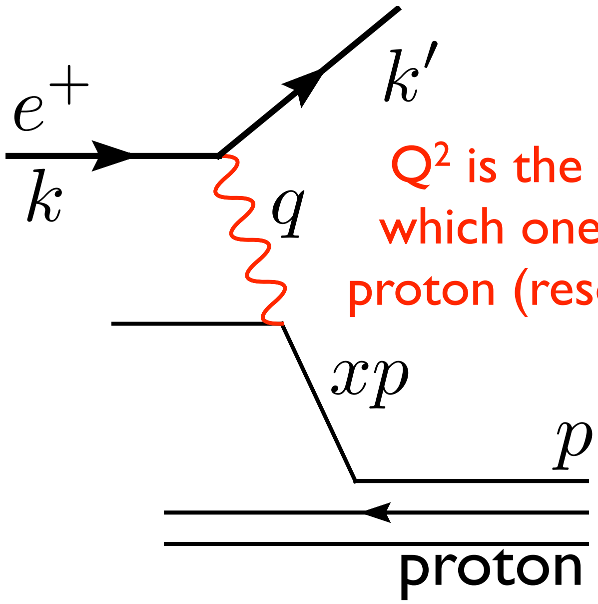
During the hard interaction, partons do not have time to interact among them, they behave as if they were free

⇒ approximate as incoherent scattering on single partons

Deep inelastic scattering

<u>Kinematics:</u>

$$Q^2 = -q^2 \quad s = (k+p)^2 \quad x_{Bj} = \frac{Q^2}{2p\cdot q} \quad y = \frac{p\cdot q}{k\cdot p}$$



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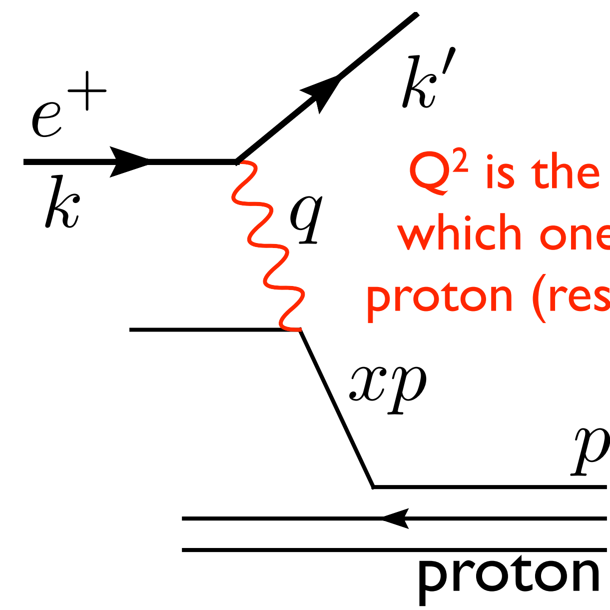

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Partonic variables:

$$\hat{p} = xp \quad \hat{s} = (k+\hat{p})^2 = 2k \cdot \hat{p} \quad \hat{y} = \frac{\hat{p} \cdot q}{k \cdot \hat{p}} = y \quad (\hat{p}+q)^2 = 2\hat{p} \cdot q - Q^2 = 0$$

$$\Rightarrow x = x_{Bj}$$

Hence at leading order, the experimentally accessible x_{Bj} coincides with the momentum fraction carried by the quark in the proton

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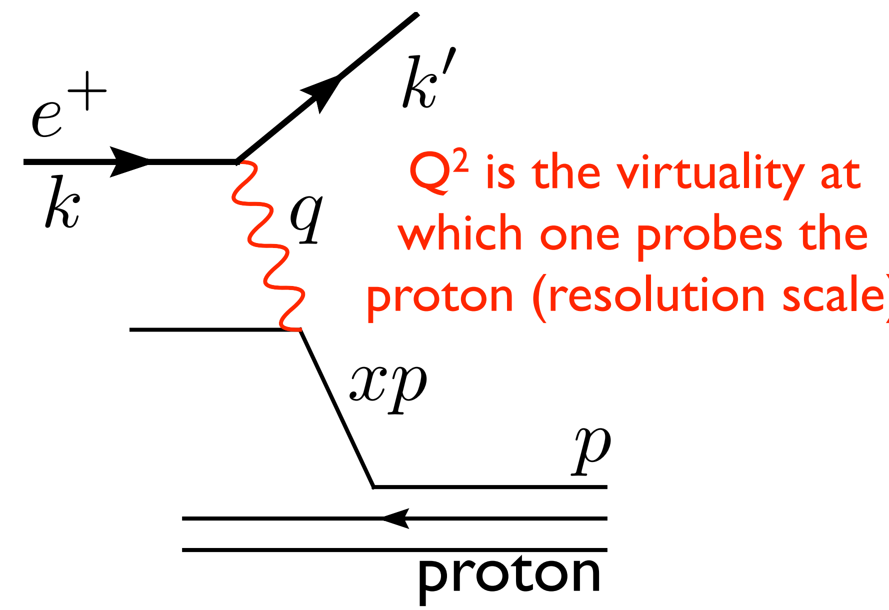

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<u>Partonic cross section:</u>

(apply QED Feynman rules and add phase space)

$$\frac{d\hat{\sigma}}{d\hat{y}} = q_l^2 \frac{\hat{s}}{Q^4} 2\pi\alpha_{em} \left(1 + (1-\hat{y})^2\right)$$

Exercise: show that in the CM frame of the electron-quark system y is given by $(1-\cos\theta_{\rm el})/2$, with $\theta_{\rm el}$ the scattering angle of the electron in this frame

Exercise:

- show that the two particle phase space is $\frac{d\phi}{16\pi}$
- show that the squared matrix element is $\frac{16\pi\alpha q_l^2}{Q^4}\hat{s}xpk\left(1+(1-y)^2\right)$
- show that the flux factor is $\frac{1}{4xpk}$

Hence derive that

$$\frac{d\hat{\sigma}}{d\hat{y}} = q_l^2\frac{\hat{s}}{Q^4}2\,\pi\,\alpha_{em}\left(1+(1-\hat{y})^2\right)$$

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Hadronic cross section (factorization):

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Using x = x_{BJ}

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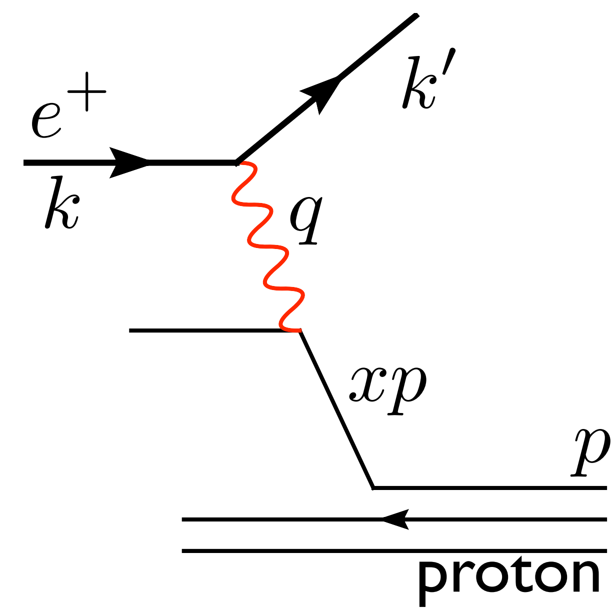
$$= \frac{2\pi\,\alpha_{em}^2 s x_{Bj}}{Q^4} \left(1 + (1-y)^2\right) \sum_l q_l^2 f_l^{(p)}(x_{Bj})$$



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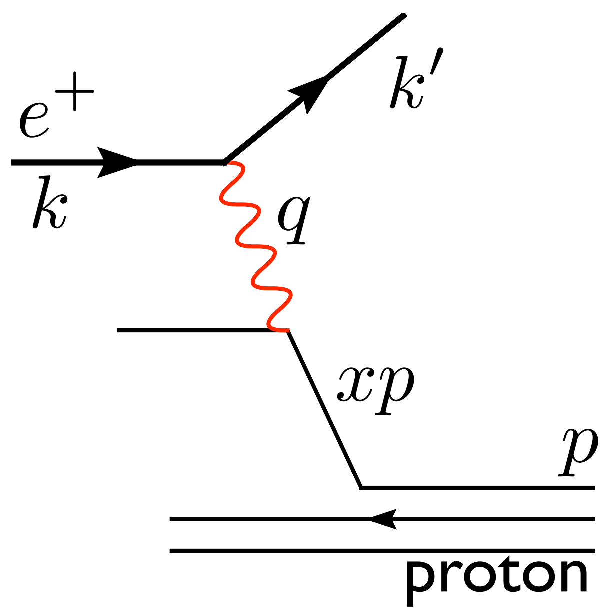

1. at fixed x_{Bj} and y the cross-section scales with s
2. the y-dependence of the cross-section is fully predicted and is typical of vector interaction with fermions ⇒ Callan-Gross relation
3. can access (sums of) parton distribution functions
4. Bjorken scaling: pdfs depend on x and not on Q^2 (violated by logarithmic radiative corrections, see later)

The structure function F_2

$$\frac{d\sigma}{dydx} = \frac{2\pi\alpha_{em}^2 s}{Q^4}\left(1+(1-y^2)\right)F_2(x) \qquad F_2(x) = \sum_l x q_l^2 f_l^{(p)}(x)$$

F_2 is called structure function (describes structure/constituents of nucleus)

For electron scattering on proton

$$F_2(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x)\right)$$

NB: use perturbative language of quarks and gluons despite the fact that parton distribution are non-perturbative

Bjorken scaling: the fact the structure functions are independent of Q is a direct evidence for the existence of point-like quarks in the proton (violated by logarithmic corrections)

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Question: F_2 gives only a linear combination of u and d. How can they be extracted separately?

Isospin

Neutron is like a proton with u & d exchanged

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F_2^n and F_2^p allow determination of u_p and d_p separately

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NB: experimentally get F_2^n from deuteron: $F_2^d(x) = F_2^p(x) + F_2^n(x)$

Sea quark distributions

Inside the proton there are fluctuations, and pairs of $u\bar{u}$,$d\bar{d}$,$c\bar{c}$,$s\bar{s}$ ... can be created

An infinite number of pairs can be created as long as they have very low momentum, because of the momentum sum rules.

We saw before that when we say that the proton is made of uud what we mean is

$$\int_0^1 dx\,(u_p(x) - \bar{u}_p(x)) = 2 \qquad \int_0^1 dx\,\left(d_p(x) - \bar{d}_p(x)\right) = 1$$

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How can one measure the difference?

Question: What interacts differently with particle and antiparticle?

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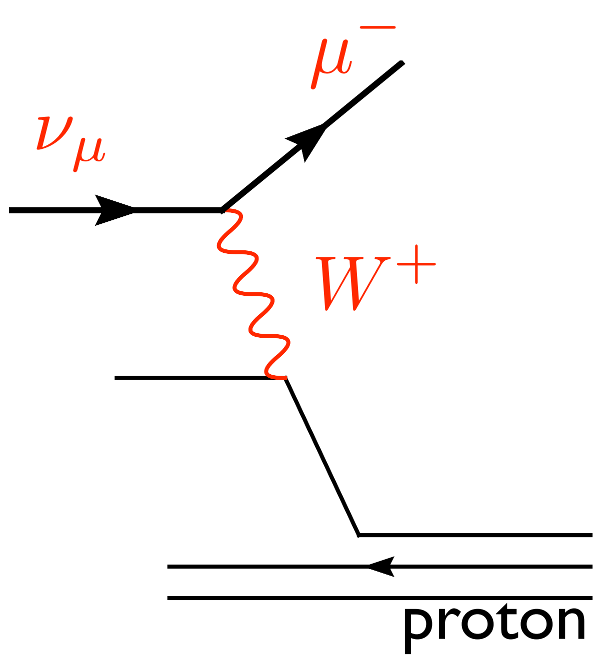
Question: What interacts differently with particle and antiparticle? W^+/W^- from neutrino scattering



Check of the momentum sum rule

$$\int_0^1 dx \sum_i x f_i^{(p)}(x) = 1$$

u_v	0,267
d_v	0,111
u_s	0,066
d_s	0,053
s_s	0,033
c_c	0,016
total	0,546

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$\gamma/W^{+/-}$ don't interact with gluons

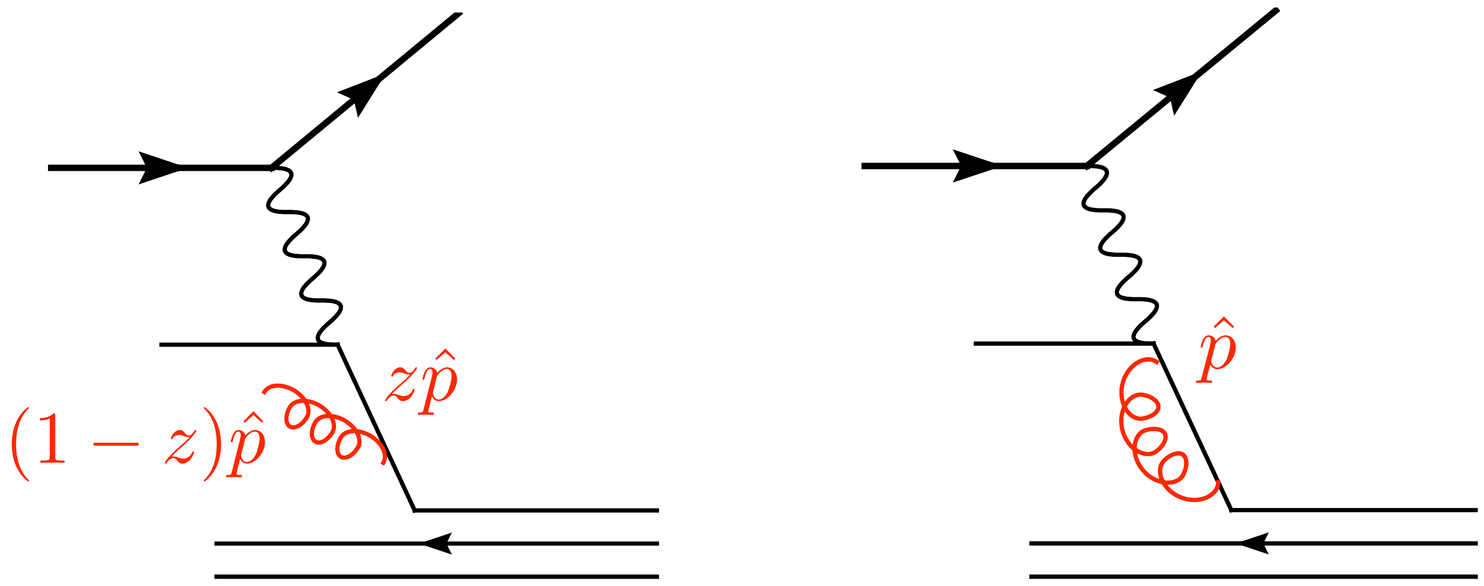
How can one measure gluon parton densities?

We need to discuss radiative effects first

Radiative corrections

To first order in the coupling:

need to consider the emission of one real gluon and a virtual one



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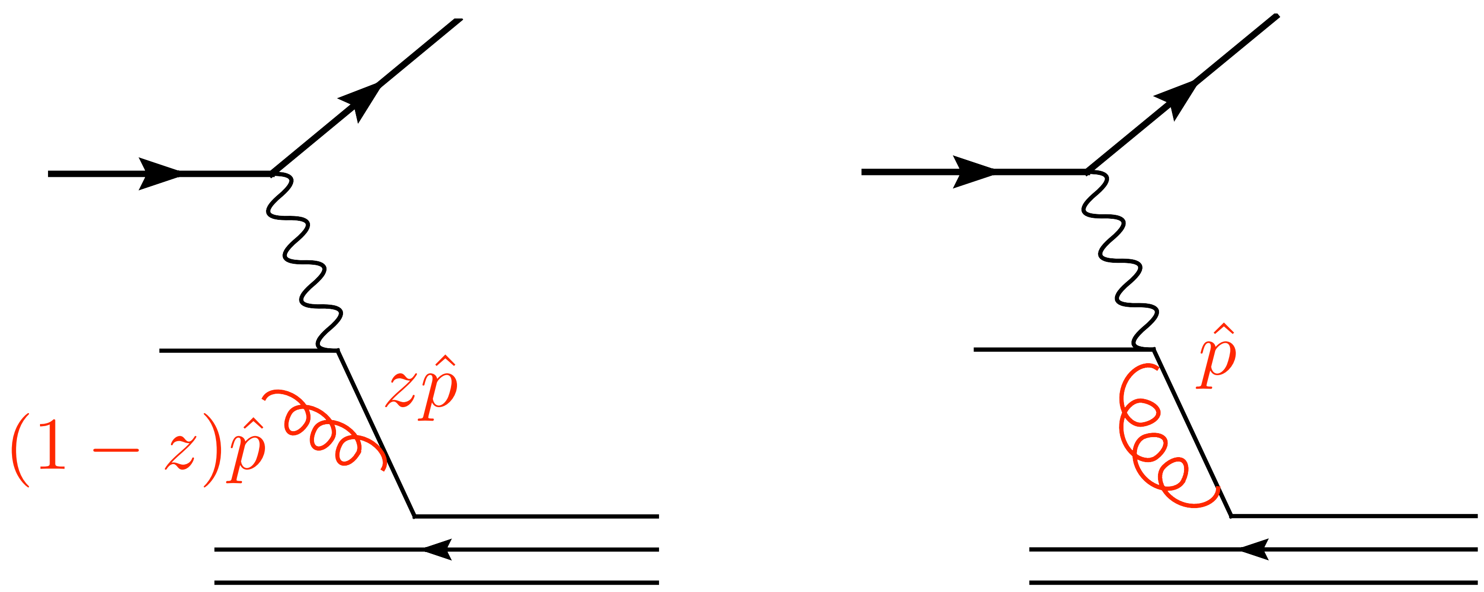

Adding real and virtual contributions, the partonic cross-section reads

$$\sigma^{(1)} = \frac{C_F \alpha_s}{2\pi} \int dz \frac{dk_\perp^2}{k_\perp^2} \frac{1+z^2}{1-z} \left(\sigma^{(0)}(z\hat{p}) - \sigma^{(0)}(\hat{p})\right)$$

Partial cancellation between real (positive), virtual (negative), but real gluon changes the energy entering the scattering, the virtual does not

Radiative corrections

Partonic cross-section:

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Soft limit: singularity at z=1 cancels between real and virtual terms

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$\Rightarrow$ naive parton model does not survive radiative corrections

Similarly to what is done when renormalizing UV divergences, collinear divergences from initial state emissions are absorbed into parton distribution functions

The plus prescription

Partonic cross-section:

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Collinear singularities still there, but they factorize.

Factorization scale

Schematically use $$\ln \frac{Q^2}{\lambda^2} = \ln \frac{Q^2}{\mu_F^2} + \ln \frac{\mu_F^2}{\lambda^2}$$

$$\sigma = \sigma^{(0)} + \sigma^{(1)} = \left(1 + \frac{\alpha_s}{2\pi} \ln \frac{\mu_F^2}{\lambda^2} P_+\right) \times \left(1 + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu_F^2} P_+\right) \sigma^{(0)}$$

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So we define

$$f_q(x, \mu_F) = f_q(x) \times \left(1 + \frac{\alpha_s}{2\pi} \ln \frac{\mu_F^2}{\lambda^2} P_{qq}^{(0)}\right) \qquad \hat{\sigma}(p, \mu_F) = \left(1 + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu_F^2} P_{qq}^{(0)}\right) \sigma^{(0)}(p)$$

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NB:

- universality, i.e. the PDF redefinition does not depend on the process
- choice of $\mu_F \sim Q$ avoids large logarithms in partonic cross-sections
- PDFs and hard cross-sections don't evolve independently
- the factorization scale acts as a cut-off, it allows to move the divergent contribution into non-perturbative parton distribution functions

Improved parton model

Naive parton model:

$$\sigma = \int dx_1 dx_2 f_1^{(P_1)}(x_1) f_2^{(P_2)}(x_2) \hat{\sigma}(x_1 x_2 s) \qquad \hat{s} = x_1 x_2 s$$

After radiative corrections:

$$\sigma = \int dx_1 dx_2 f_1^{(P_1)}(x_1, \mu^2) f_2^{(P_2)}(x_2, \mu^2) \hat{\sigma}(x_1 x_2 s, \mu^2)$$

Intermediate recap

- With initial state parton collinear singularities don't cancel
- Initial state emissions with $k_\perp$ below a given scale are included in PDFs
- This procedure introduces a scale μ_F, the so-called factorization scale which factorizes the low energy (non-perturbative) dynamics from the perturbative hard cross-section
- As for the renormalization scale, the dependence of cross-sections on μ_F is due to the fact that the perturbative expansion has been truncated
- The dependence on μ_F becomes milder when including higher orders
- The redefinition of PDFs is universal and process-independent

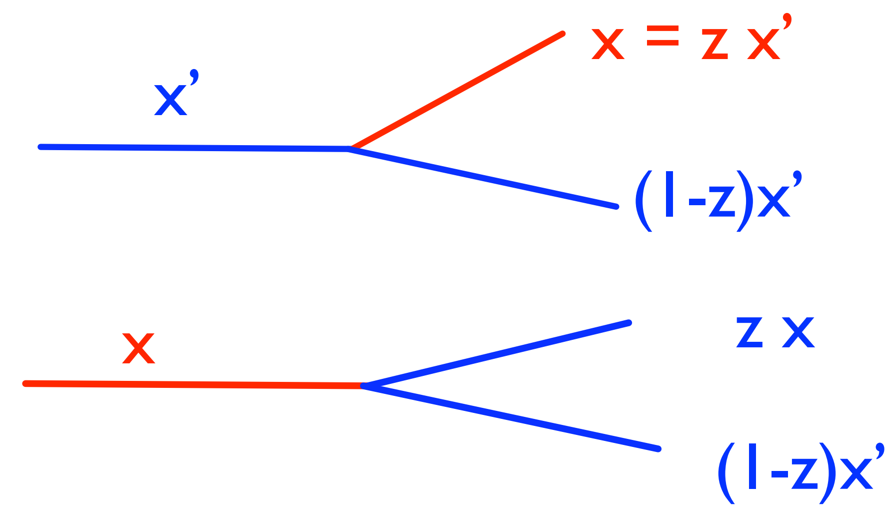
One incoming hard parton: $\sigma = \int dx f^{(P)}(x,\mu^2)\hat{\sigma}(xs,\mu^2)$

Two incoming hard partons: $\sigma = \int dx_1 dx_2 f_1^{(P_1)}(x_1,\mu^2) f_2^{(P_2)}(x_2,\mu^2)\hat{\sigma}(x_1 x_2 s,\mu^2)$

Evolution of PDFs

A parton distribution changes when

- a different parton splits and produces it
- the parton itself splits



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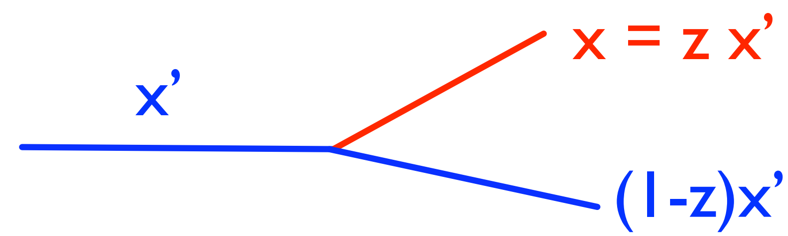
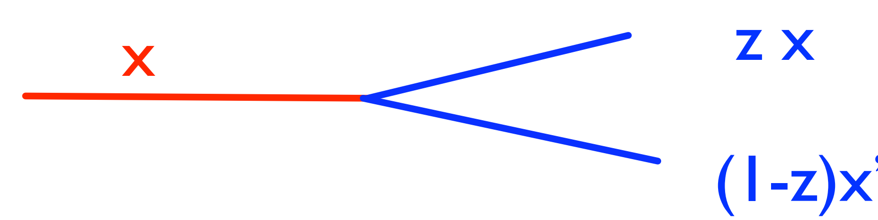


- the parton itself splits



$$\mu^2 \frac{\partial f(x,\mu^2)}{\partial \mu^2} = \int_0^1 dx' \int_x^1 dz \frac{\alpha_s}{2\pi} P(z) f(x',\mu^2) \delta(zx'-x) - \int_0^1 dz \frac{\alpha_s}{2\pi} P(z) f(x,\mu^2)$$

$$= \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P(z) f\left(\frac{x}{z},\mu^2\right) - \int_0^1 dz \frac{\alpha_s}{2\pi} P(z) f\left(x,\mu^2\right)$$

$$= \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P_+(z) f\left(\frac{x}{z},\mu^2\right)$$

<u>The plus prescription</u> $$\int_0^1 dz f_+(z) g(z) \equiv \int_0^1 dz f(z) \left(g(z) - g(1)\right)$$

DGLAP equation

$$\mu^2 \frac{\partial f(x,\mu^2)}{\partial \mu^2} = \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P(z) f\left(\frac{x}{z},\mu^2\right)$$

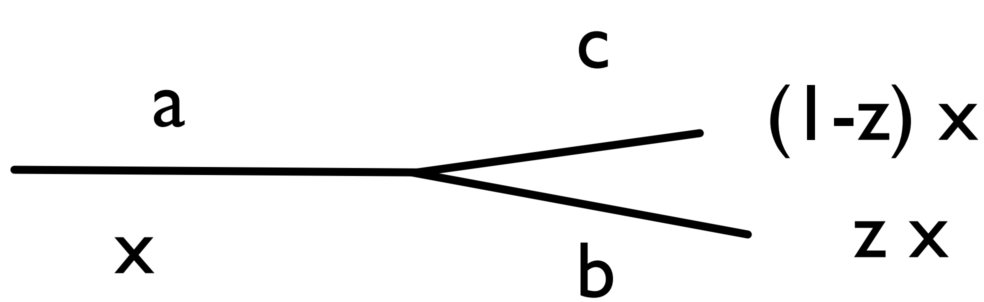
Altarelli, Parisi; Gribov-Lipatov; Dokshitzer '77

Master equation of QCD: we can not compute parton densities, but we can predict how they evolve from one scale to another

Universality of splitting functions: we can measure pdfs in one process and use them as an input for another process

Conventions for splitting functions

There are various partons types. Standard notation:



$P_{ba}(z)$

Accounting for the different species of partons the DGLAP equations become:

$$\mu^2 \frac{\partial f_i(x,\mu^2)}{\partial \mu^2} = \sum_j \int_x^1 \frac{dz}{z} P_{ij}(z) f_j\left(\frac{x}{z},\mu^2\right)$$

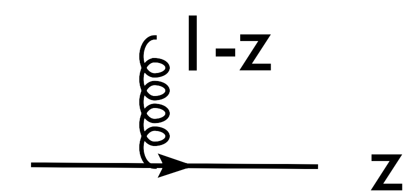
This is a system of coupled integro/differential equations

The above convolution in compact notation:

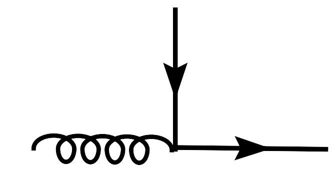
$$\mu^2 \frac{\partial f_i(x,\mu^2)}{\partial \mu^2} = \sum_j P_{ij} \otimes f_j(\mu^2)$$

Properties of splitting functions

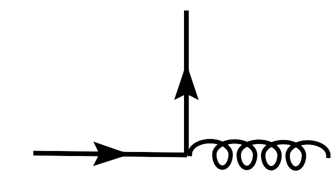
$$P_{qq}^{(0)} = P_{\bar{q}\bar{q}}^{(0)} = C_F\left[\left(\frac{1+z^2}{1-z}\right)_+\right]$$



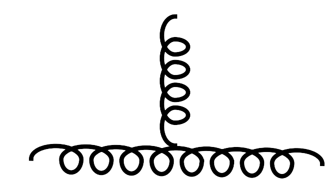
$$P_{qg}^{(0)} = P_{\bar{q}g}^{(0)} = T_R\left(z^2 + (1-z)\right)$$



$$P_{gq}^{(0)} = P_{g\bar{q}}^{(0)} = C_F\,\frac{1+(1-z)^2}{z}$$



$$P_{gg}^{(0)} = 2C_A\left[z\,\frac{1}{(1-z)}_+ + \frac{1-z}{z} + z(1-z) + b_0\delta(1-z)\right]$$



- P_{qg} anf P_{gg} symmetric under z (1-z)
- P_{qq} and P_{gg} divergence for z=1 (soft gluon)
- P_{gq} and P_{gg} divergenge for z=0 (soft gluon)
- P_{qg} no soft divergence for gluon splitting to quarks

➠ gluon PDF grows at small x

Sum rules in pQCD

Beyond the naive parton model the probabilistic picture does not hold anymore. What about basic conservation principles (e.g. sum rules)?

Exercise: show that e.g.

$$\int_0^1 dx\, \big(f_u(x,\mu^2) - f_{\bar{u}}(x,\mu^2)\big) = \mathrm{constant} \quad \text{if and only if} \quad \int_0^1 dz P_{qq}(z) = 0$$

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Solution:

1. Start from DGLAP for u

$$\mu^2 \frac{\partial f_u(x,\mu^2)}{\partial \mu^2} = \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{dz}{z} \left(P_{uu}(z) f_u\left(\frac{x}{z},\mu^2\right) + P_{ug}(z) f_g\left(\frac{x}{z},\mu^2\right)\right)$$

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2. Subtract the same equation for $\bar{u}$ and integrate over x

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3. Swap x and z integration, replace x with y = x/z

$$\mu^2 \frac{\partial}{\partial \mu^2} \int_0^1 dx \left(f_u(x,\mu^2) - f_{\bar{u}}(x,\mu^2) \right) = \frac{\alpha_s(\mu^2)}{2\pi} \int_0^1 dz P_{qq}(z) \int_0^1 dy \left(f_u\left(y,\mu^2\right) - f_{\bar{u}}\left(y,\mu^2\right) \right)$$

Sum rules in pQCD

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$$\mu^2 \frac{\partial}{\partial \mu^2} \int_0^1 dx \left(f_u(x,\mu^2) - f_{\bar{u}}(x,\mu^2)\right) = \frac{\alpha_s(\mu^2)}{2\pi} \int_0^1 dx \int_x^1 \frac{dz}{z} P_{qq}(z) \left(f_u\left(\frac{x}{z},\mu^2\right) - f_{\bar{u}}\left(\frac{x}{z},\mu^2\right)\right)$$

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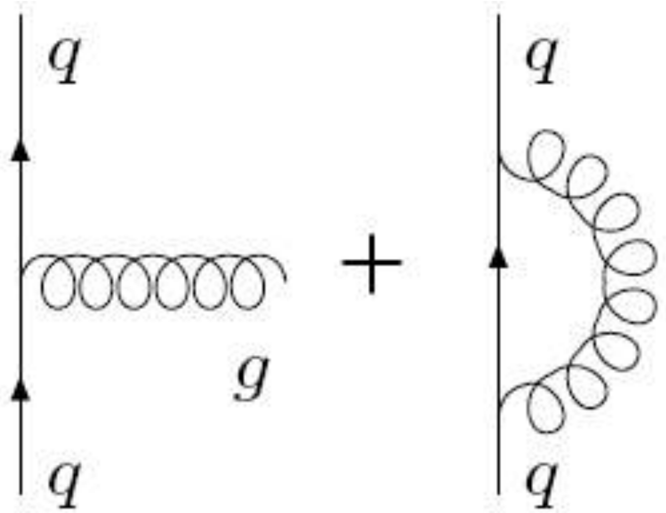
Conclusion: the integral $\int_0^1 dx \left(f_u(x,\mu^2) - f_{\bar{u}}(x,\mu^2)\right)$

does not depend on the scale if, and only if $\int_0^1 dz P_{qq}(z) = 0$

Properties of splitting functions

$$P_{qq}^{(0)} = P_{\bar{q}\bar{q}}^{(0)} = C_F \left[\left(\frac{1+z^2}{1-z} \right)_+ + \frac{3}{2}\delta(1-z) \right]$$



☛ the delta-term is the virtual correction (present only when the flavour does not change)

We have just seen that in order to conserve quark (baryon) number, the integral of the quark distribution can not vary with Q^2 ,hence, the splitting functions must integrate to zero

Exercise: use this fact to compute the coefficients of the pure delta terms in P_{qq} and P_{gg} without performing the loop integral!

History of splitting functions

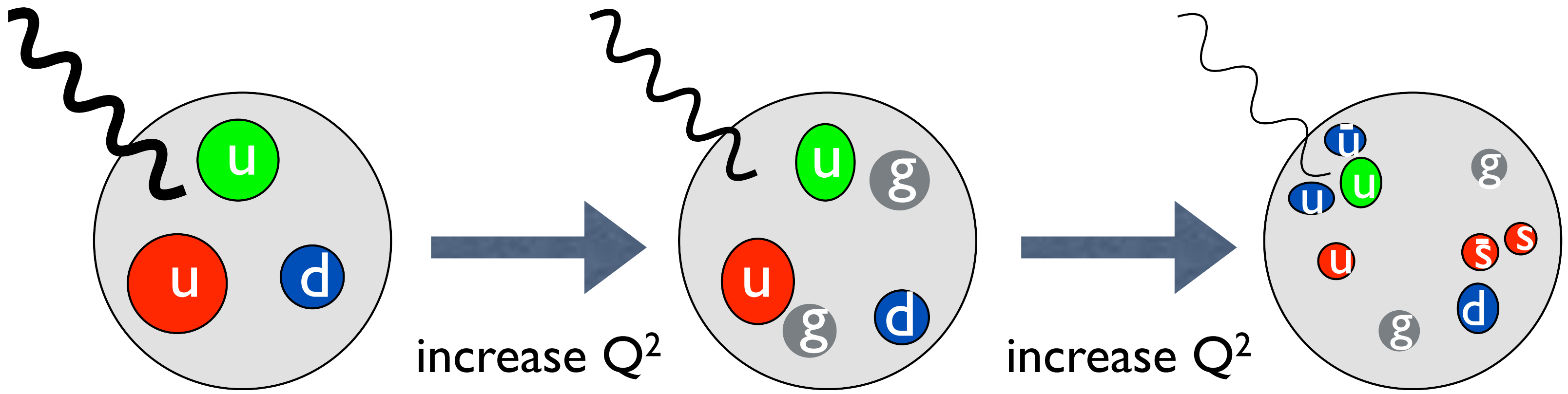
- ☑ $P_{ab}^{(0)}$: Altarelly, Parisi; Gribov-Lipatov; Dokshitzer (1977)
- ☑ $P_{ab}^{(1)}$: Curci, Furmanski, Petronzio (1980)
- ☑ $P_{ab}^{(2)}$: Moch, Vermaseren, Vogt (2004)

☛ Essential input for NNLO pdfs determination (state of the art today)

Evolution

So, in perturbative QCD we can not predict values for

- the coupling
- the masses
- the parton densities
- ...



What we can predict is the evolution with the Q^2 of those quantities. These quantities must be extracted at some scale from data.

- not only is the coupling scale-dependent, but partons have a scale dependent sub-structure
- we started with the question of how one can access the gluon pdf: Because of the DGLAP evolution, we can access the gluon pdf indirectly, through the way it changes the evolution of quark pdfs. Today also direct measurements using Tevatron jet data and LHC tt and jet data

Recap.

- Parton model: incoherent sum of all partonic cross-sections
- Sum rules (momentum, charge, flavor conservation)
- Determination of parton densities (electron & neutrino scattering)
- Radiative corrections: failure of parton model
- Factorization of initial state divergences into scale dependent parton densities
- DGLAP evolution of parton densities $\Rightarrow$ measure gluon PDF
- While PDFs loose the naive probabilistic interpretation basic conservation principle still hold (momentum sum rules, energy, flavour conservation)

DGLAP in Mellin space

How does one solve DGLAP equations?

One possibility: go to Mellin space

$$f_i(N,\mu^2) = \int_0^1 dx\, x^{N-1}\, f_i(x,\mu^2)$$

The advantage of Mellin transform: convolutions ⇒ ordinary products

Exercise: show that $(f \otimes g)(N) = f(N)g(N)$

The disadvantage of Mellin transform: need to evaluate inverse Mellin transform at the end

$$f_i(x,\mu^2) = \frac{1}{2\pi i}\int_C dN\, x^{-N}\, f_i(N,\mu^2)$$

Exercise: show that the above is indeed the inverse Mellin transform

Anomalous dimensions

Evolution equation for the non-singlet in Mellin space (for simplicity)

$$\mu^2 \frac{\partial q^{\rm NS}(N,\mu^2)}{\partial \mu^2} = \frac{\alpha_s(\mu^2)}{2\pi} \gamma_{qq}(N,\alpha_s(\mu^2))\; q^{\rm NS}(N,\mu^2)$$

Where the anomalous dimension is given by

$$\gamma_{qq}(N,\alpha_s(\mu^2)) = \int_0^1 dx\, x^{N-1}\, P_{qq}(x,\alpha_s)$$

And similarly for the gluon and singlet component. At leading order:

$$\gamma_{qq}^{(0)} = C_F \left\{ -\frac{1}{2} + \frac{1}{N(N+1)} - 2\sum_{k=2}^{N} \frac{1}{k} \right\}$$

$$\gamma_{qg}^{(0)} = T_R \left\{ \frac{2+N+N^2}{N(N+1)(N+2)} \right\} \qquad \gamma_{qg}^{(0)} = C_F \left\{ \frac{2+N+N^2}{N(N^2-1)} \right\}$$

$$\gamma_{gg}^{(0)} = 2C_A \left\{ -\frac{1}{12} + \frac{1}{N(N-1)} + \frac{1}{(N+1)(N+2)} - \sum_{k=2}^{N} \frac{1}{k} \right\} - \frac{2}{3} n_f T_R$$

Solution in Mellin space

Given the anomalous dimension, the equation for non-singlet is

$$\mu^2 \frac{\partial q^{\rm NS}(N,\mu^2)}{\partial \mu^2} = \frac{\alpha_s(\mu^2)}{2\pi} \gamma_{qq}(N,\alpha_s(\mu^2))\; q^{\rm NS}(N,\mu^2)$$

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To lowest order one has

$$\alpha_s(\mu^2) = \frac{1}{b_0 \ln \frac{\mu^2}{\Lambda^2}} \qquad \gamma_{qq}(N,\alpha_s(\mu^2)) = \gamma_{qq}^{(0)}(N)$$

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Integrate the equation

$$q^{\rm NS}(N,Q^2) = q^{\rm NS}(N,Q_0^2) \left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)} \right)^{d^{(0)}(N)} \qquad\qquad d^{(0)}(N) = \frac{\gamma^{(0)}(N)}{2\pi b_0}$$

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Finally need to take in inverse Mellin transform to go back to x-space (usually this can be done only numerically)

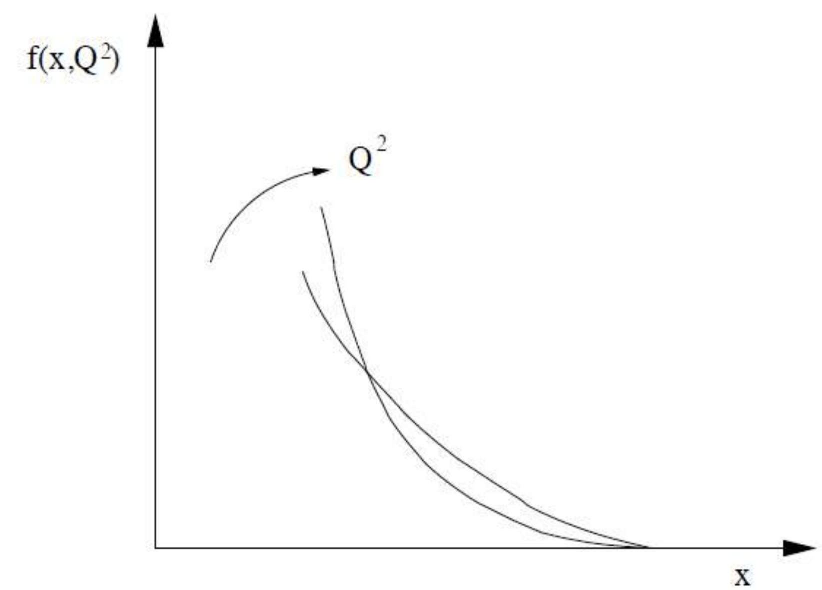
Solution in x space

$$q^{\mathrm{NS}}(N,Q^2) = q^{\mathrm{NS}}(N,Q_0^2)\left(\frac{\alpha_s(Q_0^2)}{\alpha_s(Q^2)}\right)^{d^{(0)}(N)} \quad \Rightarrow \quad q^{\mathrm{NS}}(x,Q^2) = \frac{1}{2\pi i}\int_C dN x^{-N} q^{\mathrm{NS}}(N,Q^2)$$

Explicit result shows that

$$d_{qq}^{(0)}(1) = 0 \qquad d_{qq}^{(0)}(N) < 0 \qquad N > 1$$

$$q^{\mathrm{NS}}(N,Q^2) < q^{\mathrm{NS}}(N,Q_0^2) \qquad Q^2 > Q_0^2 \qquad N > 1$$



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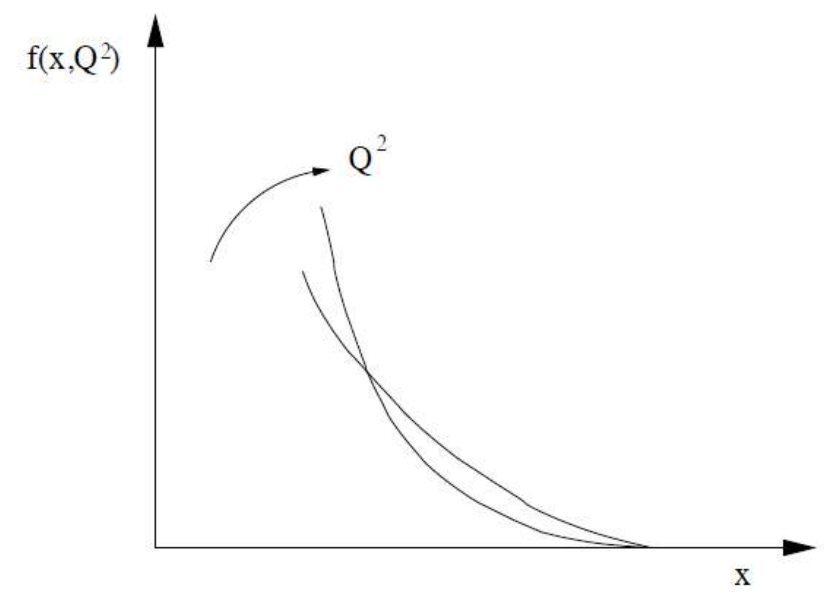

Large N ↔ small x (and viceversa)

Solution in x space

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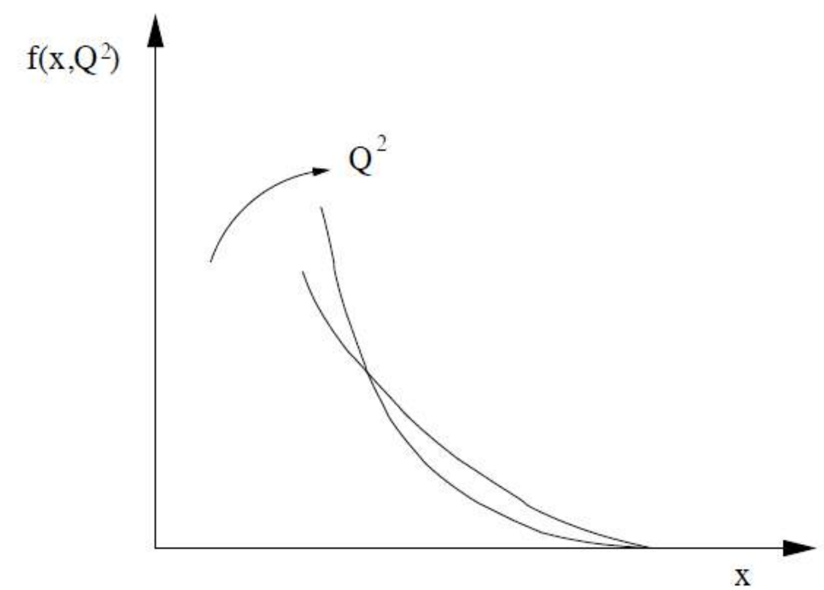


Large N ↔ small x (and viceversa)

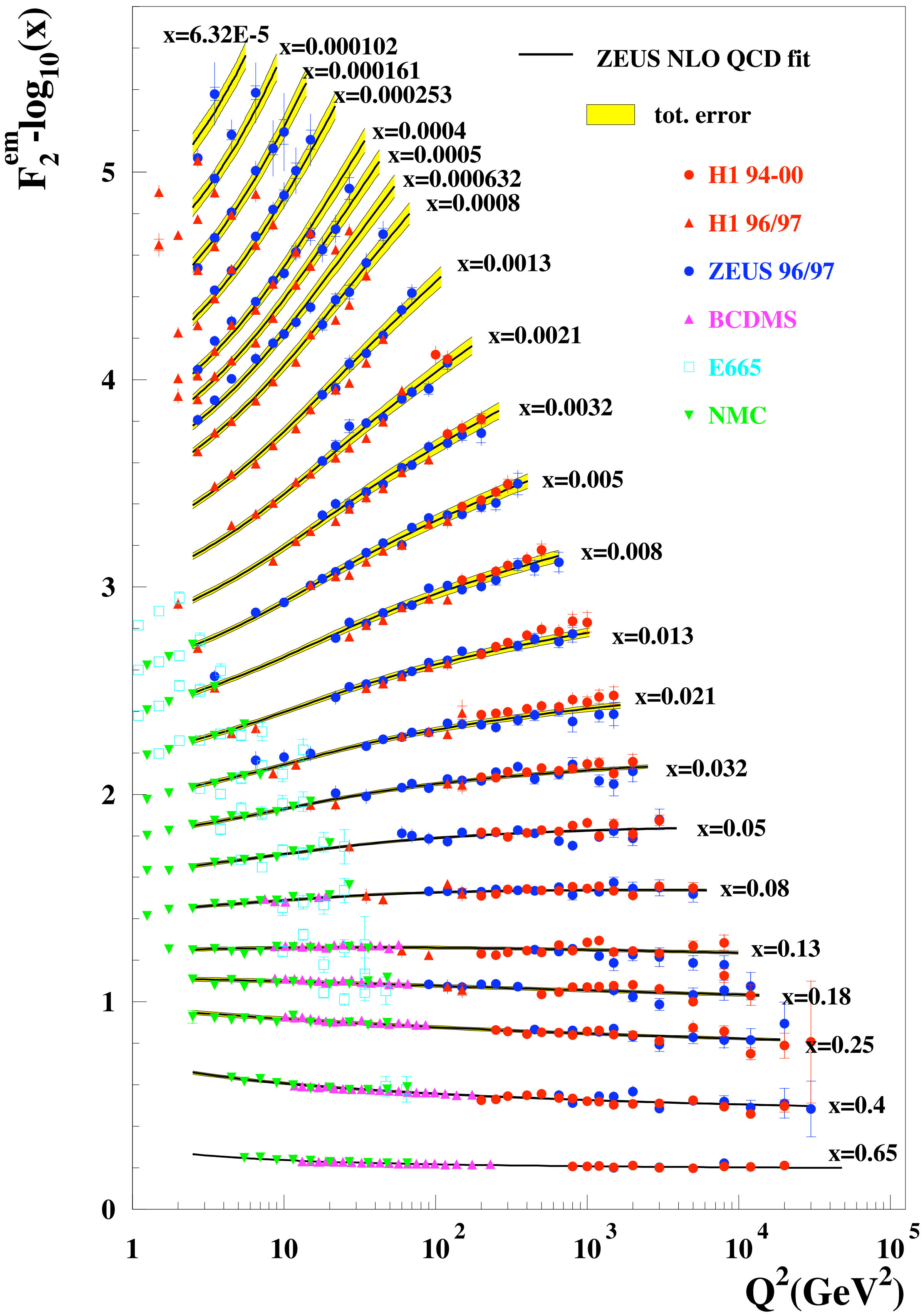
Increasing Q² $q^{NS}(x,Q^2)$ decreases at large x and increases at small x

Physically: at larger x more phase space for gluon emission ⇒ reduction of quark momentum

Main effect of increasing Q² is to shift partons from larger to smaller x

Data: F_2

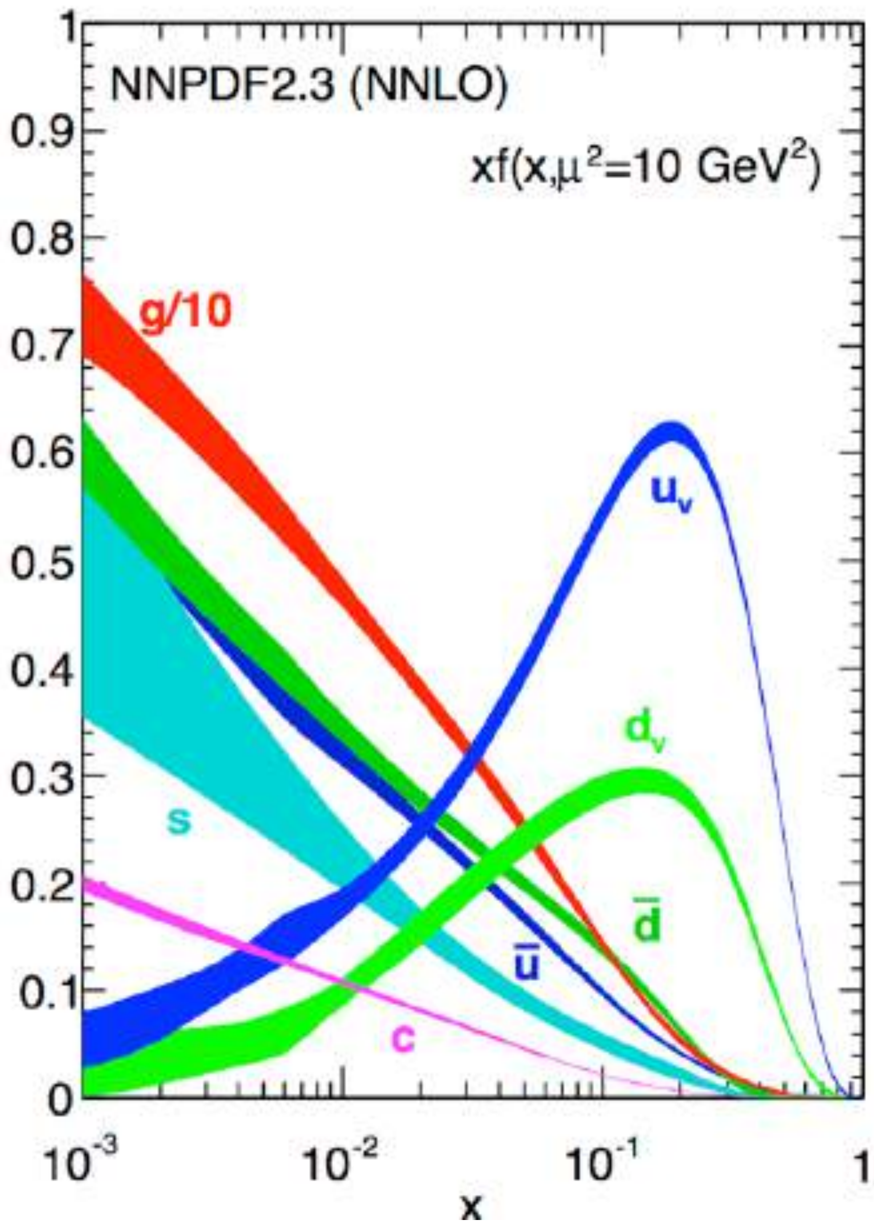
- DGLAP evolution equations allow to predict the Q^2 dependence of DIS data
- gluons crucial in driving the evolution



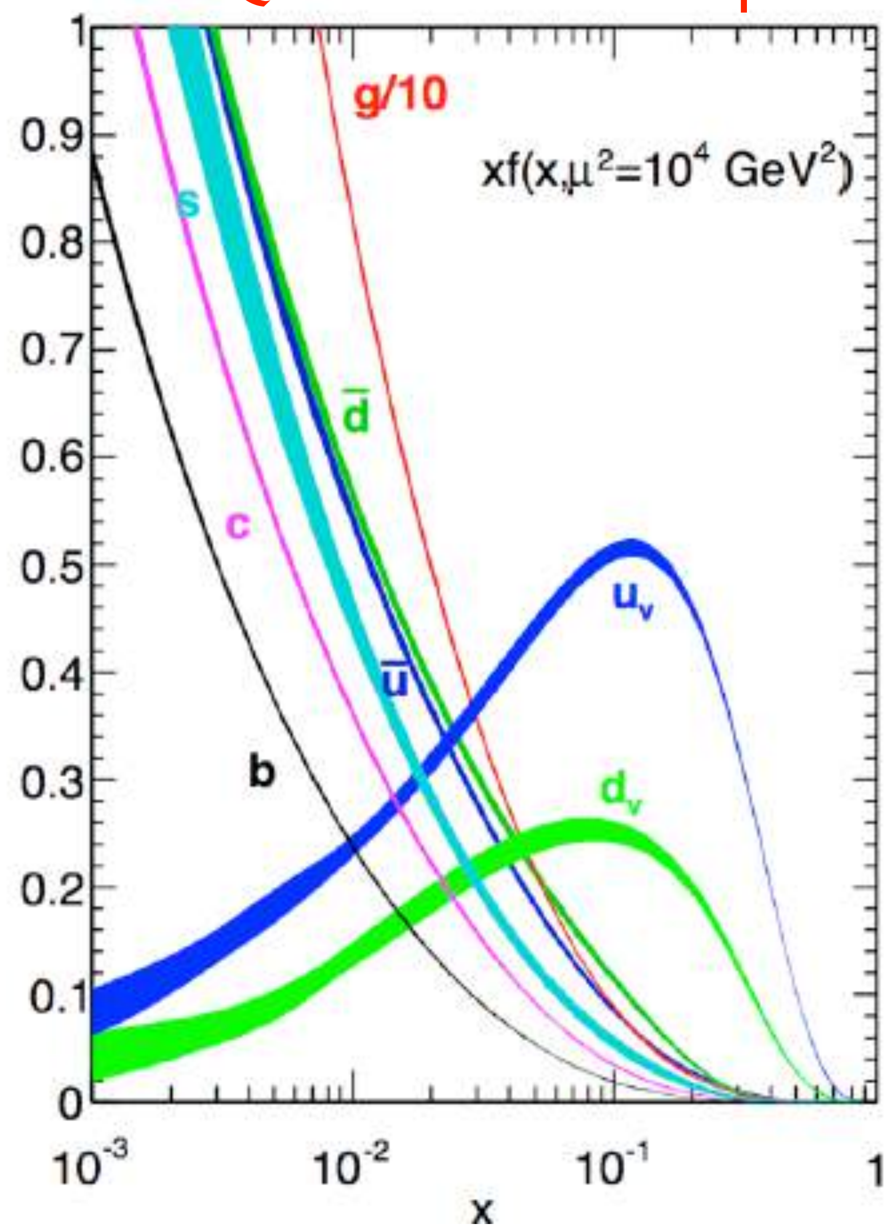
DGLAP Evolution

The DGLAP evolution is a key to precision LHC phenomenology: it allows to measure PDFs at some scale (say in DIS) and evolve upwards to make LHC (7, 8, 13, 14, 33, 100.... TeV) predictions

Measure PDFs at 10 GeV



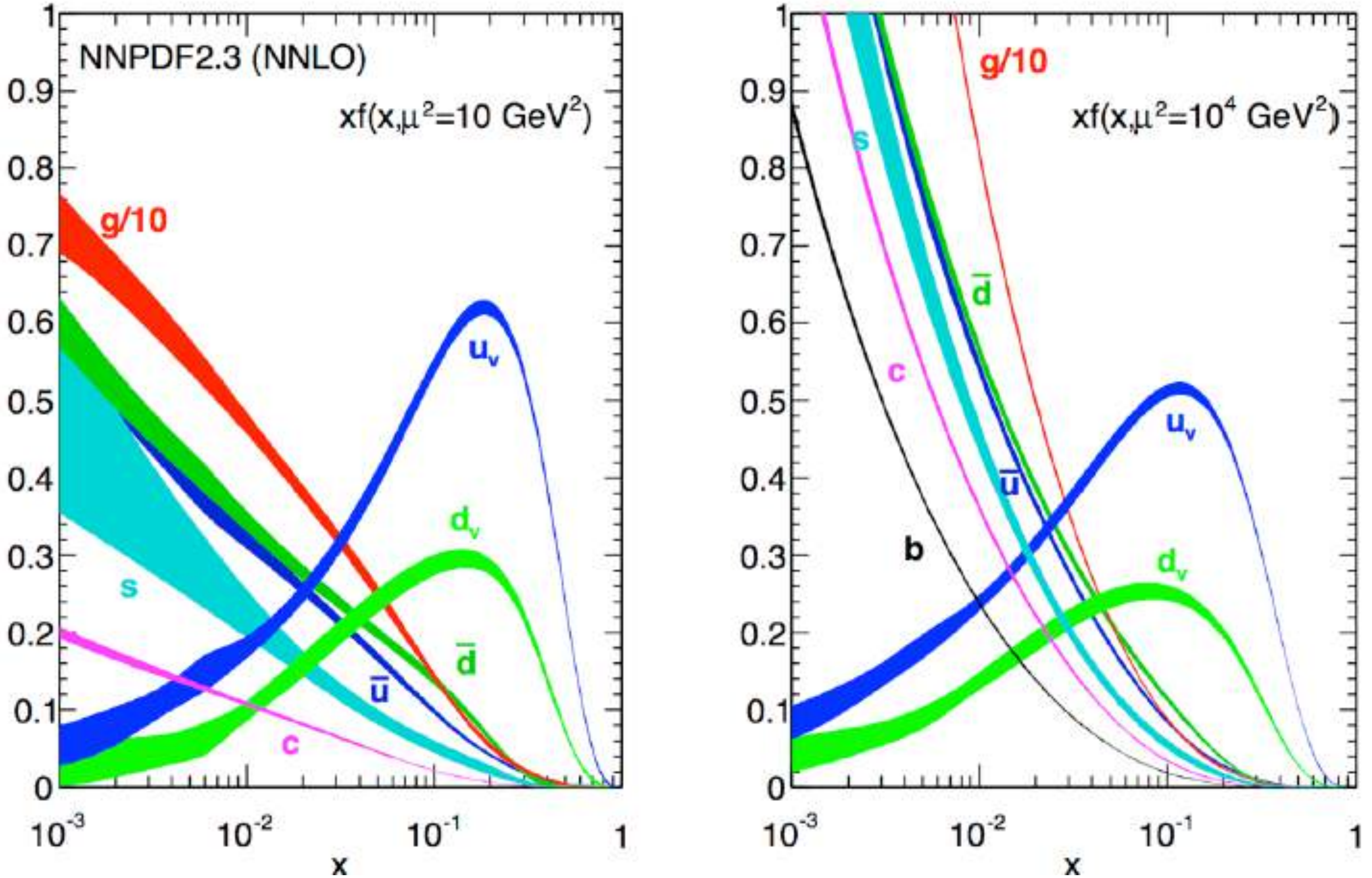
Evolve in Q^2 and make LHC predictions



Different PDFs evolve in different ways (different equations + unitarity constraint)

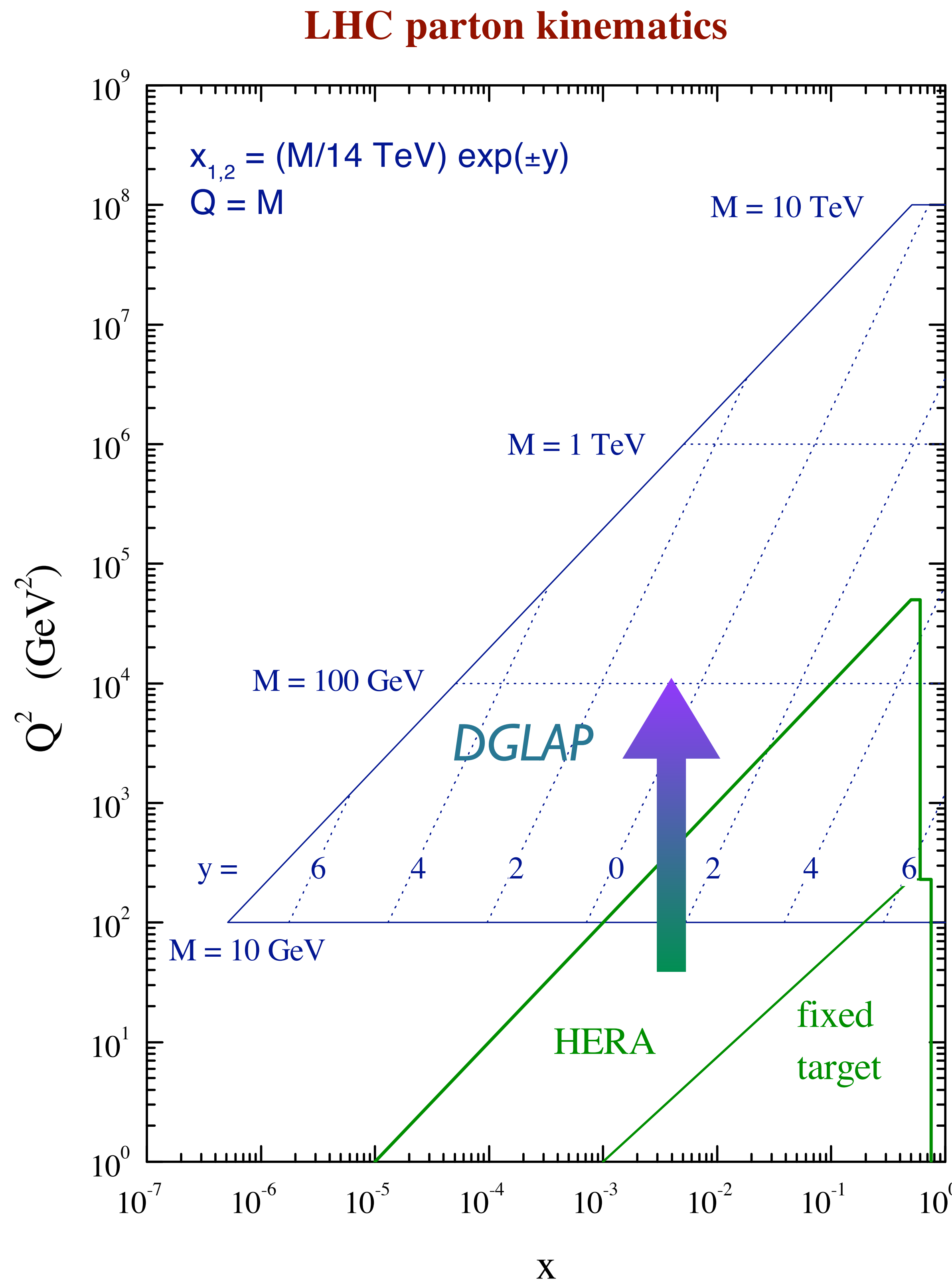
Typical features of PDFs

- vanish at $x \to 1$
- valence quarks peak at $x \simeq 1/3$
- gluon and sea distribution rise for $x \to 0$ (region dominated by gluons)



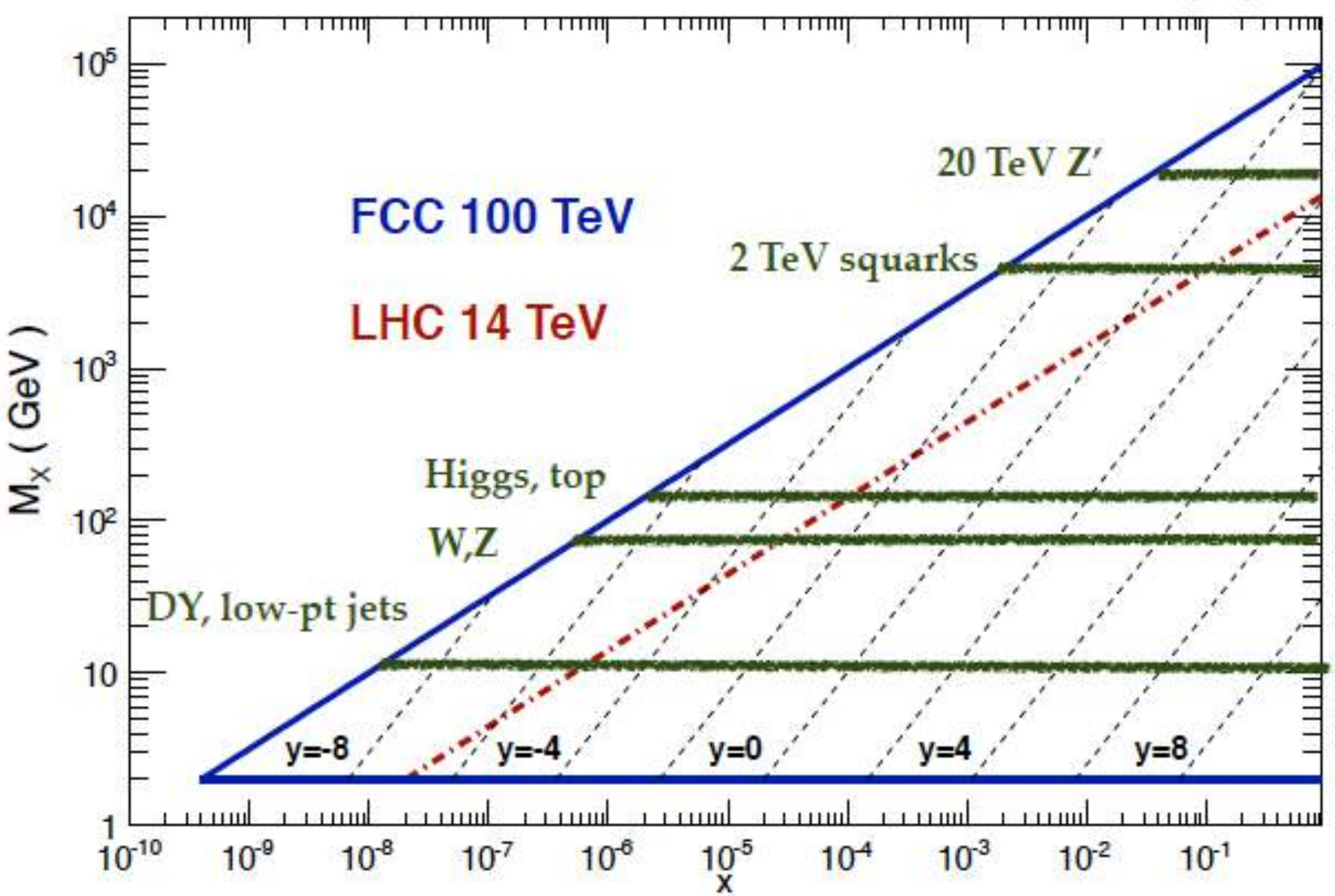
Parton density coverage

- most of the LHC x-range covered by Hera
- need 2-3 orders of magnitude Q^2-evolution
- rapidity distributions probe extreme x-values
- 100 GeV physics at LHC: small-x, sea partons
- TeV physics: large x



Parton density coverage

Coverage of 14 TeV LHC with respect to 100 TeV FCC



Progress in PDFs

PDFs are an essential ingredient for the LHC program.

Recent progress includes

- better assessment of uncertainties (e.g. different groups now agree at the 1σ level where data is available)
- exploit wealth of new information from LHC Run I and Run II measurements
- progress in tools and methods to include these data in the fits
- inclusion of PDFs for photons

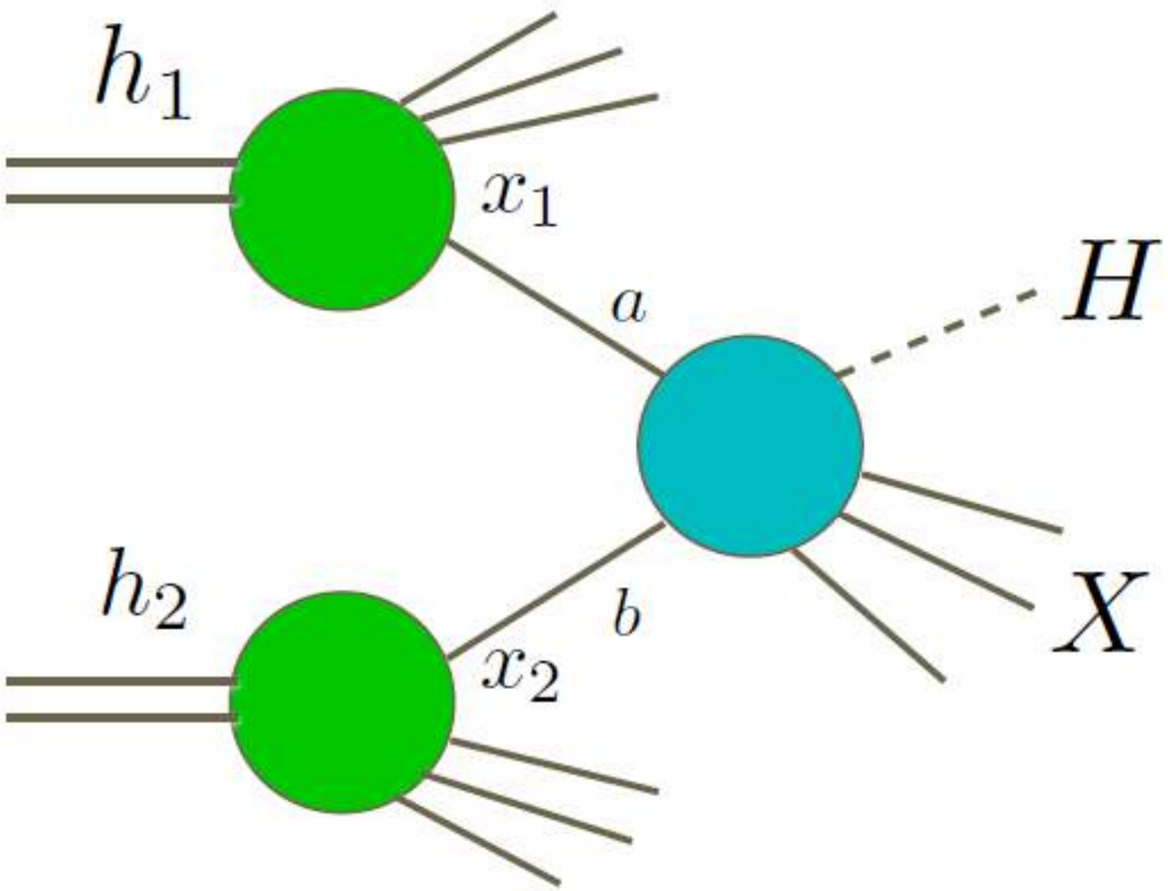
Progress in PDFs

Some issues

- which data to include in the fits (and how to deal with incompatible data)
- enhance relevance of some data (reduce effect of inconsistent data sets)
- heavy-quark treatment and masses
- parametrization for PDFs (theoretical bias, reduced in Neural Network PDFs)
- include theoretical improvement (e.g. resummation) for some observables
- unphysical behaviour close to x=0 and x=1
- meaning of uncertainties
- α_s as external input or fitted with PDFs

Parton luminosities

Even more interesting that PDFs are parton luminosities for each production channel

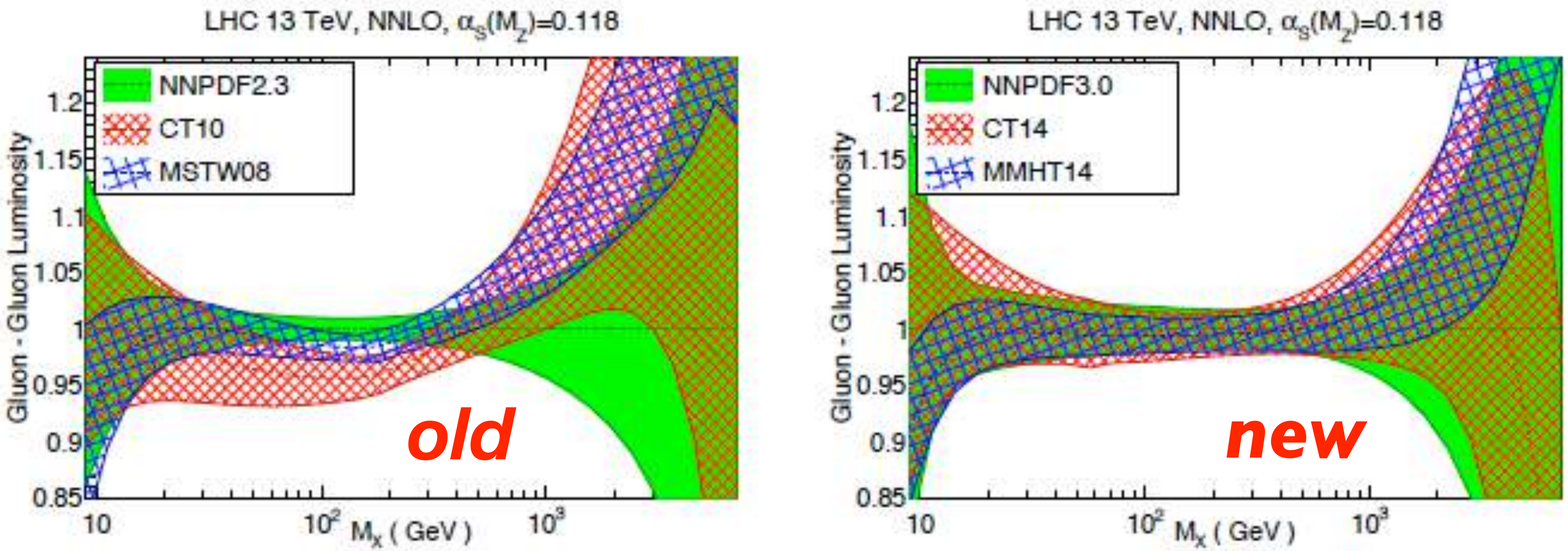


$$\sigma(S) = \sum_{i,j} \int d\tau \left[\frac{1}{S} \frac{dL_{ij}}{d\tau} \right] [\hat{s}\hat{\sigma_{ij}}]$$

$$\tau \frac{dL_{ij}}{d\tau} = \int_0^1 dx_1 dx_2 x_1 f_i(x_1, \mu_F^2) \times x_2 f_j(x_2, \mu_F^2) \delta(\tau - x_1 x_2)$$

Progress in PDFs: gluon luminosity

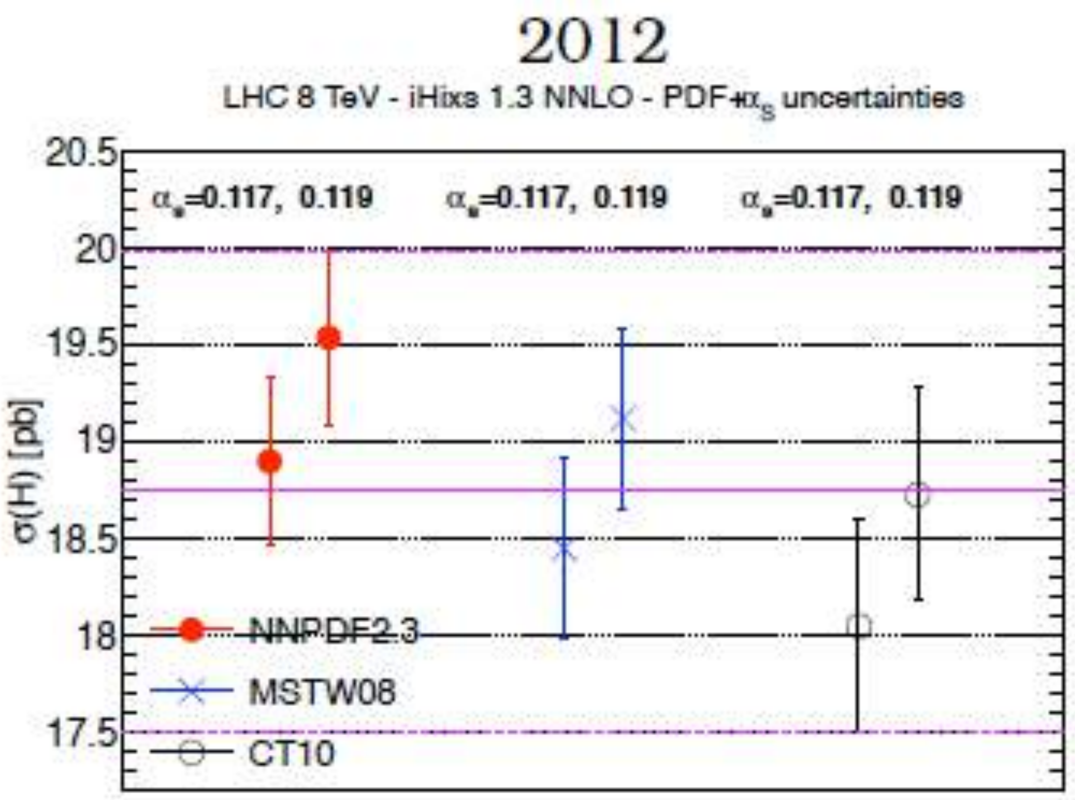
<u>Example</u>: gluon-gluon luminosity as needed for Higgs measurements

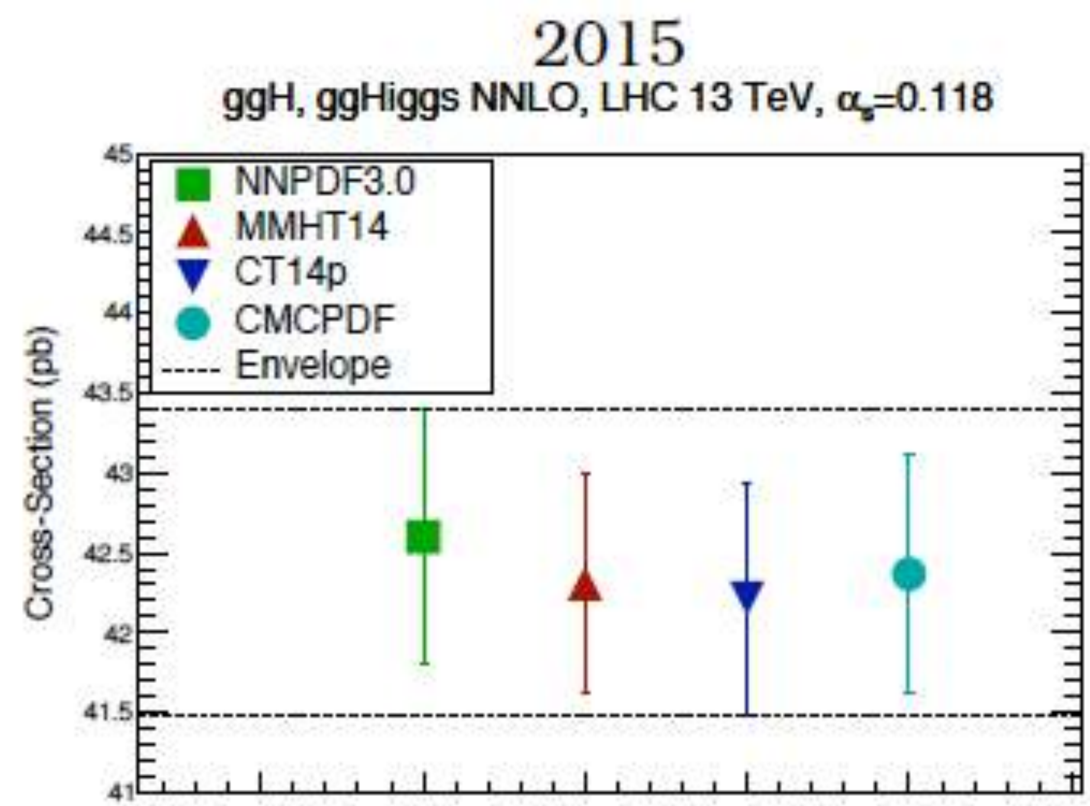


- obvious improvement from older sets to newer ones
- agreement at 1σ between different PDFs in the intermediate mass region relevant for Higgs studies (but larger differences at large M, key-region for NP searches)

Progress in PDFs: Higgs case

Improved control on gluon distributions results in more consistent Higgs production cross-sections





- PDF uncertainty in the Higgs cross-section down to about 2-3%
- envelope of 3 PDFs (previous recommendation) no longer needed

Summary

In this lecture we have learnt that

- In the QCD parton model, hadrons are treated as bound states of quasi-fee point-like quarks is very successful to explain DIS measurements
- In this model, the probability to find a parton with a given momentum fraction is given by the parton distribution function, which, in the model, are scale independent
- The model breaks down as soon as one includes radiative corrections from the initial state since collinear divergences do not cancel
- This leads to scale dependent parton distribution functions
- The dependence is governed by the DGLAP evolution equations
- QCD factorisation means that PDFs are universal and process-independent quantities: they can be measured in some process, at some scale, and use in a different process at a different scale
- PDFs are today determined by global fits to data