Quantum Gravity from the QFT perspective

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Lecture 4.

Renormalization in QG

- **Gauge symmetry and renormalization. Power counting.**
- **Quantum GR vs higher derivative theory (HDQG).**
- **Superrenormalizable QG.**
- **Ghosts in general HDQG.**
- **Ambiguities and 1-loop results.**

Bibliography

K. Stelle, Phys. Rev. D (1977).

I.L. Buchbinder, S.D. Odintsov, I.Sh., Effective Action in Quantum Gravity. (1992 - IOPP).

M. Asorey, J.L. Lopez, I.Sh., hep-th/9610006; In.J.M.Ph. A12(1997).

We start from some covariant action of gravity,

$$S = \int d^4x\sqrt{-g}\;\mathcal{L}(g_{\mu\nu})\,.$$

$\mathcal{L}(g_{\mu\nu})$ **can be of GR,** $\mathcal{L}(g_{\mu\nu}) = -\kappa^{-2}(R+2\Lambda)$ **or some other.**

Gauge transformation $x'^{\mu} = x^{\mu} + \xi^{\mu}$**. The metric transforms as**

$$\delta g_{\mu\nu} = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -\nabla_\mu\xi_\nu - \nabla_\nu\xi_\mu\,.$$

It is customary to parametrize metric as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)\,,$$

then

$$\delta h_{\mu\nu} = -\frac{1}{\kappa}\,(\partial_\mu\xi_\nu - \partial_\nu\xi_\mu) - h_{\mu\alpha}\partial_\nu\xi^\alpha - h_{\nu\alpha}\partial_\mu\xi^\alpha - \xi^\alpha\partial_\alpha h_{\mu\nu} = R_{\mu\nu\,,\,\alpha}\xi^\alpha\,.$$

The gauge invariance of the action means

$$\frac{\delta S}{\delta h_{\mu\nu}}\;\cdot\;R_{\mu\nu\,,\,\alpha}\,\xi^\alpha = 0\,.$$

It proves useful to divide $h_{\mu\nu}$ into irreducible components

$$\omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}\,, \qquad \theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu}\,.$$

The (Rivers) projectors are

$$P^{(2)}_{\mu\nu\,,\,\alpha\beta} = \frac{1}{2}(\theta_{\mu\alpha}\theta_{\nu\beta} - \theta_{\mu\beta}\theta_{\nu\alpha}) - \frac{1}{3}\,\theta_{\mu\nu}\theta_{\alpha\beta}\,;$$

$$P^{(1)}_{\mu\nu\,,\,\alpha\beta} = \frac{1}{2}\,(\theta_{\mu\alpha}\omega_{\nu\beta} + \theta_{\nu\alpha}\omega_{\mu\beta} + \theta_{\mu\beta}\omega_{\nu\alpha} + \theta_{\nu\beta}\omega_{\mu\alpha})\,;$$

$$P^{(0-s)}_{\mu\nu\,,\,\alpha\beta} = \frac{1}{3}\,\theta_{\mu\nu}\theta_{\alpha\beta}\,, \qquad P^{(0-w)}_{\mu\nu\,,\,\alpha\beta} = \omega_{\mu\nu}\omega_{\alpha\beta}\,;$$

and the transfer operators

$$P^{(ws)}_{\mu\nu\,,\,\alpha\beta} = \frac{1}{\sqrt{3}}\,\theta_{\mu\nu}\omega_{\alpha\beta}\,, \qquad P^{(sw)}_{\mu\nu\,,\,\alpha\beta} = \frac{1}{\sqrt{3}}\,\omega_{\mu\nu}\theta_{\alpha\beta}\,.$$

Then for the propagator of the gravitational field we get

$$\langle h_{\mu\nu}\; h_{\alpha\beta}\rangle \;=\; \begin{pmatrix} P^{(2)} & 0 & 0 & 0 \\ 0 & P^{(1)} & 0 & 0 \\ 0 & 0 & P^{(0-s)} & P^{(sw)} \\ 0 & 0 & P^{(sw)} & P^{(0-s)} \end{pmatrix}$$

Also, one can present the quantum metric as

$$h_{\mu\nu} = h^{\perp}_{\mu\nu} + \frac{1}{4}\,\eta_{\mu\nu}\varphi + 2\nabla_{(\mu}\varepsilon_{\nu)}\,,$$

where

$$h^{\perp}_{\mu\nu} = P^{(2)\;\alpha\beta}_{\mu\nu}\,h_{\alpha\beta}\,,\quad 2\nabla_{(\mu}\varepsilon_{\nu)} = P^{(1)}{}_{\mu\nu}{}^{\alpha\beta}\,h_{\alpha\beta}\,,$$

and

$$\varepsilon_{\mu} = \varepsilon^{\perp}_{\mu} + \partial_{\mu}\sigma\,,\quad \varphi = h - \Box\sigma\,,\quad h = h^{\mu}_{\mu}\,.$$

Performing gauge transformation, we get

$\delta h^{\perp}_{\mu\nu} = 0\,,\quad \delta\varphi = 0\,,\quad \delta\sigma = -2\xi\,,\quad \delta\varepsilon^{\perp}_{\mu} = -\xi^{\perp}_{\mu}\,,$ **where** $\xi_{\mu} = \xi^{\perp}_{\mu} + \partial\xi\,.$

An important observation is that one can use another parametrization for quantum metric, e.g.,

$$\kappa \Phi^{\mu\nu} = \sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu} \,,$$

or even quantize in a very different variables. Here we shall use the parametrization with $h_{\mu\nu}$.

An important requirement is that relevant observables should not depend on the choice of parametrization and, in particular, on the gauge fixing condition (gauge dependence).

In general, this dependence is not easy to deal with, but in some cases (one-loop divergences) we can do it easily.

Gauge invariant renormalizability

Gauge invariant renormalizability of the theory means that the divergences (and therefore counterterms), in all loop orders, have the same symmetry as the classical action. It does not guarantee the multiplicative renormalizability, that requires also a "correct" power counting.

The papers on renormalizability in quantum Gravity (QG) include

K. Stelle, Phys. Rev. D (1977).

I.L. Buchbinder, S.D. Odintsov, I.Sh., *Effective Action in Quantum Gravity* (IOPP, 1992).

B.L. Voronov and I.V. Tyutin, Sov. Nucl. Phys. (1981,1984).

P.M. Lavrov, I.Sh., Gauge invariant renormalizability of quantum gravity, arXiv:1902.04687

The last work deals with the QG theory of a general form and use Batalin-Vilkovisky formalism. In what follows we use notations of Stelle.

As far as we intend to arrive at the Feynman rules, we can start from the Faddeev-Popov approach,

$$Z(J) = \int DhDCD\bar{C} \; \mathrm{Det}\,(Y_{\alpha\beta})$$

$$\times \; \exp\Big\{ iS(h) + \frac{i}{2}\chi^\alpha \, Y_{\alpha\beta} \, \chi^\beta + \frac{i}{2}\bar{C}^\alpha \, M_\alpha^\beta \, C^\beta + iJ^{\mu\nu}h_{\mu\nu} \Big\} .$$

where the ghost part is $\quad M_\alpha^\beta = \frac{\delta\chi^\alpha}{\delta h_{\mu\nu}} R_{\mu\nu\,,\,\alpha} \,.$

The useful choice of the gauge fixing condition and the weight function depends on the theory.

The most popular gauges are the Fock-deDonder one

$$\chi^\mu = \partial_\nu \Phi^{\mu\nu} \,, \qquad \kappa\Phi^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu} \,,$$

harmonic gauge $\quad \chi_\mu = \partial^\nu h_{\mu\nu} - \beta\partial_\mu h \,, \qquad \beta = \frac{1}{2} \,,$

and background gauges

$$\chi_\mu = \nabla^\nu h_{\mu\nu} - \beta\nabla_\mu h \,, \quad \text{where} \quad g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \,.$$

Consider the total action with $g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)$ **,**

$$S_t \;=\; S(h) + \frac{1}{2}\chi^\alpha\, Y_{\alpha\beta}\, \chi^\beta + \frac{1}{2}\bar{C}^\alpha\, M^\beta_\alpha\, C^\beta\,.$$

It is not gauge invariant, but possesses BRST invariance,

$$\frac{\delta R_{\mu\nu\,,\,\alpha}}{\delta h_{\rho\sigma}}\, R_{\rho\sigma\,,\,\beta} - \frac{\delta R_{\mu\nu\,,\,\beta}}{\delta h_{\rho\sigma}}\, R_{\rho\sigma\,,\,\alpha} = R_{\mu\nu\,,\,\gamma}\, f^\gamma{}_{\alpha\beta}\,,$$

where for a gravity theory

$$f^\gamma{}_{\alpha\beta}(x,y,z) \;=\; \delta^\gamma_\alpha\delta(x-z)\partial_\beta\delta(x-y) + \delta^\gamma_\beta\delta(x-y)\partial_\alpha\delta(x-z)\,.$$

Then,

$$\delta_{BRST} h_{\mu\nu} = \kappa R_{\mu\nu\,,\,\alpha}\, C^\alpha\, \delta\mu\,,$$

$$\delta_{BRST} C^\alpha = \frac{1}{2} f^\alpha{}_{\beta\gamma}\, C^\beta\, C^\gamma\, \delta\mu \;=\; C^\beta\, \partial_\alpha C^\alpha \cdot \delta\mu\,.$$

$$\delta_{BRST} \bar{C}_\alpha = Y_{\alpha\beta}\, \chi^\beta(h)\, \delta\mu\,,$$

It proves useful to introduce two extra sources. Consider the total action

$$\bar{S} \,=\, S_t + K^{\mu\nu} R_{\mu\nu\,,\,\alpha} \bar{C}^{\alpha} + L_{\sigma} \partial_{\beta}\, C^{\sigma}\, C^{\beta}\,.$$

One can easily prove that it satisfies the Noether identity

$$\frac{\delta \bar{S}}{\delta K^{\mu\nu}}\,\frac{\delta \bar{S}}{\delta h_{\mu\nu}} \,+\, \frac{\delta \bar{S}}{\delta L_{\sigma}}\,\frac{\delta \bar{S}}{\delta \bar{C}^{\sigma}} \,+\, Y_{\alpha\beta}\,\chi^{\beta}\,\frac{\delta \bar{S}}{\delta \bar{C}_{\alpha}} \,=\, 0\,.$$

One can perform the BRST transformation in Z and since it does not change, we arrive at the Slavnov-Taylor identities for Z,

$$Z(J,\bar{\beta},\beta,K,L) \,=\, \int DhDCD\bar{C}\ \mathrm{Det}\,(Y_{\alpha\beta})$$

$$\times\ \exp\Big\{ i\bar{S} + iJ^{\mu\nu}h_{\mu\nu} + i\bar{C}_{\mu}\beta^{\mu} + i\bar{\beta}_{\mu}C\mu \Big\}\,.$$

As a next step we define $W = -i \ln Z$ and the mean fields

$$\langle h_{\mu\nu} \rangle = \frac{\delta W}{\delta J^{\mu\nu}}\,, \qquad \langle C^{\sigma} \rangle = \frac{\delta W}{\delta \bar{\beta}_{\sigma}}\,, \qquad \langle \bar{C}_{\rho} \rangle = \frac{\delta W}{\delta \beta^{\rho}}\,.$$

The corresponding effective action

$$\Gamma(h_{\mu\nu},\, C^{\sigma},\, \bar{C}_{\rho},\, K^{\mu\nu},\, L_{\sigma})$$

$$= W(J^{\mu\nu},\, \bar{\beta}_{\sigma},\, \beta^{\rho},\, K^{\mu\nu},\, L_{\sigma}) - h_{\mu\nu}J^{\mu\nu} - \bar{C}_{\rho}\beta^{\rho} - C^{\sigma}\bar{\beta}^{\sigma}\,.$$

One can prove that it satisfies the identity

$$\frac{\delta \Gamma}{\delta K^{\mu\nu}}\,\frac{\delta \Gamma}{\delta h_{\mu\nu}} + \frac{\delta \Gamma}{\delta L_{\sigma}}\,\frac{\delta \Gamma}{\delta \bar{C}^{\sigma}} + Y_{\alpha\beta}\,\chi^{\beta}\,\frac{\delta \tilde{\Gamma}}{\delta \bar{C}_{\alpha}} = 0\,.$$

Finally, we introduce modified quantities

$$\tilde{S} \,=\, \bar{S} - \frac{1}{2}\,\chi^\alpha\, Y_{\alpha\beta}\,\chi^\beta$$

and

$$\tilde{\Gamma} \,=\, \Gamma - \frac{1}{2}\,\chi^\alpha\, Y_{\alpha\beta}\,\chi^\beta\,,$$

which satisfy the identities

$$\frac{\delta \tilde{S}}{\delta K^{\mu\nu}}\,\frac{\delta \tilde{S}}{\delta h_{\mu\nu}} + \frac{\delta \tilde{S}}{\delta L_\sigma}\,\frac{\delta \tilde{S}}{\delta \bar{C}^\sigma} \,=\, 0\,.$$

and

$$\frac{\delta \tilde{\Gamma}}{\delta K^{\mu\nu}}\,\frac{\delta \tilde{\Gamma}}{\delta h_{\mu\nu}} + \frac{\delta \tilde{\Gamma}}{\delta L_\sigma}\,\frac{\delta \tilde{\Gamma}}{\delta \bar{C}^\sigma} \,=\, 0\,,$$

along with the so-called ghost equation

$$\frac{\delta \chi^\alpha}{\delta h_{\rho\sigma}}\,\frac{\delta \tilde{\Gamma}}{\delta K^{\rho\sigma}} - \frac{\delta \tilde{\Gamma}}{\delta \bar{C}_\alpha} \,=\, 0\,.$$

Consider the loop expansion

$$\tilde{\Gamma} = \tilde{S} + \sum_{k=1}^{\infty} \hbar^k \, \tilde{\Gamma}^{(k)}, \quad \text{where} \quad \tilde{\Gamma}^{(k)} = \tilde{\Gamma}^{(k)}_{div} + \tilde{\Gamma}^{(k)}_{fin}\,.$$

Assuming that the orders $1, 2, ..., k-1$ **are already renormalized, one can show that** k **- order terms do satisfy the identities**

$$\mathcal{E}\tilde{\Gamma}^{(k)}_{div} = \sum_{i=0}^{k} \left\{ \frac{\delta\tilde{\Gamma}^{(k-i)}}{\delta K^{\mu\nu}} \, \frac{\delta\tilde{\Gamma}^{(i)}}{\delta h_{\mu\nu}} + \frac{\delta\tilde{\Gamma}^{(k-i)}}{\delta L_{\sigma}} \, \frac{\delta\tilde{\Gamma}^{(i)}}{\delta \bar{C}^{\sigma}} \right\}, \qquad (*)$$

where the nilpotent $\mathcal{E}^2 = 0$ **operator is defined as**

$$\mathcal{E} = \frac{\delta\tilde{S}}{\delta h_{\mu\nu}} \frac{\delta}{\delta K^{\mu\nu}} + \frac{\delta\tilde{S}}{\delta K^{\mu\nu}} \frac{\delta}{\delta h_{\mu\nu}} + \frac{\delta\tilde{S}}{\delta L_{\sigma}} \, \frac{\delta}{\delta \bar{C}^{\sigma}} + \frac{\delta\tilde{S}}{\delta \bar{C}^{\sigma}} \frac{\delta}{\delta L_{\sigma}}\,.$$

Obviously, (*) is equivalent to

$$\mathcal{E}\tilde{\Gamma}^{(k)}_{div} = 0\,.$$

Due to the nilpotence of $\mathcal{E}$ and

$$\mathcal{E}\tilde{\Gamma}^{(k)}_{div} = 0$$

the most general local solution is

$$\tilde{\Gamma}^{(k)}_{div} = A(h_{\mu\nu}) + \mathcal{E}X(h_{\mu\nu}, C^{\sigma}, \bar{C}_{\rho}, K^{\mu\nu}, L_{\sigma}),$$

where A is some covariant functional and X is an arbitrary local functional of its variables.

One can prove that the $\mathcal{E}X$ -terms can be removed by

- **Renormalization of the field $h_{\mu\nu}$ together with some gauge transformation.**
- **Renormalization of the FP ghosts C_{μ} and $\bar{C}^{\nu}$ together with some BRST transformation.**

Finally, the problem is reduced to the possible form of the covariant local functional $A(h_{\mu\nu})$.

Let us use the notion of power counting to explore $A(h_{\mu\nu})$. The universal formula for the superficial degree of divergence is

$$D + d \;=\; \sum_{l_{int}} (4 - r_l) \;-\; 4n \;+\; 4 \;+\; \sum_{\nu} K_\nu \,.$$

Here

D **is the superficial degree of divergence for a given diagram,**
d **is the number of derivatives on external lines of the diagram,**
r_l **is the power of momenta in the propagator of internal line,**
n **is the number of vertices and**
K_ν **is the power of momenta in a given vertex.**

On the top of that one can use topological relation between number of loops p, vertices n, and internal lines

$$l_{int} \;=\; p + n - 1\,.$$

As the first example consider quantum GR.

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R+2\Lambda)\,.$$

For the sake of simplicity we consider only vertices with maximal K_ν. **Then we have** $r_l = K_\nu = 2$ **and, combining**

$$D+d \,=\, \sum_{l_{int}}(4-r_l)\,-\,4n\,+\,4\,+\,\sum_\nu K_\nu$$

with

$$l_{int} = p+n-1$$

we arrive at the estimate ($D=0$ **means log. divergences)**

$$D+d \,=\, 2+2p\,.$$

This output means that quantum GR is not renormalizable.

More details: What means the relation

$$D + d \; = \; 2 + 2p \quad ?$$

Remember that $D = 0$ means logarithmic divergences.

At the 1-loop level we can expect the divergences like

$$\mathcal{O}(R^2_{...}) = R^2_{\mu\nu\alpha\beta}\,, \;\; R^2_{\mu\nu}\,, \;\; R^2\,.$$

t'Hooft and Veltman; Deser and van Nieuwenhuisen, (1974); ...

At the 2-loop level we have

$$\mathcal{O}(R^3_{...}) = R_{\mu\nu}\Box R^{\mu\nu}\,, \,...R^3\,, \;\; R_{\mu\nu}R^{\mu}_{\alpha}R^{\alpha\nu}\,, \;\; R_{\mu\nu\alpha\beta}R^{\mu\nu}{}_{\rho\sigma}R^{\mu\nu\rho\sigma}\,.$$

M.H. Goroff and A. Sagnotti, NPB 266 (1986).

The last structure does not vanish on-shell and this proves that the theory is not renormalizable, at least within the standard perturbative approach.

Within the standard perturbative approach non-renormalizability means the theory has no predictive power.

Every time we introduce a new type of a counterterm, it is necessary to fix renormalization condition and this means a measurement. So, before making a single predictions, it is necessary to have an infinite amount of experimental data.

What are the possible solutions?

- **Change standard perturbative approach to something else.**

There are many options, but their consistency or their relation to the QG program are not clear, in all cases.

- **Change the theory, i.e., take another theory to construct QG.**

The first option is widely explores in the asymptotic safety scenarios, in the effective approaches to QG, induced gravity approach (including string theory) and so on.

Let us concentrate on the second idea.

The most natural choice is four derivative model, because we need four derivatives anyway for quantum matter field.

Already known action: $S_{gravity} = S_{EH} + S_{HD}$

where S_{HD} **includes square of the Weyl tensor and** R

$$S_{HD} = -\int d^4x\sqrt{-g}\left\{\frac{1}{2\lambda}C^2 + \frac{\omega}{3\lambda}R^2 + \text{ surface terms}\right\},$$

$$C^2(4) = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + R^2/3\,,$$

Propagators of metric and ghosts behave like $\mathcal{O}(k^{-4})$ **and we have** K_4, K_2, K_0 **vertices.**

The superficial degree of divergence

$$D + d = 4 - 2K_2 - 4K_0.$$

Dimensions of counterterms are 4, 2, 0.
This theory is definitely renormalizable.

K. Stelle, Phys. Rev. D (1977).

However there is a price to pay: massive ghosts

$$G_{\rm spin-2}(k) \sim \frac{1}{m^2}\left(\frac{1}{k^2} - \frac{1}{k^2+m^2}\right), \quad m \propto M_P\,.$$

The tree-level spectrum includes massless graviton and massive spin-2 "ghost" with negative kinetic energy and a huge mass.

Particle with negative energy means instability of vacuum state.

Even Minkowski space is not protected from spontaneous creation of massive ghost and many gravitons from vacuum.

Different sides of the HDQG problems with massive ghosts:

- **In classical systems higher derivatives generate exploding instabilities at the non-linear level** *(M.V. Ostrogradsky, 1850).*
- **Interaction between ghost and gravitons may violate energy conservation in the massless sector** *(M.J.G. Veltman, 1963).*
- **Ghost produce violation of unitarity of the S -matrix.**

One can include more than four derivatives,

$$S = S_{EH} + \sum_{n=0}^{N} \int d^4x \sqrt{-g} \Big\{ \omega_n^C C_{\mu\nu\alpha\beta} \Box^n C_{\mu\nu\alpha\beta} + \omega_n^R R \Box^n R \Big\} + \mathcal{O}\big(R^3_{\dots}\big).$$

Simple analysis shows that this theory is superrenormalizable, BUT massive ghost-like states are still present.

For the real poles case:

$$G_2(k) = \frac{A_0}{k^2} + \frac{A_1}{k^2+m_1^2} + \frac{A_2}{k^2+m_2^2} + \cdots + \frac{A_{N+1}}{k^2+m_{N+1}^2}\,.$$

For any sequence $0 < m_1^2 < m_2^2 < m_3^2 < \cdots < m_{N+1}^2$,
the signs of the corresponding terms alternate: $A_j \cdot A_{j+1} < 0$.

M. Asorey, J.-L. Lopez & I. Sh., IJMPhA (1997), hep-th/9610006.

Exact β-functions in QG

In the superrenormalizable QG one can derive exact RG equations by working at the one-loop level !

M. Asorey, J.-L. Lopez & I. Sh., IJMPhA (1997), hep-th/9610006.

$$\beta_\Lambda = \mu \frac{d\rho_\Lambda}{d\mu} = \frac{1}{(4\pi)^2}\left(\frac{5\omega_{N-2,C}}{\omega_{N,C}} + \frac{\omega_{N-2,R}}{\omega_{N,R}} - \frac{5\omega^2_{N-1,C}}{2\omega^2_{N,C}} - \frac{\omega^2_{N-1,R}}{2\omega^2_{N,R}}\right).$$

L. Modesto, L. Rachwal & I.Sh., arXiv:1704.03988, EJPC (2018).

$$\beta_G = \mu \frac{d}{d\mu}\left(-\frac{1}{16\pi G}\right) = -\frac{1}{6(4\pi)^2}\left(\frac{5\omega_{N-1,C}}{\omega_{N,C}} + \frac{\omega_{N-1,R}}{\omega_{N,R}}\right).$$

Different from four-derivative quantum gravity these β-functions do not depend on the choice of a gauge-fixing condition.

And for $N \geq 3$ **they are exact.**

Two sides of higher derivatives in QG.

The consistent theory which is supposed to work at arbitrary energy scale can not be constructed without at least fourth derivatives.

If the higher derivative terms are included, then the tree-level spectrum includes massless graviton and massive spin-2 "ghost" with negative kinetic energy and huge mass.

If we do not include the higher derivative terms into classical action, they will emerge with infinite coefficients and (most relevant) with logarithmically running parameters. In any case, the unphysical ghosts come back.

No way to live with ghosts and no way to live without ghosts.

Still we can live, so there must be some explanation, of course.

An alternative to Zweibach transformation

In the non-local theory

$$S = -\frac{1}{2\kappa}\int d^4x\sqrt{-g}\,\Big\{R + G_{\mu\nu}\,\frac{a(\Box)-1}{\Box}\,R^{\mu\nu}\Big\}, \quad a(\Box) = e^{-\Box/m^2}\,.$$

A. Tseytlin, PLB, hep-th/9509050.

In this and similar theories propagator of metric perturbations has a single massless pole, corresponding to gravitons.

With this choice there are no ghosts!

The idea is to use Zweibach-like transformation, but arrive at the non-local theory which is non-polynomial in derivatives, instead of "killing" all higher derivatives that one can kill.

One more ambiguity in the (super)string theory.

There was a proposal to use the same kind of non-local models to construct superrenormalizable and unitary models of QG.

E.T. Tomboulis, hep-th/9702146; PRD (2015), arXiv:1507.00981.

. . .

L. Modesto, L. Rachwal, NPB (2014), arXiv:1407.8036.

The propagator is defined by the terms bilinear in curvature's,

$$S = \int_x \Big\{ -\frac{1}{\kappa^2} R + \frac{1}{2} C_{\mu\nu\alpha\beta}\, \Phi(\Box)\, C^{\mu\nu\alpha\beta} + \frac{1}{2} R\, \Psi(\Box)\, R \Big\} .$$

The equation for defining the poles:

$$p^2 \Big[1 + \kappa^2 p^2 \Phi(-p^2) \Big] = p^2\, e^{\alpha p^2} = 0.$$

In this particular case there is only a massless pole corresponding to gravitons. But unfortunately, it is impossible to preserve the ghost-free structure at the quantum level.

I.Sh., PLB, arXive:1502.00106.

Typically there are infinitely many poles on the complex plane.

Complex ghosts and Lee-Wick unitarity in QG

Starting from *Tomboulis (1977)* **and** *Salam and Strathdee (1978)* **the main hope in the “minimal” fourth-derivative QG was that the real ghost pole splits into a couple of complex conjugate poles under the effect of quantum corrections.**

One-loop effects, large-*N* **approximation and lattice-based considerations indicated an optimistic picture, but unfortunately all of them are not conclusive, as shown by** *Johnston (1988)*.

However, for six- or more- derivative theory of QG, one can just start from the theory which has only complex massive poles.

L. Modesto, and I.Sh. PLB (2016), arXiv:1512.07600.

It turns out that such a theory is unitary and, moreover, this property may probably hold even at the quantum level.

Quantum consistency

There is yet another difficulty of non-local gravity, which is possibly shared by other e.g. polynomial models.

In the recent paper

M. Asorey, L. Rachwal, I.Sh., Galaxies - 2018; arXiv:1802.01036

it was shown that within the non-local models of exponential type the reflection positivity condition is not satisfied.

The Euclidean 2-point function $S_2(x,y)$ **should satisfy Osterwalder-Schrader reflection positivity property**

$$\int \theta f(x) S_2(x,y) f(y) \geqslant 0.$$

For the non-local gravity this is not true.

This means that this theory has unphysical modes regardless of the absence of massive pole in the tree-level propagator.

The main issue is stability

Certainly, the unitarity of the S- matrix is not the unique condition of consistency of the quantum gravity theory.

The most important feature is the stability of physically relevant solutions of classical general relativity in the presence of higher derivatives and massive ghosts.

The problem is well explored for the cosmological backgrounds. Gravitational waves on de Sitter space (energy $\ll M_p$):

A. A. Starobinsky, Let. Astr. Journ. (in Russian) (1983).

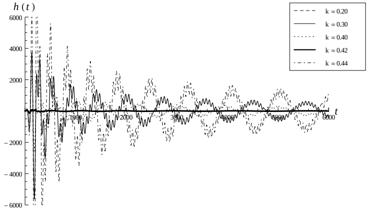
S. Hawking, T. Hertog, and H.S. Real, PRD (2001).
J. Fabris, A. Pelinson and I.Sh., NPB (2001).

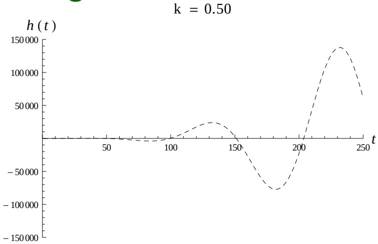
J. Fabris, A. Pelinson, F. Salles and I.Sh., JCAP, arXiv:1112.5202.

More general FRW-backgrounds:

F. Salles and I.Sh., PRD, arXiv:1401.4583.

More general cosmological backgrounds





Example: radiation-dominated Universe. There are no growing modes until the frequency k achieves the value ≈ 0.5 in Planck units. Starting from this value, we observe instability as an effect of massive ghost.

The anomaly-induced quantum correction is $\mathcal{O}(R^3_{....})$. Until the energy is not of the Planck order of magnitude, these corrections can not compete with classical $\mathcal{O}(R^2_{....})$ - terms.

Massive ghosts are present only in the vacuum state. We just do not observe them “alive” until the energy scale M_P .

What can we do Planck or greater frequencies?

The simplest possible equation is for the fourth-derivative gravity without quantum (semiclassical) corrections,

$$\frac{1}{3}\overset{....}{h} + 2H\dddot{h} + \Big(H^2 + \frac{M_{\text{P}}^2}{32\pi a_1}\Big)\ddot{h} + \frac{1}{6}\frac{\nabla^4 h}{a^4} - \frac{2}{3}\frac{\nabla^2 \ddot{h}}{a^2} - \frac{2H}{3}\frac{\nabla^2 \dot{h}}{a^2}$$
$$-\Big(H\dot{H} + \ddot{H} + 6H^3 - \frac{3M_{\text{P}}^2 H}{32\pi a_1}\Big)\dot{h} - \Big[\frac{M_{\text{P}}^2}{32\pi a_1} - \frac{4}{3}\Big(\dot{H} + 2H^2\Big)\Big]\frac{\nabla^2 h}{a^2}$$
$$-\Big[24\dot{H}H^2 + 12\dot{H}^2 + 16H\ddot{H} + \frac{8}{3}\dddot{H} - \frac{M_{\text{P}}^2}{16\pi a_1}\Big(2\dot{H} + 3H^2\Big)\Big]h = 0.$$

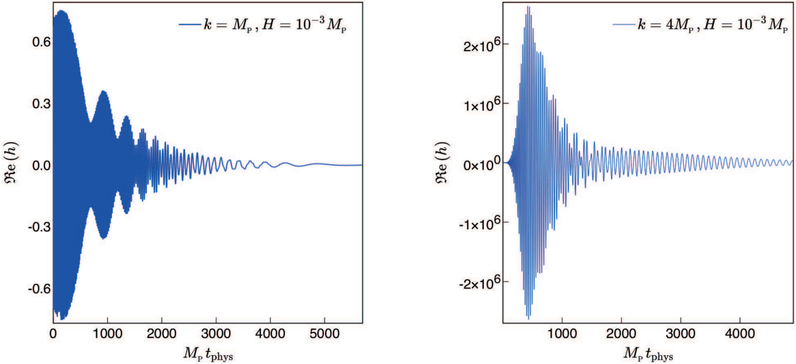
It is easy to note that the space derivatives ∇ and hence the wave vector $\vec{k}$ enter this equation only in the combination

$$\vec{q} = \frac{\vec{k}}{a(t)}.$$

When universe expands, each frequency becomes smaller!

Filipe de O. Salles, Patrick Peter, I.Sh., On the ghost-induced instability on de Sitter background. PRD (2018), arXiv:1801.00063

The qualitative conclusion is perfectly well supported by numerical analysis, including the case when the semiclassical corrections are taken into account.



The growth of the waves really stops at some point. At least in the cosmological setting this may be a solution of the problem.

General qualitative situation.

1) We know there is no way to have semiclassical or quantum gravity without higher derivatives.

2) Higher derivatives mean ghosts and instabilities. But in the closed system the problem can be solved because there is no energy to provide a global and total explosion of ghost or even tachyonic ghost modes (Lee-Wick approach).

G. Dvali, S. Folkerts, C. Germani, PRD (2011), arXiv:1006.0984;
G. Dvali and C. Gomez, Fortschr. Phys. (2013), arXiv:1112.3359.

May be there is some general unknown principle which forbids Planck-scale concentration of gravitons.

3) Then this restriction can be violated only for the Planck-scale background, which "opens" the phase space of quantum states and enables the production of instabilities. But after that the expansion of the universe reduce the frequencies and the instabilities do stabilize.

- **Ambiguities and one-loop results.**

Before we start to discuss the results of the existing loop calculations (mainly 1-loop), it is worthwhile to see whether and to which extent these results are ambiguous.

As we already know, one can use, for quantum calculations, different choices of parametrization of quantum field (metric, in the case) and also different gauge fixing conditions.

As an example, consider the 1-loop expression for the non-gauge theory with the classical action $S(\Phi_i)$. We know that

$$\bar{\Gamma}^{(1)} = \frac{i}{2}\,\text{Log Det } S''_{ij}\,, \qquad S''_{ij} = \frac{\delta^2 S}{\delta\Phi_i\,\delta\Phi_j}\,.$$

Let us change the variables according to $\Phi_i = \Phi'_k$. Obviously,

$$\bar{\Gamma}^{(1)} = \text{Log Det}\left(\frac{\delta^2 S}{\delta\Phi'_l\,\delta\Phi'_k}\right) = \text{Log Det}\left(S''_{ij}\cdot\frac{\delta\Phi_i}{\delta\Phi'_k}\,\frac{\delta\Phi_j}{\delta\Phi'_l} + \frac{\delta S}{\delta\Phi_i}\,\frac{\delta^2\Phi_i}{\delta\Phi'_l\,\delta\Phi'_k}\right).$$

Looking at the formula

$$\bar{\Gamma}^{(1)} \;=\; \text{Log Det}\left(\frac{\delta^2 S}{\delta\Phi'_l\,\delta\Phi'_k}\right) \;=\; \text{Log Det}\left(S''_{ij}\cdot\frac{\delta\Phi_i}{\delta\Phi'_k}\,\frac{\delta\Phi_j}{\delta\Phi'_l} + \frac{\delta S}{\delta\Phi_i}\,\frac{\delta^2\Phi_i}{\delta\Phi'_l\,\delta\Phi'_k}\right).$$

one can immediately note that the two 1-loop results do coincide on classical equation of motion (on-shell), when

$$\varepsilon^i \;=\; \frac{\delta S}{\delta\Phi_i} \;=\; 0\,.$$

The same is true for the Faddeev-Popov action in the gauge theory. In particular, for QG, the 1-loop contribution

$$\bar{\Gamma}^{(1)} \;=\; \frac{i}{2}\,\text{Log Det}\,\left[\left(S + S_{gf}\right)''_{ij}\right] - i\,\text{Log Det}\,\left[S''_{ghost}\right]$$

may depend on the gauge fixing condition and weight function

$$S_{FP} \;=\; S(h) + \frac{1}{2}\chi^\alpha\,Y_{\alpha\beta}\,\chi^\beta + \frac{1}{2}\bar{C}^\alpha\,M^\beta_\alpha\,C^\beta\,.$$

What does it mean for the one-loop divergences in QG?

Obviously, there may be some ambiguity, proportional to

$$\varepsilon^i = \frac{\delta S}{\delta \Phi_i} \,.$$

Another very important aspect which has to be taken into account is the locality of UV divergences (Weinberg's theorem).

The level of ambiguity will always depend on the given theory, namely on the relation between power counting-allowed divergences and the form of ε^i.

I. Quantum GR with the cosmological constant term.

$$S = -\frac{1}{\kappa^2} \int d^4x \sqrt{-g} \, \big(R + 2\Lambda\big) \, .$$

Then

$$\varepsilon^{\mu\nu} \, = \, \frac{1}{\sqrt{-g}} \, \frac{\delta S}{\delta g_{\mu\nu}} \, = \, R^{\mu\nu} - \frac{1}{2} \, \big(R + 2\Lambda\big) g^{\mu\nu} \, .$$

Using the power-counting arguments we learn that

$$\Gamma^{(1)}_{div} \, = \, \frac{1}{\epsilon} \int d^4x \sqrt{-g} \, \big\{ a_1 \, C^2 + a_2 E + a_3 \Box R + a_4 R^2 + a_5 R + a_6 \big\} \, , \tag{1}$$

where

$$E = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 \, R_{\alpha\beta} R^{\alpha\beta} + R^2 \, ,$$

$$C^2(4) = E + 2W \, , \qquad W = R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{3} \, R^2 \, .$$

It is easy to check that the ambiguity in ΔS has the form

$$\delta\Gamma^{(1)}_{div} = \frac{1}{\epsilon}\int d^4x\sqrt{-g}\,\varepsilon^{\mu\nu}\Big(b_1R_{\mu\nu} + b_2Rg_{\mu\nu} + b_3g_{\mu\nu}\Big)\,. \qquad (2)$$

$$\varepsilon^{\mu\nu} = \frac{1}{\sqrt{-g}}\,\frac{\delta S}{\delta g_{\mu\nu}} = const \times \Big[R^{\mu\nu} - \frac{1}{2}\,(R + 2\Lambda)g^{\mu\nu}\Big]\,.$$

As a result only three of the six coefficients in (1) are gauge-fixing independent.

For example, without the cosmological constant term only the topological Gauss-Bonnet counterterm can not be set to zero by the choice of the gauge fixing condition.

This result was first discovered by direct calculation in

R. Kallosh, O.V. Tarasov and I. Tyutin, Nucl.Phys. B137 (1978).

The first one-loop calculation in quantum GR without the cosmological constant term was done in

G. 'tHooft, M. Veltman, Ann. Inst. H. Poincare XX (1974) 69.
S. Deser, P. van Nieuwenhuizen, PRD 10 (1974) 401; 411.
S. Deser, H-S. Tsao, P. van Nieuwenhuizen, PRD 10 (1974) 3337.

The result for the pure QG is

$$\Gamma^{(1)}_{div} = \frac{1}{\epsilon} \int d^4x \sqrt{-g} \left\{ \frac{7}{20} R^2_{\mu\nu} + \frac{1}{120} R^2 - \frac{13}{3} \Lambda R + 10\Lambda^2 \right\}$$

but, as we already know, all but one combination of these coefficients are irrelevant.

Conclusion: 1-loop S - matrix in the QG based on GR without cosmological constant is finite. If we introduce the cosmological constant, the flat space is not a classical solution and the sense of the S - matrix approach becomes unclear.

In the same papers by 'tHooft, Veltman, Deser et al it was established that the metric-scalar, metric-vector and metric-spinor theories are not finite at the 1-loop level.

Calculations of this sort were repeated many times after 1974.

For example, in the Einstein-scalar system with non-minimal coupling the expression for divergences is

$$\Gamma^{(1)}_{div} = \frac{1}{\epsilon}\int d^4x\sqrt{-g}\Big\{c_w C^2 + c_r R^2 + c_4 R(\nabla\phi)^2 + c_5 R(\Box\phi) + c_6 R^{\mu\nu}\phi_{;\mu}\phi_{;\nu}$$

$$+ c_7 R + c_8(\nabla\phi)^4 + c_9(\nabla\phi)^2(\Box\phi) + c_{10}(\Box\phi)^2 + c_{11}(\nabla\phi)^2 + c_{12}\Big\} + (s.t.)$$

where $''s.t.''$ **means "surface terms".**

The expressions for $c_1, ..., c_{12}$ **require a few pages, and there's no agreement between different calculations, perhaps due to the ambiguity of gauge and parametrization (not checked yet!).**

One-loop calculations in HDQG (4 derivatives) and discussion of gauge ambiguity has been done by several groups:

J. Julve & M. Tonin (1978),
E.S. Fradkin & A.A. Tseytlin (1982),
I. Avramidi & A.O. Barvinsky (1986)
I. Antoniadis and E. Mottola (1992)
G. Berredo-Peixoto & I.Sh. (2005).

Detailed discussion of gauge dependence:
I. Avramidi, Ph.D. thesis (1986), hep-th/9510140;
I.Sh., A.G. Jacksenaev, PLB 324 (1994) 284.

Also, calculation in HDQG coupled to matter was done by:

E.S. Fradkin & A.A. Tseytlin (1982),
I.L. Buchbinder, I.Sh. et al (1986-1990) ...

Recently the last subject became relevant due to some works where it's stated that the (GR-based) QG contributions can provide AF in the theory of Abelian gauge field.

Consider the gauge dependence in HDQG. According to the power counting and general covariance considerations the possible counterterms have the form:

$$\Delta S = \int d^4x\sqrt{-g}\{a_1 C^2_{\alpha\beta\mu\nu} + a_2 R^2 + a_3 E + a_4 \Box R + a_5 R + a_6\}\,,$$

where a_i **are some divergent constants.**

$\Gamma(\alpha_i)$ **is an effective action for arbitrary values of the gauge parameters** α_i **and** $\Gamma_m = \Gamma(\alpha_i^{(0)})$ **for some special values** $\alpha_i^{(0)}$**. We can write**

$$\Gamma(\alpha_i) \,=\, \Gamma_m \,+\, \int d^4x\sqrt{-g}\; \varepsilon^{\mu\nu} f_{\mu\nu}(\alpha_i)\,,$$

where

$$\varepsilon^{\mu\nu} = \delta S/\delta g_{\mu\nu}$$

and $f_{\mu\nu}(\alpha_i)$ **some unknown function.**

We are interested in the divergent part of $\Gamma(\alpha_i)$.

The divergencies are local and moreover $\Gamma(\alpha_i)$ and $\varepsilon^{\mu\nu}$ have the dimension of the classical action, hence it can be only

$$f_{\mu\nu}(\alpha_i) = g_{\mu\nu} \times f(\alpha_i)\,,$$

and therefore

$$\Gamma(\alpha_i) = \Gamma_m + f(\alpha_i) \int d^4x \sqrt{-g}\, g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}}\,.$$

The change of the gauge condition at one loop equivalent to the conformal "shift" of the classical action S.

E.g., in the Weyl gravity the effective action does not depend on the gauge parameters.

Consider the general version of HDQG.

$$\frac{1}{\sqrt{-g}}\, g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\int d^4x\sqrt{-g}R^2 \,=\, -\,6(\Box R)\,.$$

Conformal and surface terms don't contribute to $\varepsilon^{\mu\nu}$**, hence**

$$\Gamma(\alpha_i) \,=\, \Gamma_m \,+\, f(\alpha_i)\int d^4x\sqrt{-g}\,\Big\{-\frac{1}{\kappa^2}\,(R-4\Lambda)-\frac{2\omega}{\lambda}\,\Box R\Big\}\,.$$

Therefore, a_1, a_2, a_3 **do not depend on the gauge parameter values while** a_4, a_5, a_6 **do.**

One can construct gauge independent parameters

$$a_7 \,=\, a_6 \,+\, \Lambda a_5\,, \qquad \frac{\lambda}{4\omega}\,a_4 \,-\, \kappa^2 a_5\,, \qquad \frac{\lambda\,\Lambda}{4\omega}\,a_4 + \kappa^2\,a_6\,.$$

An important consequence is that one can define the ren. group beta-functions only for the dimensionless ratio, $\kappa^2\,\Lambda$**. This is related to the known feature of induced gravity (***S. Adler, 1981***).**

Writing the classical action of HDQG in the form

$$S = -\mu^{n-4} \int d^n x \sqrt{g} \left\{ \frac{1}{2\lambda} C^2 - \frac{1}{\rho} E + \frac{1}{\xi} R^2 + \tau \Box R - \frac{1}{\kappa^2} (R - 2\Lambda) \right\}$$

the counterterms are, with $\epsilon = (4\pi)^2 (n-4)$**,**

$$\Delta S = \frac{\mu^{n-4}}{\epsilon} \int d^n x \sqrt{g} \left\{ \frac{133}{20} C^2 - \frac{196}{45} E + \left(\frac{10\,\lambda^2}{\xi^2} - \frac{5\,\lambda}{\xi} + \frac{5}{36} \right) R^2 + \right.$$

$$\left. + \left(\frac{\xi}{12\,\lambda} - \frac{13}{6} - \frac{10\,\lambda}{\xi} \right) \frac{\lambda}{\kappa^2} R + \left(\frac{56}{3} - \frac{2\,\xi}{9\,\lambda} \right) \frac{\lambda \Lambda}{\kappa^2} + \left(\frac{\xi^2}{72} + \frac{5\lambda^2}{2} \right) \frac{1}{\kappa^4} \right\}.$$

The surface term was not included here for brevity.

The form of the couterterms confirms our expectations, namely they have the same algebraic structure as the classical action.

The renormalization group equations are

$$(4\pi)^2\,\frac{d\lambda}{dt} = -\,\frac{133}{10}\,\lambda^2\,, \qquad (4\pi)^2\,\frac{d\rho}{dt} \,=\, -\,\frac{196}{45}\,\rho^2\,,$$

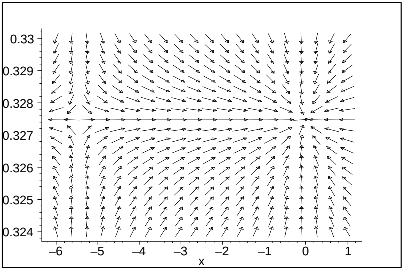
$$(4\pi)^2\,\frac{d\xi}{dt} \,=\, -\,10\,\lambda^2\,\xi^2 + 5\,\lambda\,\xi - \frac{5}{36}\,.$$

There is an AF in the coupling constant λ and parameter ρ.

The equation for ξ is better explored in other variables

$$\theta = \lambda/\rho\,, \qquad \omega = -3\,\lambda/\xi\,.$$

The phase plane looks like $(x \equiv \omega\,,\ y = \theta)$



Finally, let us comment of gauge dependence for the superrenormalizable HDQG.

$$S = \int d^4x \sqrt{-g}\,\Big\{ c_1 R_{\mu\nu\alpha\beta}\Box^k R^{\mu\nu\alpha\beta} + c_2 R_{\mu\nu}\Box^k R^{\mu\nu} + c_3 R\Box^k R + \, ... \Big\}\,.$$

In this case the counterterms have the form ($k \geq 3$ case),

$$\Delta S = \int d^4x \sqrt{-g}\big\{ a_1 C^2_{\alpha\beta\mu\nu} + a_2 R^2 + a_3 E + a_4 \Box R + a_5 R + a_6 \big\}\,,$$

but $\varepsilon^{\mu\nu}$ has much larger dimension. As a result there is no ambiguity related to gauge fixing in the countereterms in all loop orders. Practical calculations have been performed for the cosmological constant counterterm, so far.

Conclusions

- **QG can be formulated, in a natural way, as a symmetry-preserving theory. This means that all loop divergences are diffeomorphism-invariant local functionals.**

- **At the same time the simplest QG based on GR is not renormalizable by power counting, such that the requested number of counterterms eventually becomes infinite.**

- **HDQG is renormalizable theory, so one can calculate any observable. But its physical interpretation is spoiled by massive ghosts, which can violate unitarity if being removed ad hoc.**

- **It is unclear whether ghosts persist at the non-perturbative level, also there are strong indications that they are actually not generated below the Planck scale.**

- **Taking into account the role of higher derivatives, the ghost issue is one of the main challenges in the whole QG.**

Exercises and references

1. Explore the relations between Rivers projectors (page 4) and the parametrization on page 5.

2. Derive the equations of motion for the action of gravity which includes R, R^2 and $R_{\mu\nu}R^{\mu\nu}$ terms. Explain why adding $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ is not changing the physical contents of the theory. Verify this statement by direct calculation. Verify that the traces of equations of motion for these three terms are the same, up to numerical coefficients [arXiv:1507.03620].

3. Derive the action of ghosts for the three parameterizations of the metric discussed on page 8.

4. Verify BRST invariance for gravity and discuss its dependence on the action (e.g. quantum GR or fourth derivative).

5. Write down the explicit form of the generators for diffeomorphism, and the form of the gauge structure functions. Verify the closed algebra of the generators on pg. 9.

6. Using mathematical induction, prove the topological relation on page 15. Generalize the power counting formula for $D + d$ for an arbitrary dimension of spacetime. Apply these results to establish the number of derivatives required to have renormalizable QG in $n = 2$ and $n = 6$ dimensions. Try to construct the gravity action in $2D$, leading to a consistent QG theory.

7. Consider six-derivative scalar model

$$S = \int d^n x \Big\{ a\varphi\Box\varphi + b\varphi\Box^2\varphi + c\varphi\Box^3\varphi. \Big\}$$

Find the conditions for the constants a, b, c which are needed to have positively defined kinetic energy of the massless mode and (i) two real massive poles. (ii) complex conjugate poles. Prove that if the masses of the massive particles in (i) are real and different, the lightest of these particles is ghost and the heaviest one has positive kinetic energy [hep-th/9610006, arXiv:1604.07348].

8. Derive the general form of the gauge fixing dependence in the one-loop counterterms in quantum GR. Extract the combinations of the coefficients of six possible counterterms which are invariant [arXiv:1712.03338 and references therein]. Which of the famous coefficients on page 38 is invariant?

9. As a follow-up of the previous exercise, use the results of the paper of ’tHooft and Veltman (1974) to derive the coefficient of the pole of the Gauss-Bonnet terms in pure QG and in the theory with additional scalar field. Without explicit calculations show that this sum does not depend on whether the quantum calculations are performed in Einstein or Jordan frame.

10. Verify the gauge fixing independence of the three combinations of parameters given on page 43. Show that this list can not be extended.

11. Prove that in the theory of QG with six or more derivatives all logarithmic divergences are gauge-fixing independent (pg. 46).