α_s from QCD Vertices

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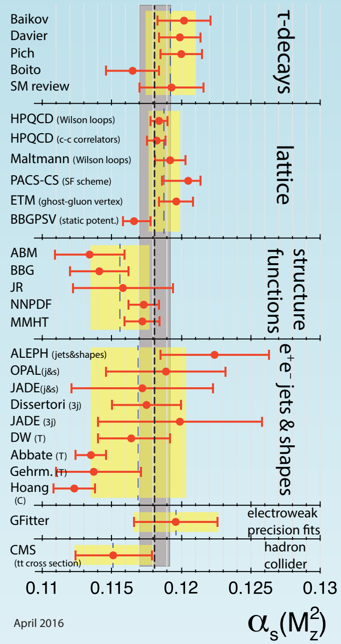
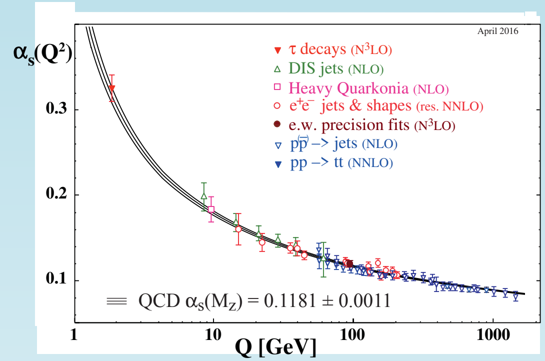
INTRODUCTION

Running Coupling Constant α_s





The strong coupling constant $\alpha_s(\overline{\mu})$, taken at a fixed reference scale $\overline{\mu}$, "plays a key role in the understanding of QCD and in its application to collider physics":

- $\alpha_s(\overline{\mu})$ is an important source of uncertainty in the Standard Model predictions,
- $\alpha_s(\overline{\mu})$ "yields one of the essential boundary conditions for completions of the Standard Model at high energy".

The value of α_s should also be determined with good accuracy over as large a range of scales as possible, in order to reveal potential anomalous running in the strength of the strong interaction.

Determination of α_s using Lattice QCD (I)

The Lattice QCD average (FLAG2019)

$$\alpha^{(5)}_{\overline{MS}}(M_Z) = 0.11823(81) ,$$

yielding

$$\Lambda^{(5)}_{\overline{MS}}(M_Z) = 211(10) \text{ MeV} ,$$

(30% error reduction from 2016, error about 4 times smaller than 15 years ago) is by now a factor two more precise than the nonlattice world average (PDG 18)

$$\alpha^{(5)}_{\overline{MS}}(M_Z) = 0.1174(16) .$$

The present world average is (PDG 18)

$$\alpha^{(5)}_{\overline{MS}}(M_Z) = 0.1181(11) \quad \text{with} \quad \Lambda^{(5)}_{\overline{MS}}(M_Z) = 210(14) \text{ MeV}.$$

Determination of α_s using Lattice QCD (II)

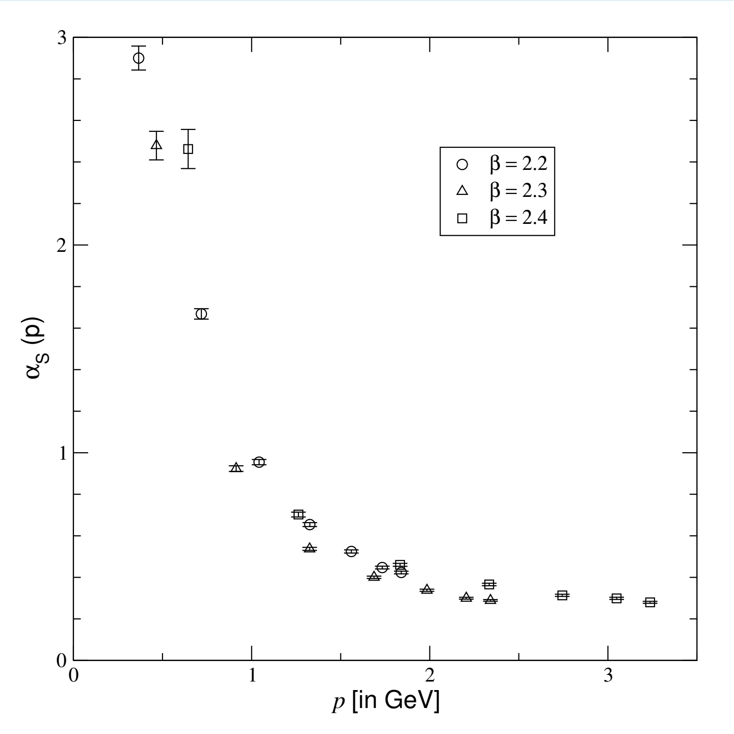
FLAG considers the following lattice evaluations of the strong-coupling constant $\alpha_s(\overline{\mu})$:

- step-scaling methods (talk by Tomasz Korzec),
- $q\,\overline{q}$ potential (talks by Yuichiro Kiyo and Johannes Weber),
- short-distance lattice quantities, e.g. Wilson loops,
- heavy-quark-current two-point functions (talk by Peter Petreczky),
- eigenvalue spectrum of the Dirac operator (talk by Shoji Hashimoto),
- ghost-gluon vertex (review, this talk).

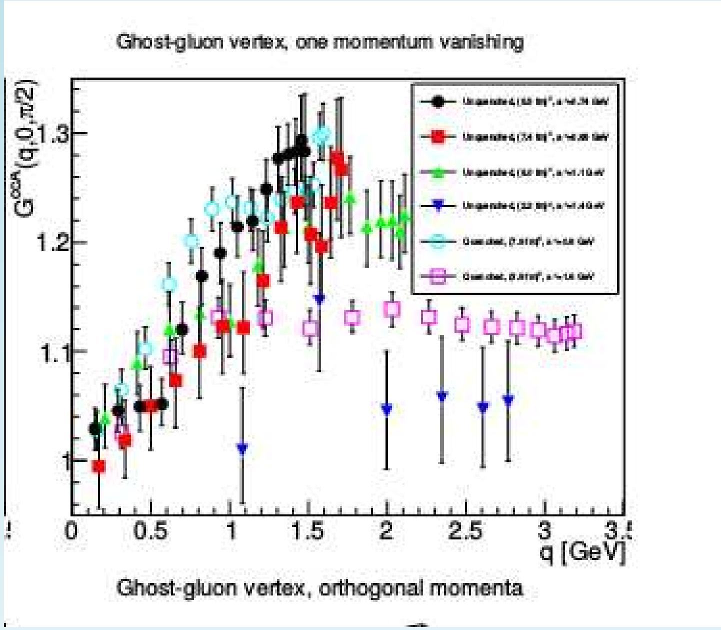
α_s from Vertices

- "The most intuitive and in principle direct way to determine the coupling constant in QCD"
- Consider one of the vertices and a suitable combination of renormalization constants to relate bare (lattice) and renormalized coupling constant (textbook definitions)
- Requires gauge fixing and a nonperturbative renormalization condition: usually, Landau gauge and the vertex equal to its tree-level value at some scale μ (various MOM schemes)
- Possible (IR) Gribov-copy effects

α_s from the Ghost-Gluon Vertex



The running coupling constant $\alpha_s(p^2) = \alpha_0 \, Z_3(p^2) \widetilde{Z}_3^2(p^2) / \widetilde{Z}_1^2(p^2)$ from the ghost-gluon vertex (A.C. et al., 2004).



Dressing function of the ghost-gluon vertex (T. Boz et al., 2019). Small unquenching effects in the ghost sector.

Problem: vertices are usually noisy ⇒ use propagators!

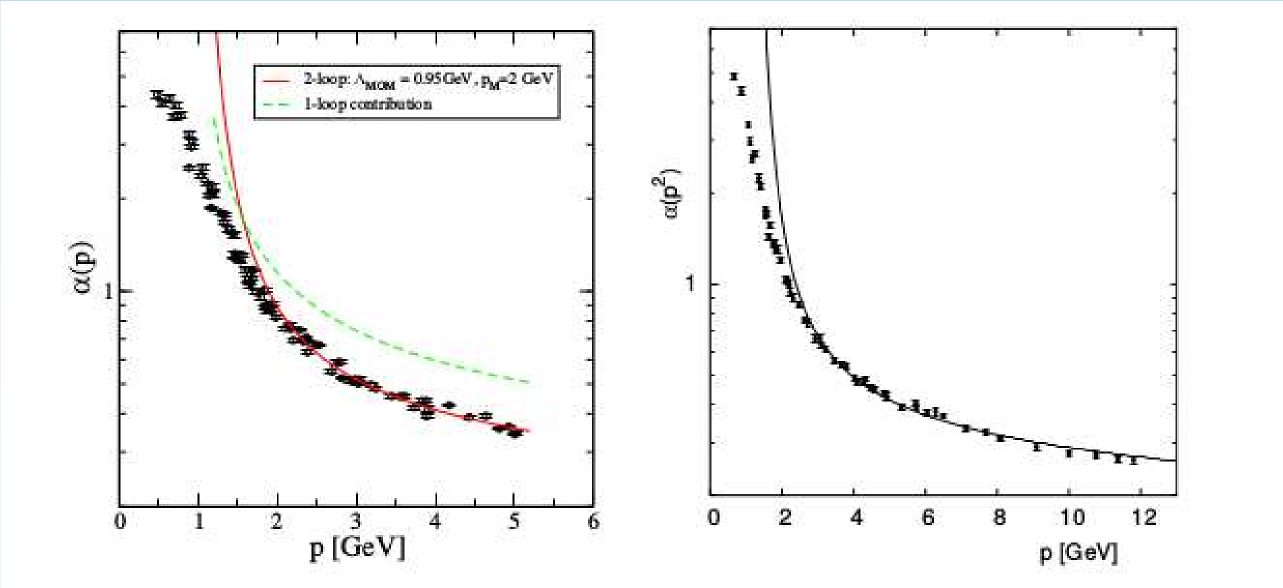
Different MOM Schemes

The MOM-schemes require the values of properly chosen Green functions to be fixed (usually to their tree-level values) at a given (μ-dependent) configuration of external momenta (subtraction point).

Considering gluon and ghost propagators and ghost-gluon vertex, the most common ones are:

- MOM scheme: the vertex reduces to the tree-level one at a symmetric subtraction point $q_1^2 = q_2^2 = q_3^2 = \mu^2$;
- $\widetilde{\text{MOM}}$ scheme: the vertex reduces to the tree-level one at an asymmetric subtraction point $q_1^2 = q_2^2 = \mu^2,\ q_3^2 = 0$;
- MOM-Taylor scheme: with a zero-momentum incoming ghost the renormalization constant $\widetilde{Z}_1(\mu^2)$ for the proper ghost-gluon vertex (in Landau gauge) is equal to one.
- miniMOM scheme: one requires $\widetilde{Z}_1(\mu^2) = \widetilde{Z}_1^{\overline{MS}}(\mu^2)$ and uses $\widetilde{Z}_1^{\overline{MS}}(\mu^2) = 1$ in Landau gauge (non-renormalization of the ghost-gluon vertex).

α_s from Gluon and Ghost Propagators

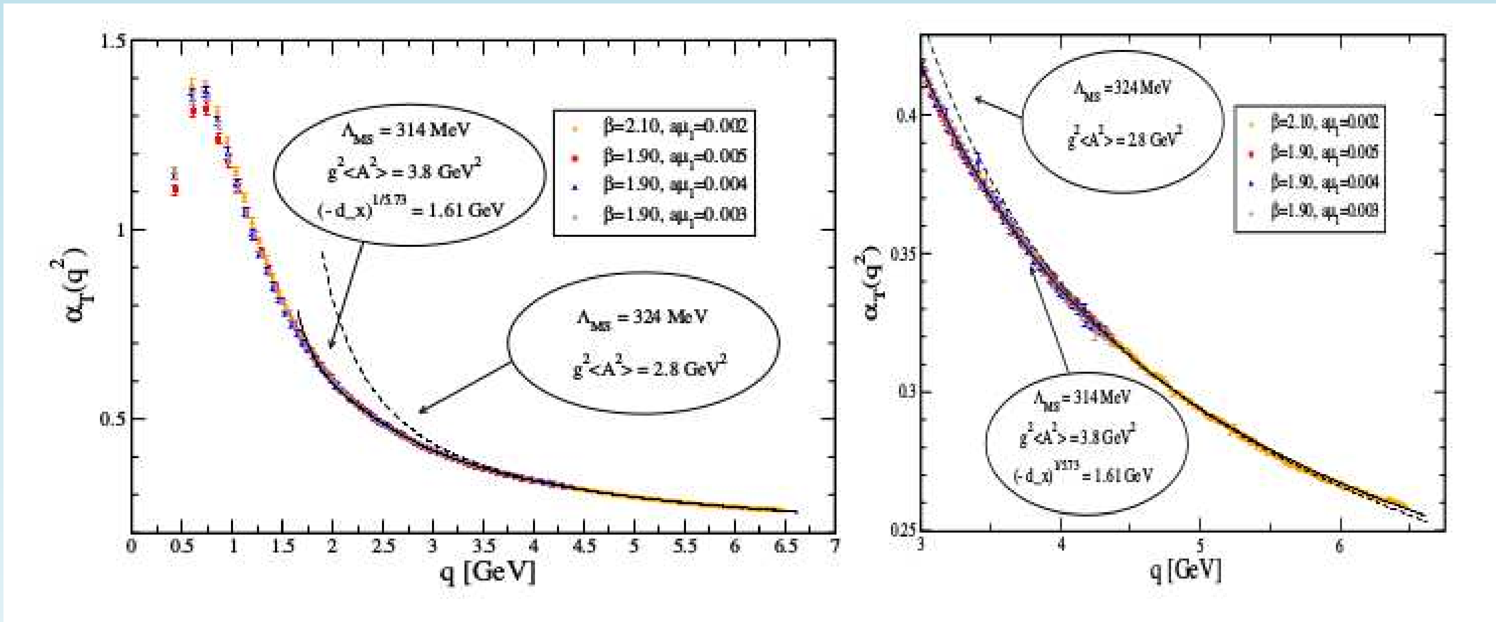


(Bloch et al., 2004)

The strong coupling constant (in Landau gauge) $\alpha_R(p^2)$ defined as the $a \to 0$ limit of $\alpha_0(a) F_B(p^2, a) J_B^2(p^2, a)$, where $F_B(p^2, a)$ and $J_B(p^2, a)$ are the (bare) gluon and ghost form factors.

General questions to be addressed: non-perturbative effects, breaking of rotational symmetry.

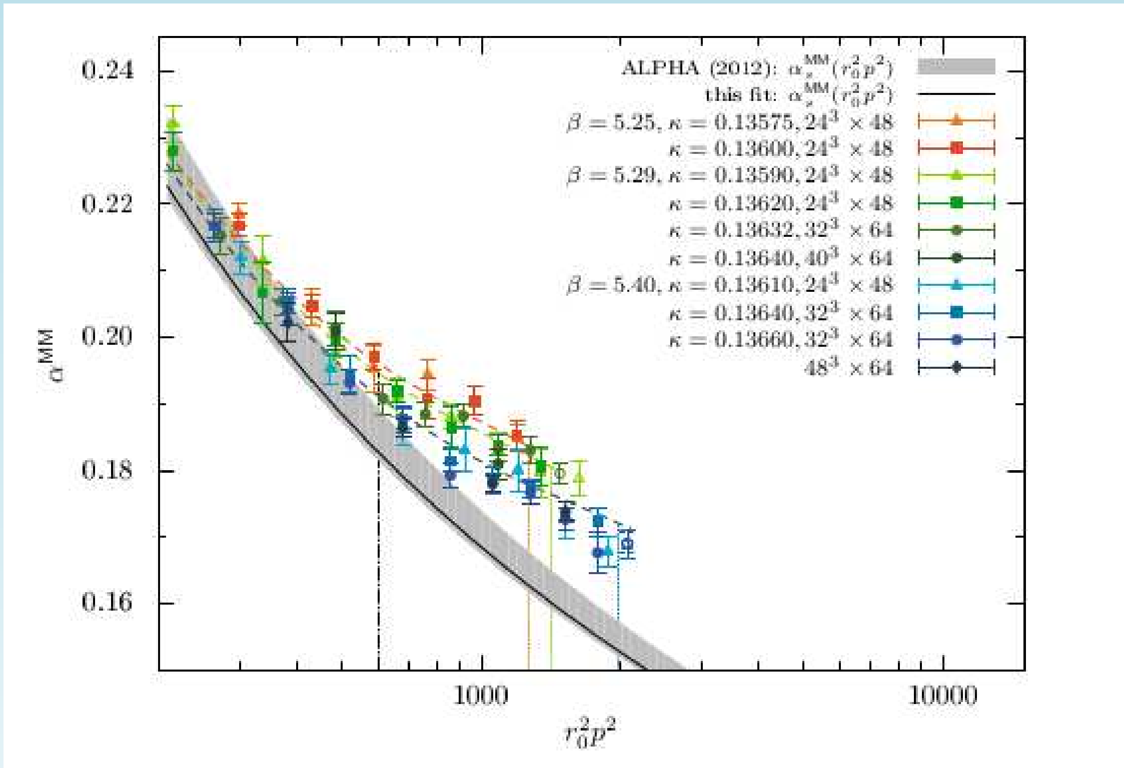
α_s in the MOM-Taylor Scheme



(Blossier et al., 2014; 4-loop expression)

The Taylor coupling $\alpha_T(q^2)$ (ghost-gluon vertex in Landau gauge) is defined as the $\Lambda \to +\infty$ limit of $\frac{g_0^2(\Lambda^2)}{4\pi} G(q^2, \Lambda^2) F^2(q^2, \Lambda^2)$, where $G(q^2, \Lambda^2)$ and $F(q^2, \Lambda^2)$ are the (bare) gluon and ghost form factors (and Λ is an UV cutoff).

α_s in the miniMOM Scheme



(Sternbeck et al., 2012; 4-loop expression)

The strong coupling constant (in Landau gauge) $\alpha_s^{MM}(p)$ is defined as the $a \to 0$ limit of $\frac{g_0^2(a)}{4\pi} Z_D(p,a) Z_G^2(p,a)$, where $Z_D(p,a)$ and $Z_G(p,a)$ are the (bare) gluon and ghost form factors. Data fitted without A^2 and $1/p^x$ contributions.

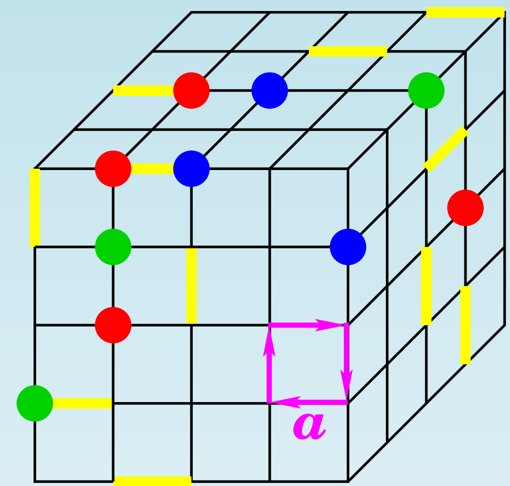
LATTICE SETUP

(how to evaluate propagators and vertices on the lattice)

QCD on a Lattice

Three ingredients:

1. Quantization by path integrals $\Rightarrow$ sum over configurations with "weights" $e^{i\,S/\hbar}$.
2. Euclidean formulation (analytic continuation to imaginary time) $\Rightarrow$ weight becomes $e^{-S/\hbar}$.
3. Discrete space-time (lattice spacing a) and finite-size (L) lattices.



- The UV cutoff a goes to zero in the continuum (= physical) limit: rigorous formulation of quantum field theory.
- The IR cutoff L implies an IR cut at small momenta $p_{min} \sim 1/L$.
- Connection to continuum physics: $L \to +\infty$ (infinite-volume limit), $a \to 0$ (continuum limit), $m_q \to$ physical values.

Gauge-Related Lattice Features

- Gauge action

$$S = \frac{\beta}{3} \sum_{\square} \text{Re}\,\text{Tr}\, U_{\square}$$

written in terms of oriented plaquettes $U_{\square}$, formed by the link variables $U_{x,\mu} \equiv e^{ig_0 a A^b_\mu(x) T_b}$ (group elements).

- Under gauge transformations $U_{x,\mu} \to g(x)\, U_{x,\mu}\, g^\dagger(x+\mu)$, where $g \in SU(N) \Rightarrow$ closed loops are gauge-invariant.
- Reduces to the usual action for $a \to 0$ (with $\beta = 2N/g_0^2$).
- Integration volume is finite: no need for gauge-fixing.
- When gauge fixing, procedure is incorporated in the simulation, no need to consider Faddeev-Popov matrix and ghost fields explicitly.

Numerical Simulations

When we are interested in gauge-dependent quantities we consider the following steps:

1. Choose an initial configuration $\mathcal{C}_0 = U_\mu(x) \in$ SU(N_c)
2. Thermalize the initial configuration (heat-bath, etc.) $\mathcal{C}_0 \to \mathcal{C}_1$
3. Fix the gauge for the configuration $\mathcal{C}_i$ with $i = 1, 2, \ldots$
4. Evaluate (gauge-dependent) quantities using the configuration $\mathcal{C}_i$
5. Produce a new (independent) configuration $\mathcal{C}_i \to \mathcal{C}_{i+1}$
6. Go back to step 3

We do not need to simulate anti-commuting variables or to evaluate the determinant of the Faddeev-Popov matrix!

Lattice Landau Gauge

In the continuum: $\partial_\mu A_\mu(x) = 0$. On the lattice the (minimal) Landau gauge is imposed by minimizing the functional

$$S[U;g] = -\sum_{x,\mu} Tr\, U^g_\mu(x) \; ,$$

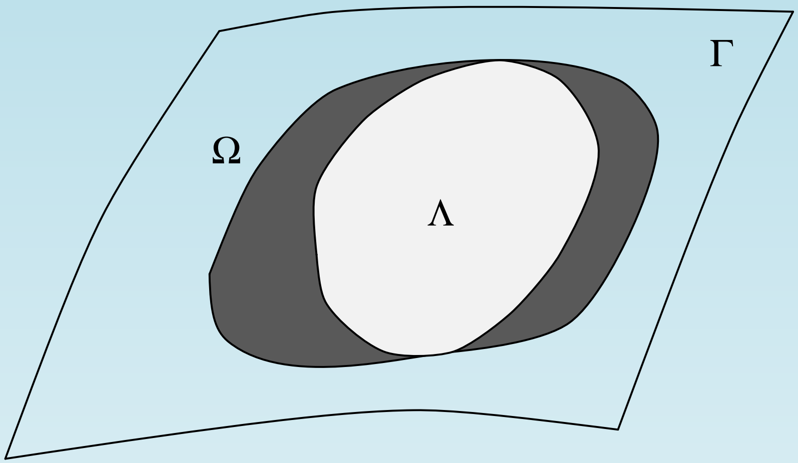
where $g(x) \in SU(N)$ and $U^g_\mu(x) = g(x)\, U_\mu(x)\, g^\dagger(x + ae_\mu)$ is the lattice gauge transformation. By considering the relations $U_\mu(x) = e^{iag_0 A_\mu(x)}$ and $g(x) = e^{i\tau\theta(x)}$, we can expand $S[U;g]$ (for small τ):

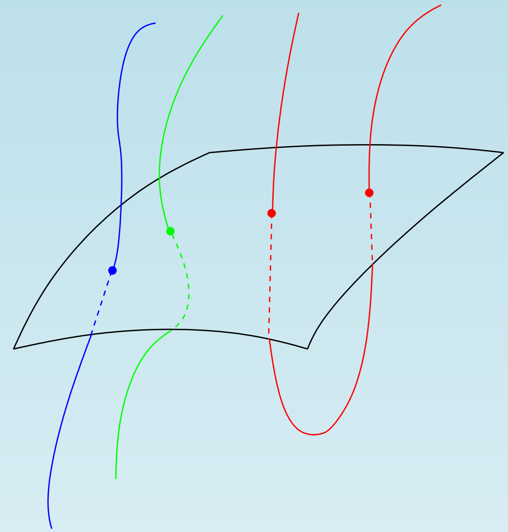
$$S[U;g] = S[U;1\!\!1] + \tau\, S^{'}[U;1\!\!1](b,x)\,\theta^b(x)$$

$$+ \frac{\tau^2}{2}\,\theta^b(x)\, S^{''}[U;1\!\!1](b,x;c,y)\,\theta^c(y) + \ldots$$

where $S^{''}[U;1\!\!1](b,x;c,y) = \mathcal{M}(b,x;c,y)[A]$ is a lattice discretization of the Faddeev-Popov operator $-D \cdot \partial$ with $A_\mu(x) = \left[U_\mu(x) - U^\dagger_\mu(x)\right]_{\text{traceless}}/(2i)$.

Constraining the Functional Integral





- At a stationary point $S^{'}[U;\mathbb{1}](b,x) = 0$ one has $\sum_{\mu} A_{\mu}^{b}(x) - A_{\mu}^{b}(x - a e_{\mu}) = 0$, which is a discretized version of the (continuum) Landau gauge condition. At a local minimum one also has $\mathcal{M}(b,x;c,y)[A] \geq 0$. This defines the first Gribov region $\Omega \equiv \{U : \partial \cdot A = 0, \mathcal{M} \geq 0\} \equiv$ local minima of $S[U;\omega]$ (V.N. Gribov, 1978).
- All gauge orbits intersect Ω (Dell'Antonio et al., 1991) but the gauge fixing is not unique (Gribov copies).
- Absolute minima of $S[U;\omega] \equiv$ fundamental modular region Λ, free of Gribov copies in its interior. (Finding the absolute minimum is a spin-glass problem.)
- Perturbation theory sits at the "center" of Ω.

Gluon and Ghost Propagators

As a consequence of the restriction of the measure to the region Ω:

- In minimal Landau gauge the gluon propagator

$$D^{ab}_{\mu\nu}(p) = \sum_x e^{-2i\pi k\cdot x} \langle A^a_\mu(x)\, A^b_\nu(0)\rangle = \delta^{ab} \left(g_{\mu\nu} - \frac{p_\mu\, p_\nu}{p^2} \right) D(p^2)$$

  is suppressed in the IR limit, i.e. $D(0)$ is finite (and nonzero) $\Rightarrow$ change of concavity at intermediate momenta.

- Infinite volume favors configurations on the first Gribov horizon, where λ_{min} of $\mathcal{M}$ goes to zero. In turn, the ghost propagator

$$G(p) = \frac{1}{N_c^2 - 1} \sum_{x,\, y,\, a} \frac{e^{-2\pi i\, k\cdot(x-y)}}{V} \langle\, \mathcal{M}^{-1}(a, x; a, y)\, \rangle\,,$$

  is IR enhanced at intermediate momenta, but it is free-like in the IR limit.

Consider momenta $p \gtrsim 1.5$ GeV (at least), in order to avoid complicated nonperturbative effects on the (gluon, ghost) propagators.

Momenta on the Lattice

The components of the lattice momenta are given by

$$p_\mu = 2\,\sin\left(\frac{\pi\,p_\mu}{N}\right), \qquad p_\mu = 0, 1, \ldots, N/2\,.$$

For a given lattice side N and lattice spacing a we have (in $4d$)

$$p_{min} = \frac{2}{a}\sin\left(\frac{\pi}{N}\right) \sim \frac{2\pi}{L}\,, \qquad p_{max} = \frac{4}{a}\sin\left(\frac{\pi}{2}\right) = \frac{4}{a}\,,$$

where $L = Na$.

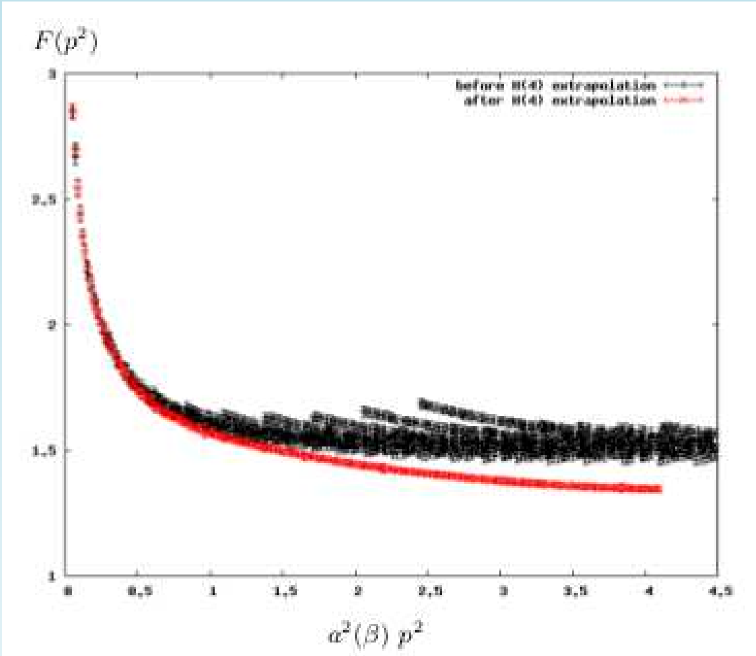
Constraint: with $Lm_\pi \geq 4$, i.e. $L \gtrsim 5.66$ fermi, one has $p_{min} \approx 0.22$ GeV. Also a is small enough to allow simulating the heaviest quark mass considered.

For example: $a = 0.1$ fermi (and $N \approx 56$) $\Rightarrow p_{max} \approx 7.9$ GeV.

In Blossier et al. (2014): $a \approx 0.06$ fermi and $V = 48^3 \times 96$.

FLAG requirements for the continuum limit: for $\alpha = 0.3$ one should have $ap < c$, where c depends on the type of $\mathcal{O}(a)$ improvement used in the lattice setup.

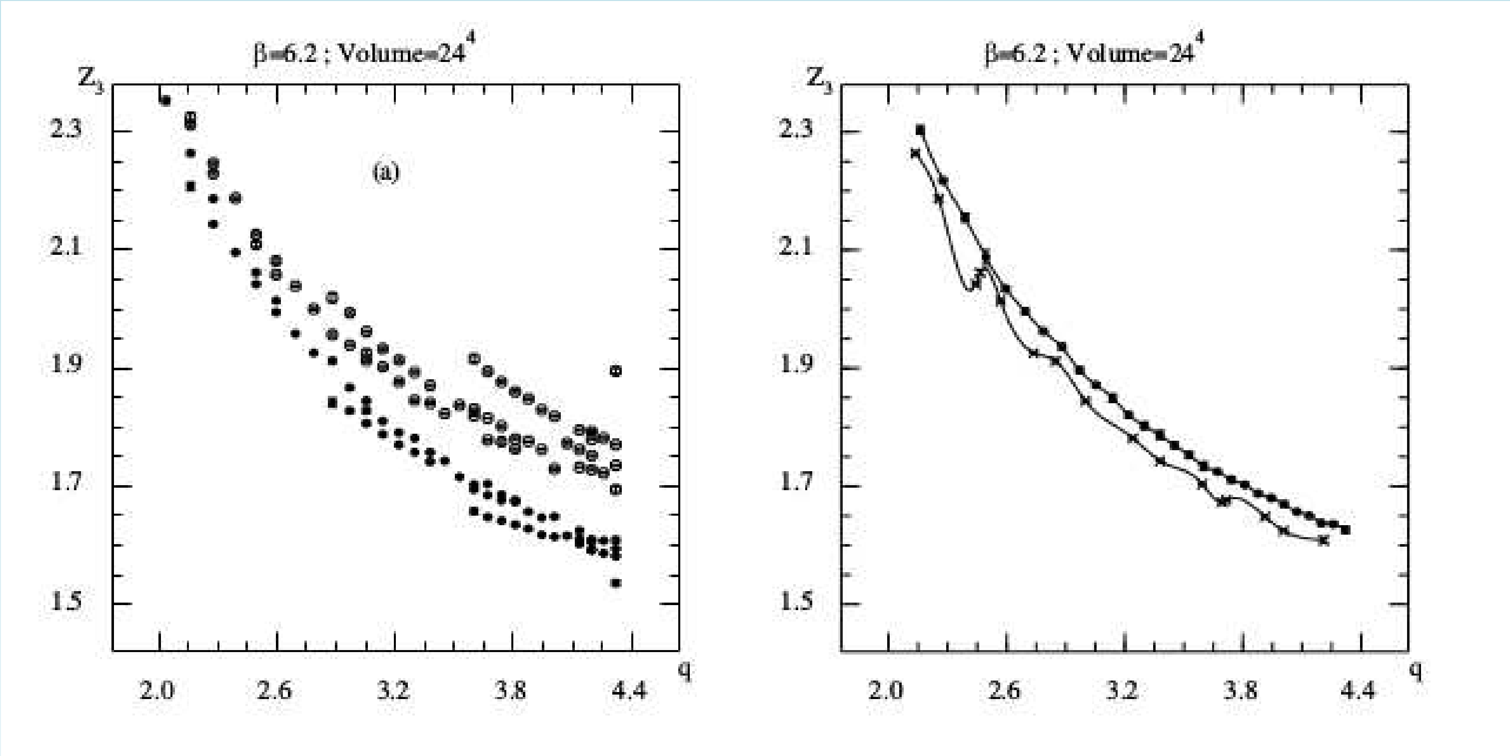
Breaking of Rotational Invariance (I)



$H(4)$ artifacts for the ghost form factor in Landau gauge (Blossier et al., 2014): a lattice quantity $Q(a^2p^2)$, as a function of momenta p^2, depends on $p^2 = \sum_\mu p_\mu^2$ but also on $p^{[4]} = \sum_\mu p_\mu^4$, $p^{[6]}$, etc. Global fit of the data using

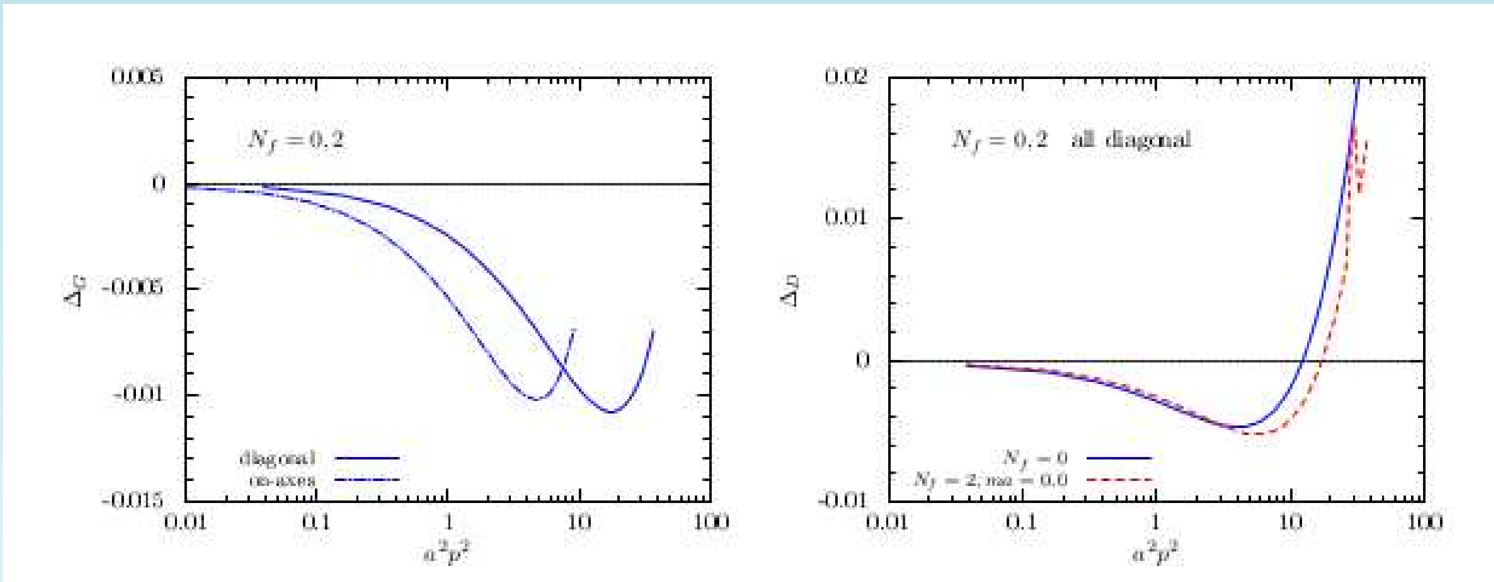
$$Q(a^2p^2, a^4p^{[4]}, a^6p^{[6]}, \ldots) = Q(a^2p^2) + \left(r_0 + r_1 a^2p^2\right) a^2p^{[4]}/p^2 + \ldots$$

Breaking of Rotational Invariance (II)



$H(4)$ artifacts for the gluon form factor in Landau gauge (Becirevic et al., 1999): comparison of $H(4)$ extrapolation with the "democratic" selection and the use of two different lattice momenta.

Breaking of Rotational Invariance (III)



Hypercubic corrections (at one loop) for the ghost form factor (left) and for the gluon form factor (right) in Landau gauge (Sternbeck et al., 2012).

POSSIBLE IMPROVEMENT(S)

α_s from the Ghost Propagator (I)

Evaluate $\alpha_s(\overline{\mu})$ considering only the ghost propagator:

1. rotational analysis for one quantity (not two!), which is also better behaved;
2. small unquenching effect $\Rightarrow$ setup the analysis in the quenched case;
3. not necessary to take $\widetilde{Z}_1(\mu^2) = 1 \Rightarrow$ one may consider other gauges ($\xi \neq 0$) and use different determinations of $\alpha_s(\overline{\mu})$;
4. all the necessary ingredients are available!

Following (Becirevic et al., 1999, gluon case) one may consider the bare ghost propagator (A.C. et al., 2004; Boucaud et al., 2005) $G_B(\mu)$ and the two coupled differential equations (in any renormalization scheme)

$$\frac{d \log Z_G(\mu)}{d \log \mu^2} = \Gamma_G(\alpha) \, , \qquad \frac{d\alpha}{d \log \mu} = \beta(\alpha) \, ,$$

where $Z_G(\mu) = \mu^2 G_B(\mu)$.

Note: only in a MOM scheme is the function $\Gamma_G(\alpha)$ the anomalous dimension of the ghost renormalization constant.

α_s from the Ghost Propagator (II)

For example, at two loops we can write

$$\Gamma_G\left[\alpha(\mu^2)\right] \,=\, -\left[\frac{g_0}{4\,\pi}\,\alpha \,+\, \frac{g_1}{(4\,\pi)^2}\,\alpha^2 \,+\, \mathcal{O}\left(\alpha^3\right)\right],$$

$$\frac{d}{d\log\mu}\,\alpha \,=\, \beta(\alpha) \,=\, -\left[\frac{\beta_0}{2\,\pi}\,\alpha^2 \,+\, \frac{\beta_1}{(2\,\pi)^2}\,\alpha^3 \,+\, \mathcal{O}\left(\alpha^4\right)\right]$$

and solve these equations obtaining

$$q \,=\, \mu_0\,\exp\left[\frac{2\,\pi}{\beta_0}\left(\frac{1}{\alpha(q)} - \frac{1}{\alpha(\mu_0)}\right)\right]\left[\frac{\alpha(q)}{\alpha(\mu_0)}\right]^{\beta_1/\beta_0^2}\left[\frac{2\,\pi\,\beta_0 \,+\, \beta_1\,\alpha(\mu_0)}{2\,\pi\,\beta_0 \,+\, \beta_1\,\alpha(q)}\right]^{\beta_1/\beta_0^2}$$

and

$$Z_G(q) \,=\, Z_G(\mu_0)\left[\frac{\alpha(q)}{\alpha(\mu_0)}\right]^{g_0/\beta_0}\left[\frac{2\,\pi\,\beta_0 \,+\, \beta_1\,\alpha(q)}{2\,\pi\,\beta_0 \,+\, \beta_1\,\alpha(\mu_0)}\right]^{\frac{g_1}{2\,\beta_1}\,-\,\frac{g_0}{\beta_0}}\,.$$

α_s from the Ghost Propagator (III)

A fit of the data $(q, Z_G(q))$ provides a determination of $Z_G(\mu_0)$ and $\alpha(\mu_0)$ at the scale μ_0. The fit can be done in any scheme for which the coefficients g_i and β_i are known, up to some order. The result for $\alpha(\mu_0)$ allows to evaluate Λ_{QCD} in that scheme.

Some recent results:

- complete set of vertex and wave function with N_f fermions, in an arbitrary representation, to five-loop order for a generic covariant gauge and an arbitrary simple gauge group, in the minimal subtraction scheme (Chetyrkin et al., 2017);
- self-energies and a set of three-particle vertex functions (at points where one of the momenta vanishes) at the four-loop level in the MS scheme, for a generic gauge group and with the full gauge dependence; they also derive the five-loop beta function in the miniMOM scheme (Ruijl et al., 2017).

Lattice Linear Covariant Gauge

We want to impose the gauge condition $\partial_\mu A^b_\mu(x) = \Lambda^b(x)$, for real-valued functions $\Lambda^b(x)$, generated using a Gaussian distribution with width $\sqrt{\xi}$.

The Landau gauge [$\Lambda^b(x) = 0$] is obtained on the lattice by minimizing the functional $S_{LG}[U; g] = -\sum_{x,\mu} Tr U^g_\mu(x)$.

The lattice linear covariant gauge condition $\nabla \cdot A^b(x) = \sum_\mu A^b_\mu(x + e_\mu/2) - A^b_\mu(x - e_\mu/2) = \Lambda^b(x)$ may be obtained by minimizing the functional (A.C. et al., 2009) $S_{LCG}[U, g, \Lambda] = S_{LG}[U; g] + \Re\, \mathrm{Tr} \sum_x i g(x) \Lambda(x)$.

Conceptual problem: using the standard compact discretization, the gluon field is bounded while the four-divergence of the gluon field satisfies a Gaussian distribution, i.e. it is unbounded. This may give rise to convergence problems when considering large values of ξ.

The limit $\xi \to 0$ is (numerically) well behaved (A.C. et al., 2010).

The ghost propagator in linear covariant gauge has been recently evaluated for the first time (A.C. et al., 2018).

Conclusions

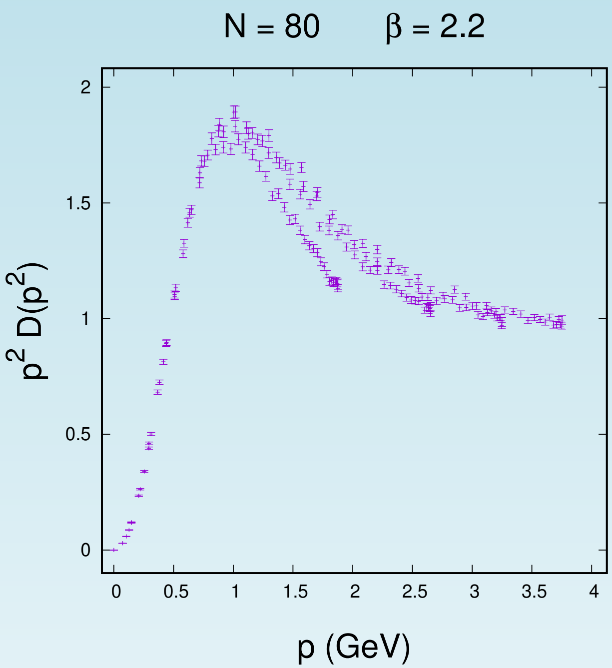
1. Using propagators/vertices for the evaluations of the strong coupling constant $\alpha_s(\overline{\mu})$ is an interesting approach which, hopefully, will again be used in the near future

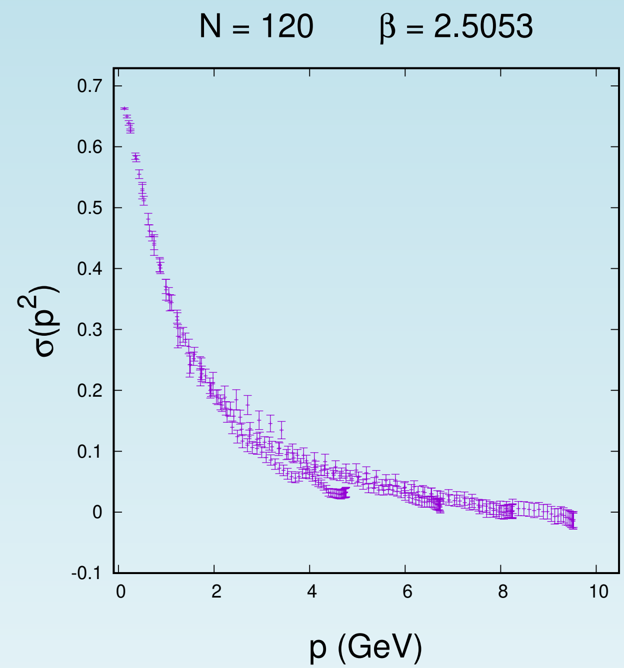
2. Recent results (in the continuum and on the lattice) allow to extend the analysis to the linear covariant gauge (at least for relatively small value of ξ)

3. One needs to attack the breaking of rotational symmetry in a more systematic way (improved gauge fixing?)

THANKS!

Breaking of Rotational Invariance (I)





Gluon dressing function $p^2 D(p^2)$ (in minimal Landau gauge, $N = 80$ and $\beta = 2.2$) for four different sets of momenta, using unimproved momenta $p^2 = 4 \sum_\mu \sin^2(\pi\, n_\mu / N)$.

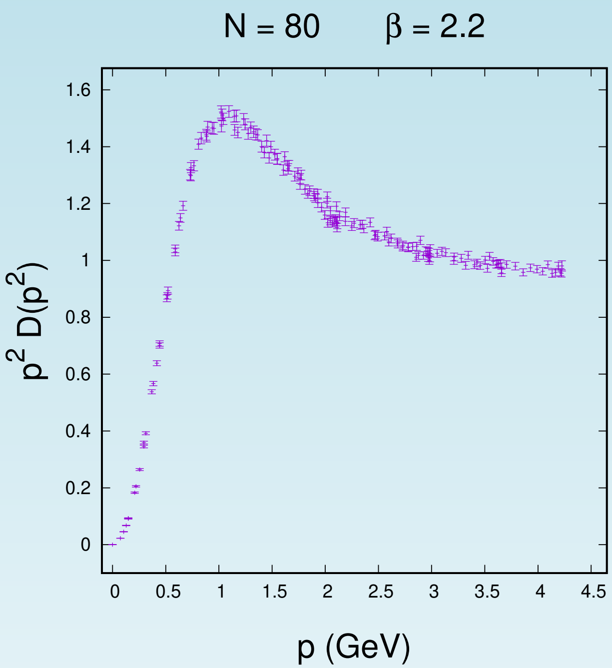
Gribov ghost form factor $\sigma(p^2)$ (in minimal Landau gauge, $N = 120$ and $\beta = 2.5053$) for four different sets of momenta, using unimproved momenta $p^2 = 4 \sum_\mu \sin^2(\pi\, n_\mu / N)$.

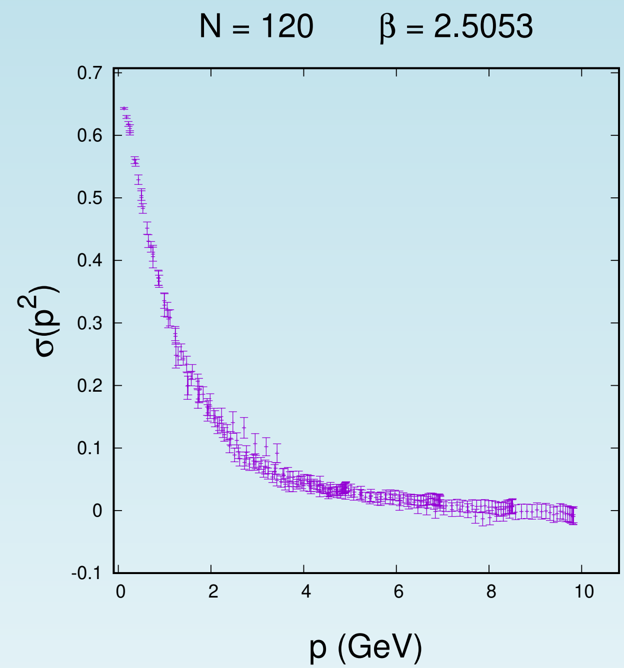
Breaking of Rotational Invariance (II)

	gluon propagator			ghost propagator		
N^4	r_4	r_6	$< \chi^2 >$	r_4	r_6	$< \chi^2 >$
48^4	0.054	0.000	2.62	0.016	—	1.30
72^4	0.084	-0.006	2.46	0.017	—	2.41
96^4	0.107	-0.015	2.35	0.014	—	0.48
120^4	0.073	-0.005	2.39	0.016	—	3.02
80^4	0.091	-0.006	2.70	0.021	—	1.70
128^4	0.059	-0.002	1.96	0.016	—	2.85
160^4	0.070	-0.006	2.67	0.019	—	2.95
192^4	0.073	-0.006	2.01	0.008	—	2.29

For each lattice volume $V = N^4$ we show the parameters r_4 and r_6 used to define improved momenta $p^2 = \sum_\mu \hat{p}_\mu^2 + r_4\,\hat{p}_\mu^4 + r_6\,\hat{p}_\mu^6$ with $\hat{p}_\mu = 2\,\sin\left(\pi\, n_\mu / N\right)$, for the gluon and ghost propagators.

Breaking of Rotational Invariance (III)

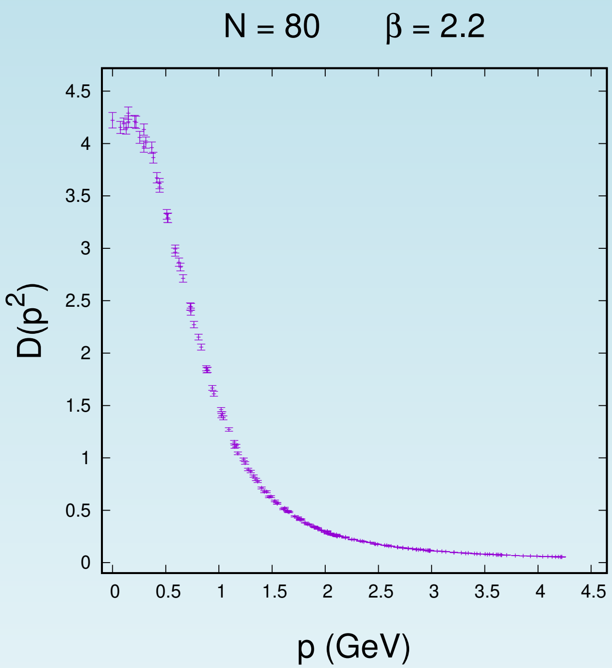


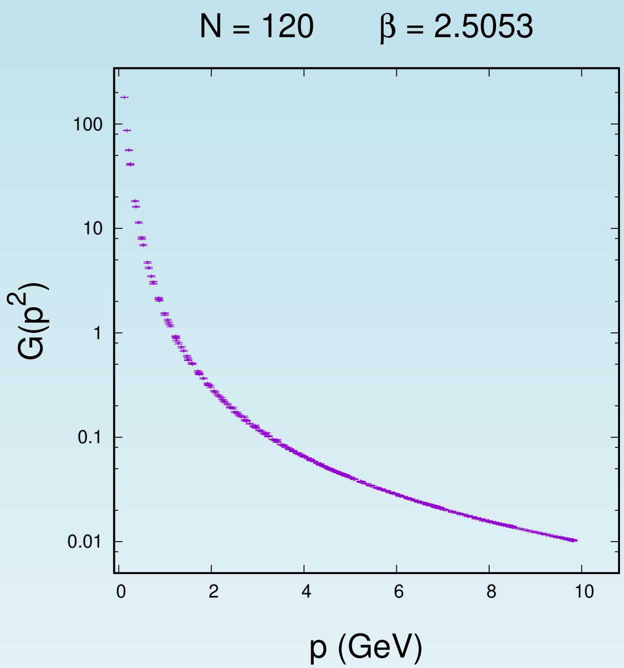


Gluon dressing function $p^2 D(p^2)$ (in minimal Landau gauge, $N = 80$ and $\beta = 2.2$) for four different sets of momenta, using improved momenta.

Gribov ghost form factor $\sigma(p^2)$ (in minimal Landau gauge $N = 120$ and $\beta = 2.5053$) for four different sets of momenta, using improved momenta.

Breaking of Rotational Invariance (IV)

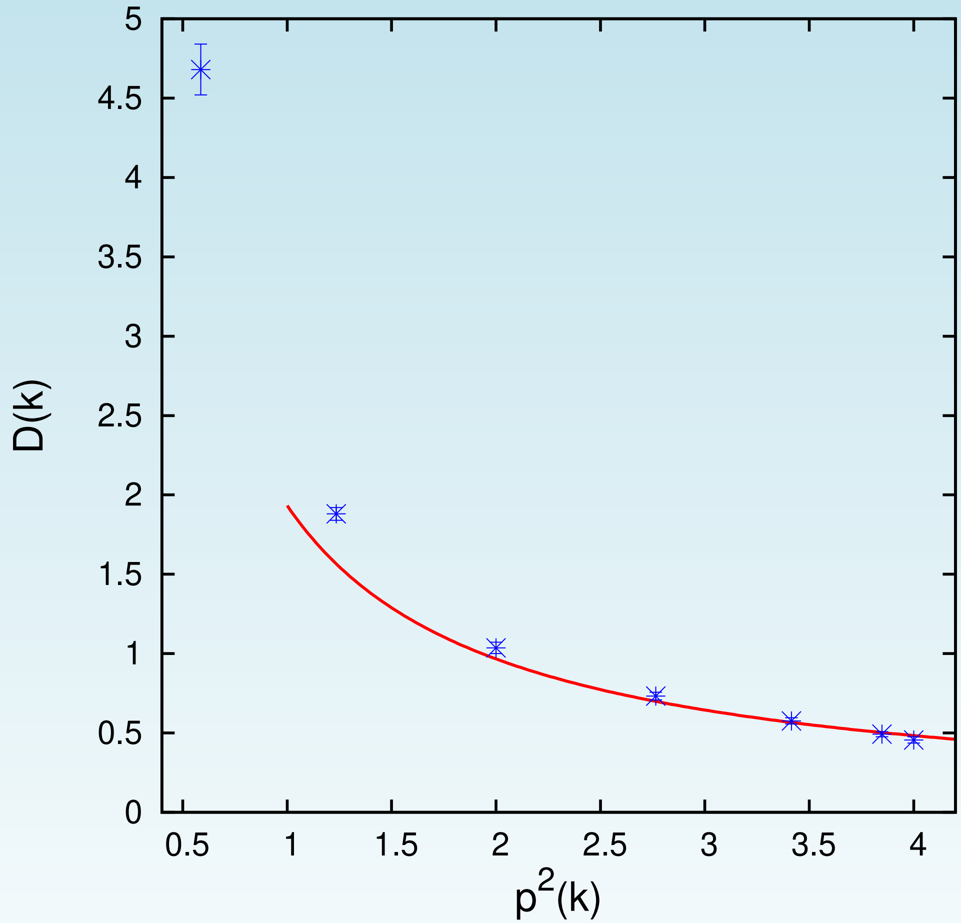




Gluon propagator $D(p^2)$ (in minimal Landau gauge, $N = 80$ and $\beta = 2.2$) for four different sets of momenta, using improved momenta.

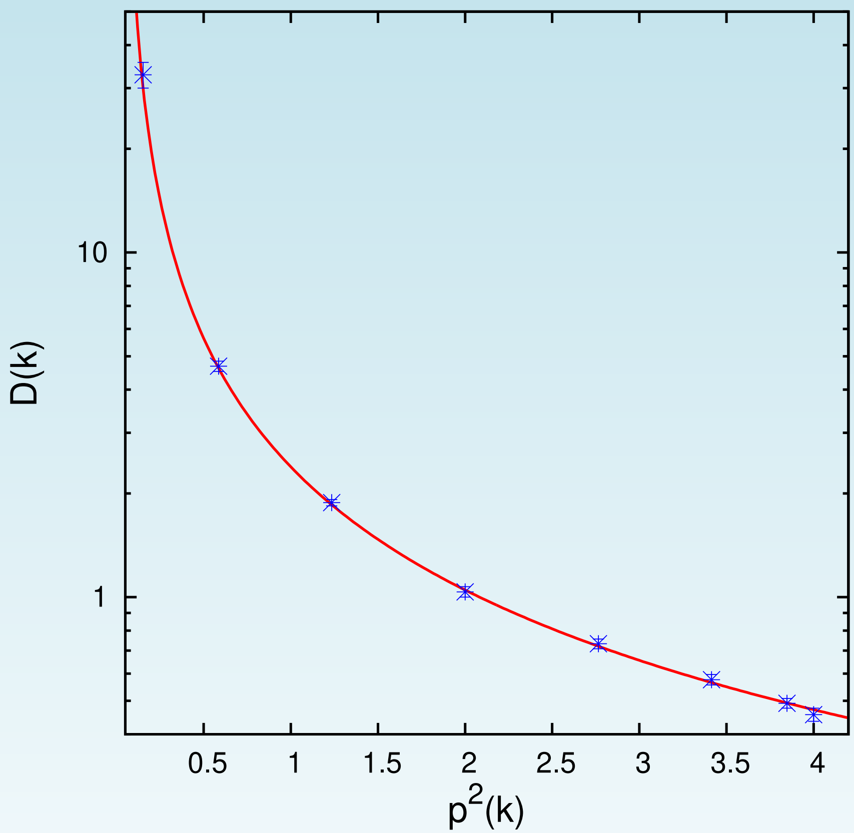
Ghost propagator $G(p^2)$ (in minimal Landau gauge, $N = 120$ and $\beta = 2.5053$) for four different sets of momenta, using improved momenta.

Comparison with Perturbation Theory



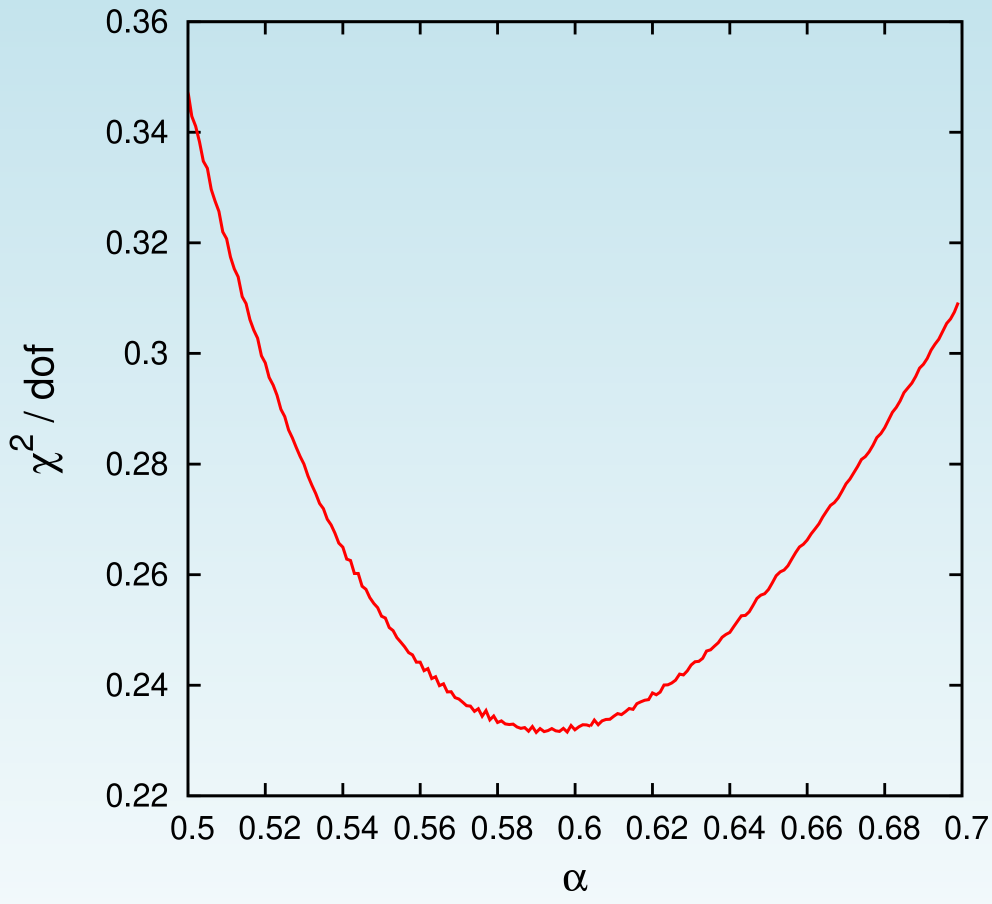
Fit of the Landau gluon propagator $D(k)$ using $c/p^2(k)$ in the SU(2) case at $\beta = 2.7$ and $V = 16^4$ ($p_{min} \approx 1.7$ GeV and $p_{max} \approx 8.7$ GeV).

Comparison with RG-Improved PT



Fit of the Landau gluon propagator $D(k)$ using $c\left[\log\left(p^2(k)/\Lambda^2\right)\right]^{-\alpha}/p^2(k)$ in the SU(2) case at $\beta = 2.7$ and $V = 16^4$. Here, $\alpha = \gamma_0/\beta_0 = 13/22$, where b_0 is (minus) the first coefficient of the beta function, and γ_0 is (minus) the first coefficient of the anomalous dimension of the gluon field.

Anomalous Dimension of the Gluon Field



Numerical evaluation of α considering the best χ^2/dof.

Theoretical value: 0.5909.

Numerical value: 0.590 ± 0.001!