Determination of the QCD coupling constant from the static energy and the free energy

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Outline

1. Introduction
2. Lattice QCD
3. Static energy
   - Static energy in perturbation theory
   - Static energy on the lattice
4. Singlet free energy
5. Summary

Conceptual idea of lattice determinations of α_s

- We compute observables on the lattice at sufficiently high scales for the weak-coupling approach to be applicable
- We compare continuum extrapolated lattice results to perturbative results in $\overline{\text{MS}}$ scheme to determine parameters

The QCD static energy of a (static) quark-antiquark pair (2010-2019)

- The scale is set by the (inverse) size of the system, $\nu = 1/r$
- Other scales are involved, i.e. the ultra-soft scale $\mu_{us} \sim \alpha_s/r$
- The static energy receives contributions from the singlet potential
  - renormalon subtraction required in the dimensional regularization
  - diverges as $\sim 1/a$ towards continuum limit of the lattice regularization

Bibliography : QCD static energy of a quark-antiquark pair

- TUM group[1] using 3 sea quark flavors (2010-now)
- Frankfurt/Jena group[2] using 2 sea quark flavors (2012-2018)
- JLQCD affiliated group[3] using 3 sea quark flavors (2018)

Extension to 4 sea quark flavors is planned by the TUMQCD collaboration

[1] Brambilla et al., Phys. Rev. Lett. 105 (2010) 212001
Bazavov et al., Phys. Rev. D86 (2012) 114031
Bazavov et al., Phys. Rev. D90 (2014) 7, 074038
Bazavov et al. [TUMQCD], arXiv:1907.11747

[2] Jansen et al. [ETMC], JHEP 1201, 025 (2012)
Karbstein et al., JHEP 1409, 114 (2014)
Karbstein et al., Phys.Rev. D98 (2018) no.11, 114506

[3] Takaura et al., JHEP 1904, 155 (2019)
Takaura et al., Phys. Lett. B789, 598-602 (2019)

Why lattice QCD?

- QCD is fraught with UV divergences that need regularization
- Regulators break symmetries or introduce unphysical properties and introduce some artificial scale μ
- We must remove the regulator before comparing to the real world

- The lattice approach regulates UV divergences through a finite lattice spacing and IR divergences through a finite box size
- The lattice explicitly respects gauge invariance, but explicitly breaks rotational symmetry and distorts chiral symmetry
- The lattice regulator can be systematically improved, i.e. discretization effects can be reduced via EFT methods and/or by brute force (smaller lattice spacing, larger box size)
- The lattice approach lends itself to perturbative and non-perturbative studies of QCD

Gauge ensembles

- We use the (rooted) Highly Improved Staggered Quark (HISQ)[4] action for two degenerate light quarks and a physical strange quark
- We use the tree-level Symanzik-improved gauge action
- Discretization errors of HISQ action scale as $\alpha_s a^2$ and a^4
- We use high statistics ensembles generated by the HotQCD[5] collaboration for a study of EoS with a pion mass of $m_\pi \approx 160\,\mathrm{MeV}$ and a kaon mass of $m_K \approx 504\,\mathrm{MeV}$ in the continuum limit.
- We also use extra ensembles generated for another study of EoS at high T with a pion mass of $m_\pi \approx 320\,\mathrm{MeV}$ in the continuum limit[6]
- We use $\left(r^2 \frac{\partial V_S}{\partial r}\right)_{r=r_1} = 1$ to fix the lattice scale, $r_1 = 0.3106(14)(8)(4)\,\mathrm{fm}$.

$N_\sigma{}^3 \times N_\tau$	a^{-1} [GeV]	# TU
48^4	$\lesssim 2.4$	8-16K
$48^3 \times 64$	$\lesssim 3.2$	8-9K
64^4	$\lesssim 4.9$	9K

$N_\sigma{}^3 \times N_\tau$	a^{-1} [Gev]	# TU
48^4	2.4	3K
64^4	$\lesssim 7.9$	8K

[4] Follana et al. [HPQCD], Phys.Rev. D75, 054502 (2007)
[5] Bazavov et al. [HotQCD], Phys.Rev. D90, 094503 (2014)
[6] Bazavov et al., Phys.Rev. D97, no. 1, 014510 (2018))

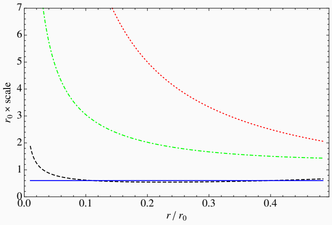
Static quark-antiquark energy

- Static energy determined from large-time behavior of Wilson loops

$$E(r) = \lim_{t\to\infty} \mathrm{i}\frac{d}{dt}\langle \ln W(t,\boldsymbol{r})\rangle\,, \qquad W(t,\boldsymbol{r}) = \exp\left\{ \mathrm{i}g \oint_{\boldsymbol{r},t} dz^\mu A_\mu \right\}$$

- Known in perturbation theory @ N^{3}LL (dimensional regularization)
- Nonperturbatively calculable in the lattice regularization
- For $r \ll 1/\Lambda_{\mathrm{QCD}}$ both schemes must agree, with hierarchy of scales

$$\frac{1}{r} \gg V_o - V_s \gg \Lambda_{\mathrm{QCD}}\,, \qquad \left\{ \begin{array}{c} V_s \\ V_o \end{array} \right\} \approx -\left\{ \begin{array}{c} -C_F \\ +\frac{1}{2N_c} \end{array} \right\} \frac{\alpha_s}{r}$$



Static quark-antiquark energy in perturbation theory

- Static energy determined from large-time behavior of Wilson loops

$$E(r) = \Lambda_s - \frac{C_F \alpha_s}{r}\left(1 + \#\alpha_s + \#\alpha_s^2 + \#\alpha_s^3 + \#\alpha_s^3 \ln \alpha_s + \#\alpha_s^4 \ln^2 \alpha_s + \#\alpha_s^4 \ln \alpha_s + \dots\right) \quad @\ \mathrm{N^3LL}$$

- US contributions to the static energy can be understood in pNRQCD

$$E(r) = \Lambda_s + \underbrace{V_s(r, \mu_{us})}_{\sim \ln^k (r\mu_{us}),\ k=1,2,\dots} - i \frac{g^2}{N_c} V_A^2 \int_0^\infty dt\, \underbrace{e^{-it(V_o - V_s)} \langle \mathrm{tr}\ \boldsymbol{r} \cdot \boldsymbol{E}(t) \boldsymbol{r} \cdot \boldsymbol{E}(0) \rangle (\mu_{us})}_{\sim \ln^k (r\mu_{us}),\ \ln^k \left(\frac{V_o - V_s}{\mu_{us}}\right),\ k=1,2,\dots} + \dots$$

  include the singlet potential and the ultra-soft contribution
- Potential nonrelativistic QCD (pNRQCD) Lagrangian[7] at NLO

$$\mathcal{L}_{\mathrm{pNRQCD}} = \mathcal{L}_{\mathrm{light}} + \int d^3 r\ \mathrm{tr}\ \left\{ S^\dagger [i\partial_0 - V_s(r, \nu, \mu_{us})] S +)^\dagger [iD_0 - V_o(r, \nu, \mu_{us})] O \right\}$$

$$+ V_A(r, \nu, \mu_{us})\ \mathrm{tr}\ \left\{ O^\dagger \boldsymbol{r} \cdot g\boldsymbol{E} S + S^\dagger \boldsymbol{r} \cdot g\boldsymbol{E} O \right\}$$

$$+ V_B(r, \nu, \mu_{us})\ \mathrm{tr}\ \left\{ O^\dagger \boldsymbol{r} \cdot g\boldsymbol{E} O + O^\dagger O \boldsymbol{r} \cdot g\boldsymbol{E} \right\} + \dots$$

[7] Brambilla et al., Nucl. Phys. B566 (2000) 275

Dealing with the mass renormalon of the potential

- The singlet potential is affected by an r-independent renormalon

$$E(r) = \Lambda_s + V_s(r,\nu,\mu_{us}) + \delta_{us}(r,\nu,\mu_{us}),$$

- May absorb renormalon into residual mass:

$$RS(\nu) = \Lambda_s - \Lambda_s^{\rm rs}(\nu), \quad V_s^{\rm rs}(r,\nu,\mu_{us}) = V_s(r,\nu,\mu_{us}) + \Lambda_s^{\rm rs}(\nu),$$
$$E(r) = RS(\nu) + V_s^{rs}(r,\nu,\mu_{us}) + \delta_{us}(r,\nu,\mu_{us}),$$

- Or, to avoid both large logs $\ln(r\nu)$ or an r-dependent renormalon term

$\Rightarrow$ compute the force, resum the logarithm via $\nu = 1/r$, and integrate[8]

$$F\left(r,\frac{1}{r}\right) = \left.\frac{\partial E(r,\nu)}{\partial r}\right|_{\nu=\frac{1}{r}}$$
$$E(r) = \int_{\bar{r}}^{r} dr' F\left(r',\frac{1}{r'}\right) + \text{const}$$

[8] Garcia i Tormo, MPLA 28 133028

Force at $\mathrm{N^3LO}$

- The force at $\mathrm{N^3LO}$ with $\nu = 1/r$

$$F\left(r, \nu = \frac{1}{r}\right) = \frac{C_F}{r^2}\alpha_s(1/r)\Bigg[1 + \frac{\alpha_s(1/r)}{4\pi}\left(\tilde{a}_1 - 2\beta_0\right) + \frac{\alpha_s^2(1/r)}{(4\pi)^2}\left(\tilde{a}_2 - 4\tilde{a}_1\beta_0 - 2\beta_1\right) + \frac{\alpha_s^3(1/r)}{(4\pi)^3}\left(\tilde{a}_3 - 6\tilde{a}_2\beta_0 - 4\tilde{a}_1\beta_1 - 2\beta_2 + a_3^L \ln\frac{C_A\alpha_s(1/r)}{2}\right) + \mathcal{O}(\alpha_s^4, \alpha_s^4\ln^2\alpha_s)\Bigg],$$

- The ultra-soft scale gives rise to the $\ln\alpha_s(1/r)$ term[9]

$$\ln\left(\frac{V_o - V_s}{\nu}\right) - \ln\left(r\nu\right) = \ln\left(\frac{C_A}{2}\,\alpha_s(1/r)\right) = \ln\left(r\mu_{us}\right)$$

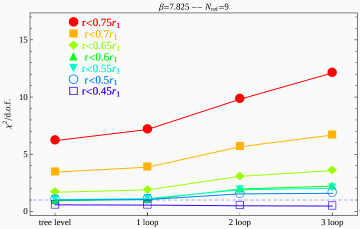
[9] Brambilla et al., Phys. Rev. D60 (1999) 091502

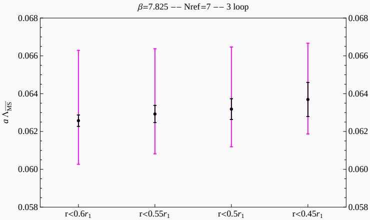
Leading ultra-soft resummation

$$
\begin{aligned}
F(r,\nu=1/r) = \frac{C_F}{r^2}\alpha_s(1/r)\Bigg[1
&+ \frac{\alpha_s(1/r)}{4\pi}\left(\tilde{a}_1 - 2\beta_0\right) \\
&+ \frac{\alpha_s^2(1/r)}{(4\pi)^2}\left(\tilde{a}_2 - 4\tilde{a}_1\beta_0 - 2\beta_1\right) \\
&- \frac{\alpha_s^2(1/r)}{(4\pi)^2}\frac{a_3^L}{2\beta_0}\ln\frac{\alpha_s(\mu_{us})}{\alpha_s(1/r)} \\
&+ \frac{\alpha_s^2(1/r)\alpha_s(\mu_{us})}{(4\pi)^3}a_3^L\ln\frac{C_A\alpha_s(1/r)}{2r\mu_{us}} \\
&+ \frac{\alpha_s^3(1/r)}{(4\pi)^3}\left(\tilde{a}_3 - 6\tilde{a}_2\beta_0 - 4\tilde{a}_1\beta_1 - 2\beta_2\right) + \mathcal{O}(\alpha_s^4)\Bigg].
\end{aligned}
$$

- Ultra-soft scale $\mu_{us} = \frac{C_A}{2}\,\frac{\alpha_s(1/r)}{r}$, recover F^{N^3LO} in limit $\mu_{us} \to 1/r$

Fitting lattice results of the static energy (2014)





Different perturbative orders

- χ^2/dof reduces for higher orders at shorter distances

$\Rightarrow$ Weak-coupling suitable for static energy for $r \lesssim 0.15\,\text{fm}$

- At shortest distances little sensitivity to perturbative order

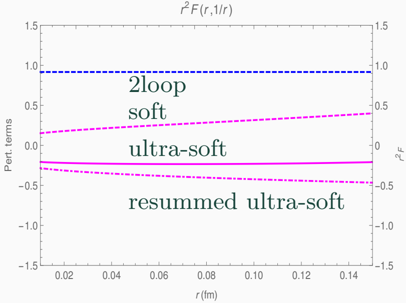
When going to shorter distances

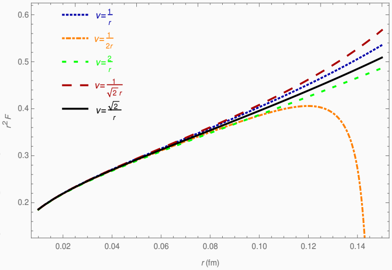
- Statistical errors increase
- Perturbative errors decrease

Perturbative errors estimated from

- scale variation $\nu = \frac{1}{\sqrt{2}r}$ to $\frac{\sqrt{2}}{r}$
- soft higher order term $\pm\# \frac{\alpha_s^4}{r}$

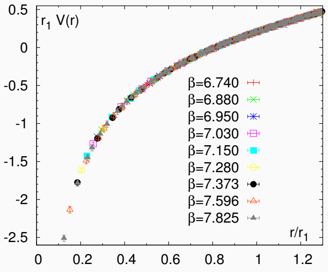
Perturbative uncertainty in the 2019 edition



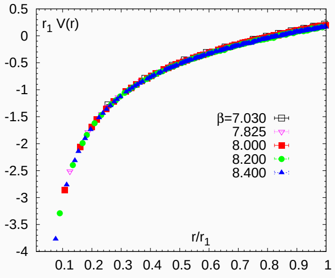


- Ultra-soft logs are small – use *three-loop with leading US resummation*
- Soft scale variation generates the dominant uncertainty at three loops
- More conservative soft scale variation in 2019 edition: $\nu = \frac{1}{2r}$ to $\frac{2}{r}$
- Nonmonotonic scale dependence is a common effect in EFTs, whenever the error is estimated from scale variation, is minimal for $\nu \approx 1/(\sqrt{2}r)$
- Variation of ultra-soft resummation included in the error budget

Static energy on the lattice: 2014 vs 2019



2014 edition[10], $a^{-1} \leq 4.9\,\mathrm{Gev}$



2019 edition[11], $a^{-1} \leq 7.9\,\mathrm{Gev}$

- Smallest distance $r = 0.04\,\mathrm{fm}$
- Perturbative errors dominant
- Very light pion $m_\pi = 160\,\mathrm{MeV}$
- Consistent with 2012 edition[12]

- Three extra fine lattice spacings at $T = 0$
- Include for shortest distances singlet free energies at $T > 0$ $\Rightarrow a^{-1} \lesssim 22\,\mathrm{GeV}$

[10] Bazavov et al., Phys. Rev. D90 (2014) 7, 074038
[11] Bazavov et al. [TUMQCD], arXiv:1907.11747
[12] Bazavov et al., Phys. Rev. D86 (2012) 114031

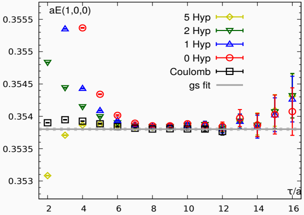
Wilson loops vs Wilson line correlators in Coulomb gauge

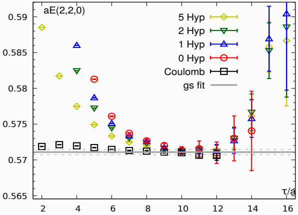
Wilson loops on the lattice

+ Explicit gauge invariance
- Cusp divergences due to corners
- Extra cusp divergences for off-axis separation
- Self-energy divergences due to spatial Wilson lines

Wilson line correlator on the lattice

- Must fix some gauge, i.e. Coulomb gauge
+ No corners, no cusps
+ On- and off-axis separation have same divergence
+ No spatial Wilson lines

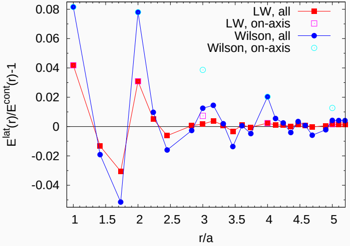


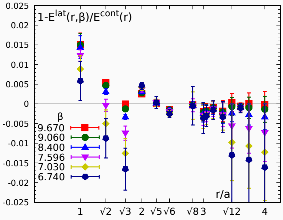


- Same ground state for both, but Wilson lines technically favorable
- Distortions at small distance and time for both operators

Lattice artifacts in the static quark-antiquark energy

- The static energy at short distances has percent-level lattice artifacts

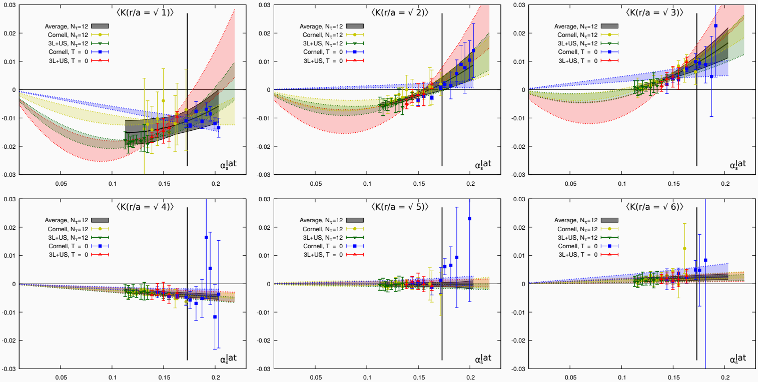




- Improved gauge action (Lüscher–Weisz) – reduced symmetry breaking
- Tree-level correction (TLC): $\frac{E^{\mathrm{lat}}(r)}{E^{\mathrm{cont}}(r)}$ for OGE without running coupling
- After tree-level correction – smaller, similar pattern of lattice artifacts[13]
- Impasse: only data with $r/a \geq \sqrt{8}$ omitting $r/a = \sqrt{12}$ are safe
- Way out: estimate the artifacts $\Rightarrow$ nonperturbative correction (NPC)

[13] Bazavov et al, Phys.Rev. D98 (2018) no.5, 054511

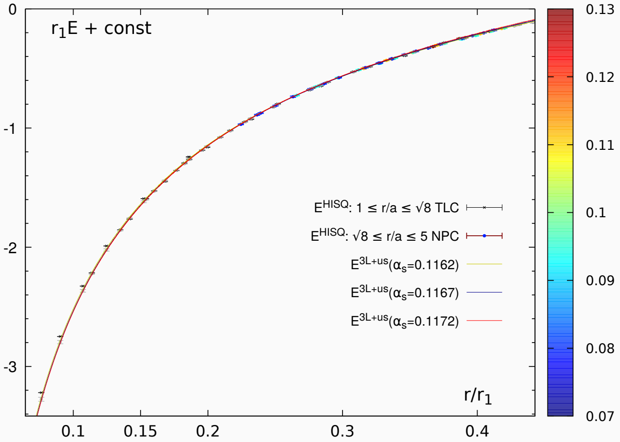
Nonperturbative correction of the static energy



- Estimate continuum static energy using fine lattice using $r/a \geq \sqrt{5}$, determine corrections for coarser lattices
- Estimate continuum static energy for $r/a \geq 1$ at N^3LO *with leading ultra-soft resummation*, marginalizing over Λ_{QCD} in some window

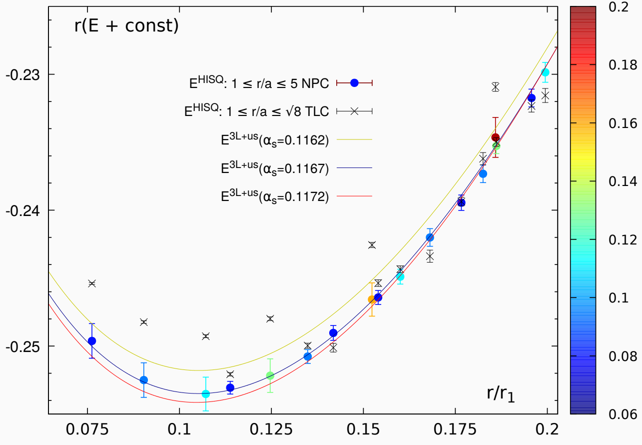
⇒ Extrapolate the corrections in boosted coupling α_s^{lat} to finer lattices

Impact of the lattice artifacts



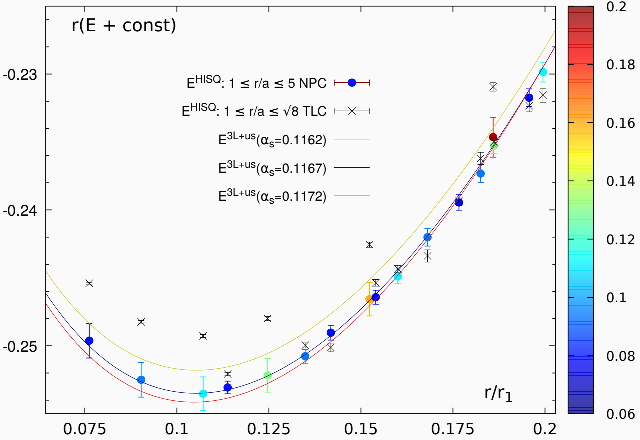
- Restrict lattice data to $r < 0.14\,\mathrm{fm} \approx 0.45 r_1$ (perturbative regime)
- Analyze TLC data at $r/a \geq \sqrt{8} \Rightarrow$ artifacts are statistically irrelevant

Impact of the lattice artifacts



- TLC data at $r/a < \sqrt{8}$ yield α_s smaller by up to 2σ at bad χ^2/d.o.f.
- NPC data at $r/a < \sqrt{8}$ are well-described by fit for $r/a \geq \sqrt{8}$

Impact of the lattice artifacts



- Combined analysis of lattice data with $a \leq 0.06$ fm, i.e., $a/r_1 \leq 0.2$
- Statistical and systematic errors reduced, while χ^2/d.o.f. is unchanged

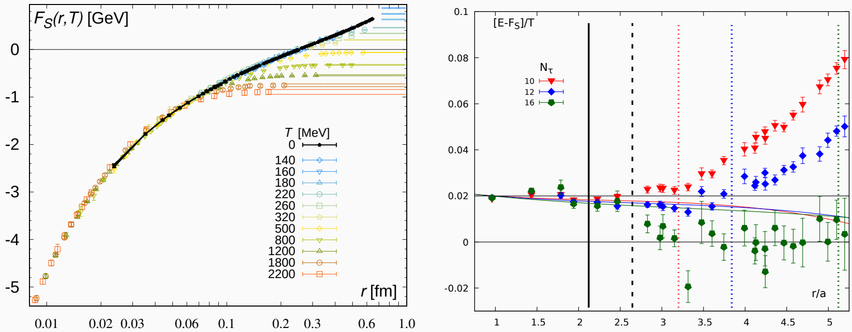
Systematic errors in the 2019 edition

$\min(r/a)$	$\max(r)$ fm	α_s^{3L}	$\delta^{\rm stat}$	$\delta^{\rm pert}_{2014}$	$\delta^{\rm pert}_{2019}$	α_s^{2L}
$\sqrt{8}$	0.097	0.1166	0.0007	$^{+0.0007}_{-0.0003}$	$^{+0.0016}_{-0.0005}$	0.1167
$\sqrt{8}$	0.131	0.1167	0.0005	$^{+0.0008}_{-0.0003}$	$^{+0.0019}_{-0.0006}$	0.1168
1	0.055	0.1164	0.0005	$^{+0.0003}_{-0.0001}$	$^{+0.0008}_{-0.0003}$	0.1164
1	0.073	0.1166	0.0004	$^{+0.0004}_{-0.0001}$	$^{+0.0010}_{-0.0003}$	0.1166
1	0.098	0.1167	0.0003	$^{+0.0005}_{-0.0002}$	$^{+0.0012}_{-0.0004}$	0.1167
1	0.131	0.1167	0.0003	$^{+0.0007}_{-0.0003}$	$^{+0.0015}_{-0.0005}$	0.1168

- Must keep $r \lesssim 0.1$ fm to enable the full soft scale variation
- For soft scale $1/(\sqrt{2}r) \le \nu \le \sqrt{2}/r$ stable against variation of $\max(r)$
- No leading ultra-soft resummation: α_s^{3L} lower by $\sim 70\%\ \delta^{\rm pert}$
- Include $r/a < \sqrt{8}$ to reduce all perturbative errors
- r_1 scale error: ± 1.7 MeV for $\Lambda_{\rm QCD}$, ± 0.0001 for $\alpha_s(M_Z, N_f = 5)$

$$\Lambda_{\rm QCD}^{N_f=3} = 314^{+16}_{-8}\,\text{MeV}, \qquad \alpha_s(M_Z, N_f = 5) = 0.11660^{+0.00110}_{-0.00056}$$

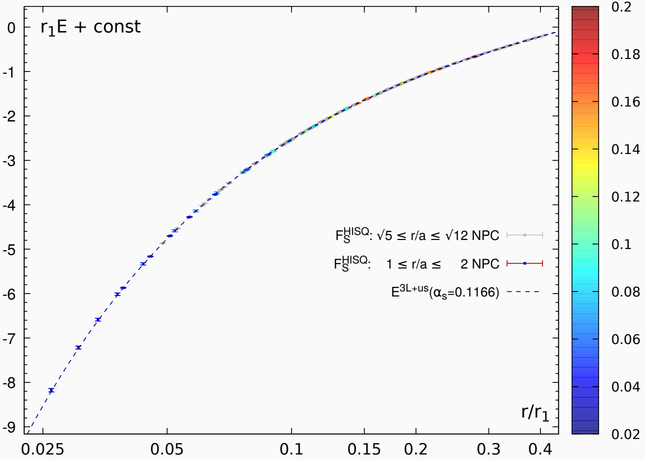
$T > 0$ data in the 2019 edition



- Singlet free energy for $T > 0$ with much finer lattice spacing[14] (no pion)
- F_S defined via Coulomb gauge thermal Wilson line correlator: $\tau = 1/T$
- $T > 0$ effects exponentially suppressed for $\alpha_s/r \gg T$, i.e., $r/a \ll \alpha_s N_\tau$
- Nonconstant $T > 0$ effects are numerically small for $r/a \lesssim 0.3 N_\tau$ due to compensation between static gluons vs nonstatic gluons and quarks

[14] Bazavov et al, Phys.Rev. D98 (2018) no.5, 054511

α_s from $T > 0$



- Restrict $N_\tau = 12$ data to $r/a \leq 2$ or 3, i.e., $r \leq 0.17/T$ or $0.25/T$
- Cannot avoid the nonperturbative correction for the lattice artifacts
- Restrict to tiny distances $r \leq 0.03\,\mathrm{fm}$ to reduce the perturbative error

$T = 0$ vs $T > 0$

N_τ	max(r/a)	max(r) fm	α_s^{3L}	$\delta^{\rm stat}$	$\delta^{\rm pert}_{2014}$	$\delta^{\rm pert}_{2019}$	α_s^{2L}
64	2	0.057	0.1165	0.0006	$^{+0.0003}_{-0.0001}$	$^{+0.0008}_{-0.0003}$	0.1164
64	2	0.078	0.1166	0.0005	$^{+0.0004}_{-0.0001}$	$^{+0.0010}_{-0.0003}$	0.1166
64	2	0.096	0.1166	0.0005	$^{+0.0004}_{-0.0002}$	$^{+0.0011}_{-0.0003}$	0.1166
12	2	0.057	0.1165	0.0007	$^{+0.0002}_{-0.0001}$	$^{+0.0006}_{-0.0002}$	0.1164
12	2	0.078	0.1166	0.0006	$^{+0.0003}_{-0.0001}$	$^{+0.0008}_{-0.0003}$	0.1166
12	2	0.091	0.1167	0.0006	$^{+0.0003}_{-0.0001}$	$^{+0.0009}_{-0.0003}$	0.1167
64	3	0.055	0.1164	0.0005	$^{+0.0003}_{-0.0001}$	$^{+0.0008}_{-0.0003}$	0.1164
64	3	0.073	0.1166	0.0004	$^{+0.0004}_{-0.0001}$	$^{+0.0010}_{-0.0003}$	0.1166
64	3	0.096	0.1167	0.0004	$^{+0.0005}_{-0.0002}$	$^{+0.0012}_{-0.0004}$	0.1167
64	3	0.134	0.1167	0.0003	$^{+0.0006}_{-0.0003}$	$^{+0.0014}_{-0.0005}$	0.1168
12	3	0.055	0.1167	0.0005	$^{+0.0002}_{-0.0001}$	$^{+0.0006}_{-0.0002}$	0.1167
12	3	0.073	0.1168	0.0005	$^{+0.0003}_{-0.0001}$	$^{+0.0008}_{-0.0003}$	0.1168
12	3	0.096	0.1168	0.0005	$^{+0.0003}_{-0.0001}$	$^{+0.0009}_{-0.0003}$	0.1168
12	3	0.133	0.1168	0.0004	$^{+0.0005}_{-0.0002}$	$^{+0.0012}_{-0.0004}$	0.1169

Complete agreement between α_s from $T = 0$ or $T > 0$

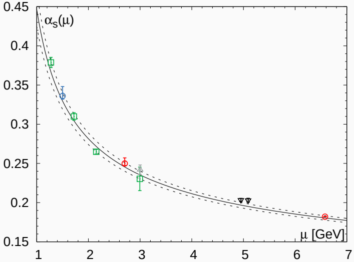
Summary

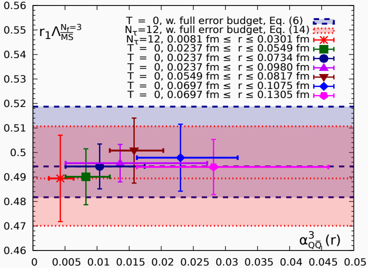
- We determine the strong coupling constant α_s from the static energy using 6 lattice spacings with more conservative perturbative errors and from the singlet free energy using 15 lattice spacings (and two N_τ)

Static energy	2014	2019
$\alpha_s(m_Z, N_f = 5)$	$0.1166^{+0.0012}_{-0.0008}$	$0.11660^{+0.00110}_{-0.00056}$
$\Lambda_{\mathrm{QCD}}(N_f = 3)$	315^{+18}_{-12} MeV	314^{+16}_{-08} MeV
Soft scale	$\mu = 1/\max(r) \gtrsim 2/r_1$	$\mu = 1/\max(r) \gtrsim 4/r_1$

Singlet free energy	past	2019
$\alpha_s(m_Z, N_f = 5)$	NA	$0.11638^{+0.00095}_{-0.00087}$
$\Lambda_{\mathrm{QCD}}(N_f = 3)$	NA	311^{+14}_{-12} MeV
Soft scale	NA	$\mu = 1/\max(r) \gtrsim 10/r_1$

Running of α_s at low scales





- 2014 HPQCD quarkonium correlators[15]
- 2019 quarkonium correlators[16]
- 2014 TUMQCD static energy[17]
- 2019 static energy and singlet free energy[18]

[15] Chakraborty et al., Phys.Rev. D91 (2015) no.5, 054508
McNeile et al., Phys.Rev. D82 (2010) 034512
Allison et al., Phys.Rev. D78 (2008) 054513
[16] PP, JHW: Phys.Rev. D100 (2019) 3, 034519
[17] Bazavov et al., Phys. Rev. D90 (2014) 7, 074038
[18] Bazavov et al., arXiv:1907.11747

Thank you!

Coefficients of the force – color factors and beta function

- Color factors: $C_F = \frac{N_c^2 - 1}{2N_c}$, $C_A = N_c$, $T_F = \frac{1}{2}$
- Beta function:

$$\frac{d\,\alpha_s(\nu)}{d\ln\nu} = \alpha_s\beta(\alpha_s) = -\frac{\alpha_s^2}{2\pi}\sum_{n=0}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n\beta_n = -2\alpha_s\left[\beta_0\frac{\alpha_s}{4\pi} + \beta_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots\right]$$

- Relevant coefficients explicitly contributing to the force:

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F N_f,$$

$$\beta_1 = \frac{34}{3}C_A^2 - \frac{20}{3}C_A N_f T_F - 4C_F N_f T_F,$$

$$\beta_2 = \frac{2857}{54}C_A^3 - \left(\frac{1415}{27}C_A^2 + \frac{205}{9}C_A C_F - 2C_F^2\right)N_f T_F + \left(\frac{158}{27}C_A + \frac{44}{9}C_F\right)N_f^2 T_F^2$$

Coefficients of the force – other coefficients (I)

- Coefficients $\tilde{a}_i$:

$$\tilde{a}_1 = a_1 + 2\gamma_E \beta_0,$$

$$\tilde{a}_2 = a_2 + \left(\frac{\pi^2}{3} + 4\gamma_E^2\right) \beta_0^2 + \gamma_E \left(4 a_1 \beta_0 + 2\beta_1\right),$$

$$\tilde{a}_3 = a_3 + \left(8\gamma_E^3 + 2\gamma_E \pi^2 + 16\zeta(3)\right)\beta_0^3 + 2\gamma_E \beta_2$$

$$+ \left[\left(12\gamma_E^2 + \pi^2\right) \beta_0^2 + 4\gamma_E \beta_1\right] a_1 + \left[6 a_2 \gamma_E + \frac{5}{2}\left(4\gamma_E^2 + \frac{\pi^2}{3}\right)\beta_1\right]\beta_0$$

- Coefficients a_i:

$$a_1 = \frac{31}{9} C_A - \frac{20}{9} T_F N_f,$$

$$a_2 = \left(\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3)\right) C_A^2 - \left(\frac{1798}{81} + \frac{56}{3}\zeta(3)\right) C_A T_F N_f$$

$$- \left(\frac{55}{3} - 16\zeta(3)\right) C_F T_F N_f + \left(\frac{20}{9} T_F N_f\right)^2$$

Coefficients of the force – other coefficients (II)

- Coefficient a_3:

$$a_3 = a_3^{(3)} N_f^3 + a_3^{(2)} N_f^2 + a_3^{(1)} N_f + a_3^{(0)},$$

$$a_3^{(3)} = -\left(\frac{20}{9}\right)^3 T_F^3,$$

$$a_3^{(2)} = \left(\frac{12541}{243} + \frac{368}{3}\zeta(3) + \frac{64\pi^4}{135}\right) C_A T_F^2 + \left(\frac{14002}{81} - \frac{416}{3}\zeta(3)\right) C_F T_F^2,$$

$$a_3^{(1)} = (-709.717)\, C_A^2 T_F + \left(-\frac{71281}{162} + 264\zeta(3) + 80\zeta(5)\right) C_A C_F T_F$$

$$+ \left(\frac{286}{9} + \frac{296}{3}\zeta(3) - 160\zeta(5)\right) C_F^2 T_F + (-56.83(1))\, \frac{18 - 6N_c^2 + N_c^4}{96 N_c^2},$$

$$a_3^{(0)} = 502.24(1)\, C_A^3 - 136.39(12)\, \frac{N_c^3 + 6N_c}{48} + \frac{8}{3}\pi^2 C_A^3 \left(-\frac{5}{3} + 2\gamma_E + 2\log 2\right)$$

- Coefficient a_3^L: $a_3^L = \frac{16\pi^2}{3} C_A^3$

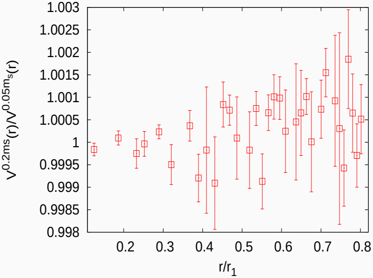
Quark mass dependence and topology

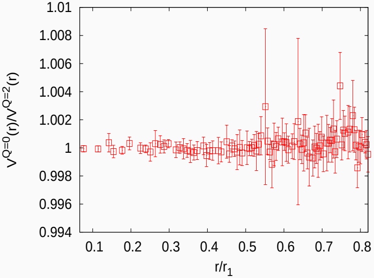
- Combine gauge ensembles with different light sea quark mass

$\Rightarrow$ No statistically significant quark mass effects up to $r \approx 0.5 r_1$

- Fine gauge ensembles with fully suppressed topological tunneling

$\Rightarrow$ No statistically significant difference between static energy in different topological sectors up to $r \approx 0.5 r_1$ observed[19]





[19] Bazavov et al., arXiv:1811.12902