Higher-Point Functions with Hexagonalization

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Outline

- Introduction and Motivations.
- Review of Hexagonalization and Two-Particle contributions.
- The constraints and the decagon.
- The higher-point correlation functions.
- Conclusions.

Introduction and Motivations

- The octagon is a big success!

  (Coronado)
  (Kostov, Petkova, Serban)
  (Bargheer, Coronado, Vieira)
  (Belitsky, Korchemsky)

  (see Korchemsky's talk)

- Possible next steps are to deform the octagon and compute the decagon.

- At the moment is very difficult to get finite g results with Hexagonalization in general. New resummation techniques.

- Look for new structures and all loop relations.
  Non-planar integrability.

- New way of computing Feynman integrals.

- At one-loop, any n-point function of Half-BPS operators is in principle known. (Drukker, Plefka)

- Very few results for higher loops and strong coupling. (Eden, Korchemsky, Sokatchev) (Gonçalves, Pereira, Zhou)

- Study several interesting limits as the light-like limit, take OPE's, …

Very Short Review

(Basso, Komatsu, Vieira)
(TF, Komatsu)
(Eden, Sfrondini)
(Bargheer, Caetano, TF, Komatsu, Vieira)
(Eden, Jiang, le Plat, Sfrondini)

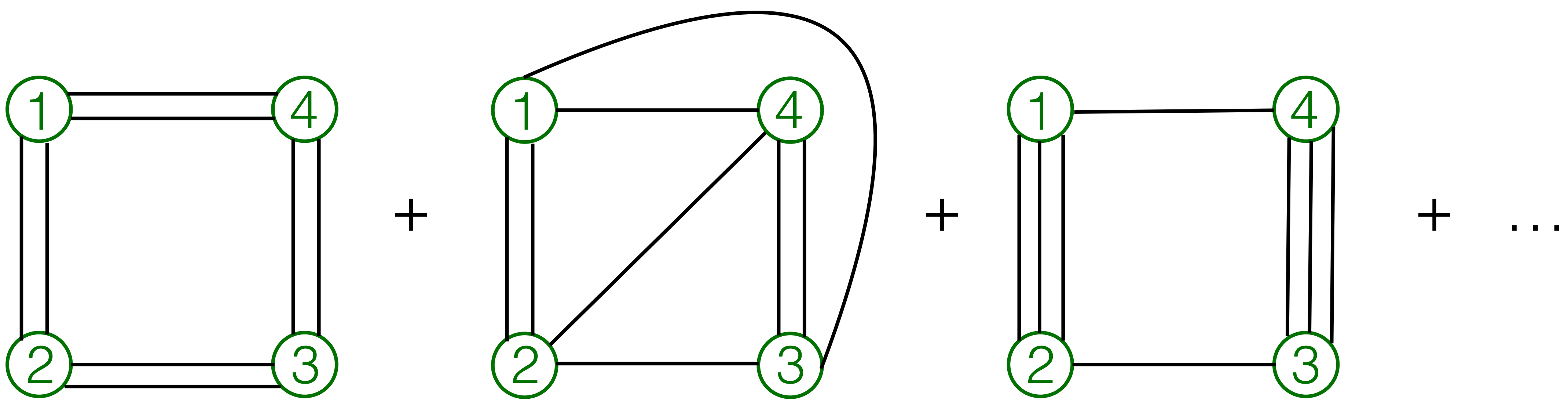
Half-BPS operators:

$$\mathcal{O}_L(x) = \mathrm{Tr}\left((y \cdot \Phi(x))^L\right),$$

with $y \cdot y = 0$ and $\Phi(x)^I$ six scalars

Step 1:

Draw all tree-level graphs



Propagator: $d_{ij} = \frac{y_{ij}}{x_{ij}}$

Bridge length (homotopically equivalent): l_{ij}

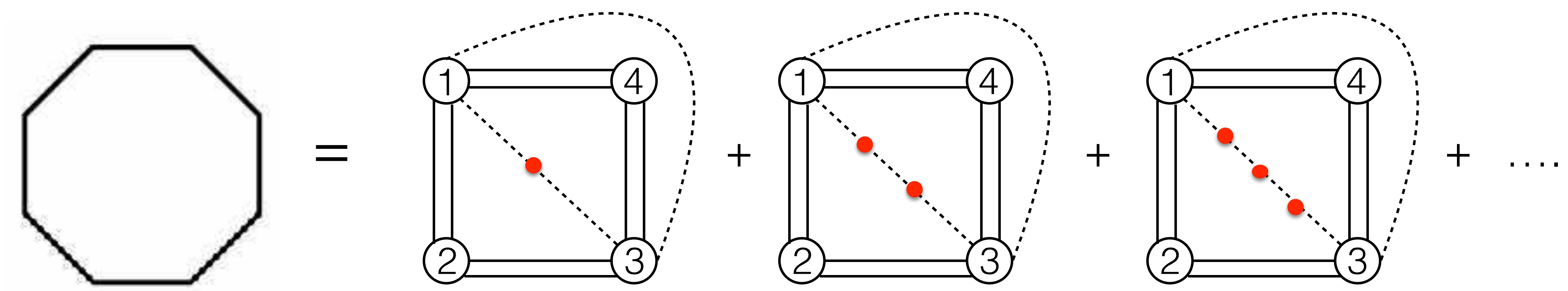
Step 2:

Divide the surfaces into hexagons.

Promote each hexagonal patch to a hexagon form-factor.

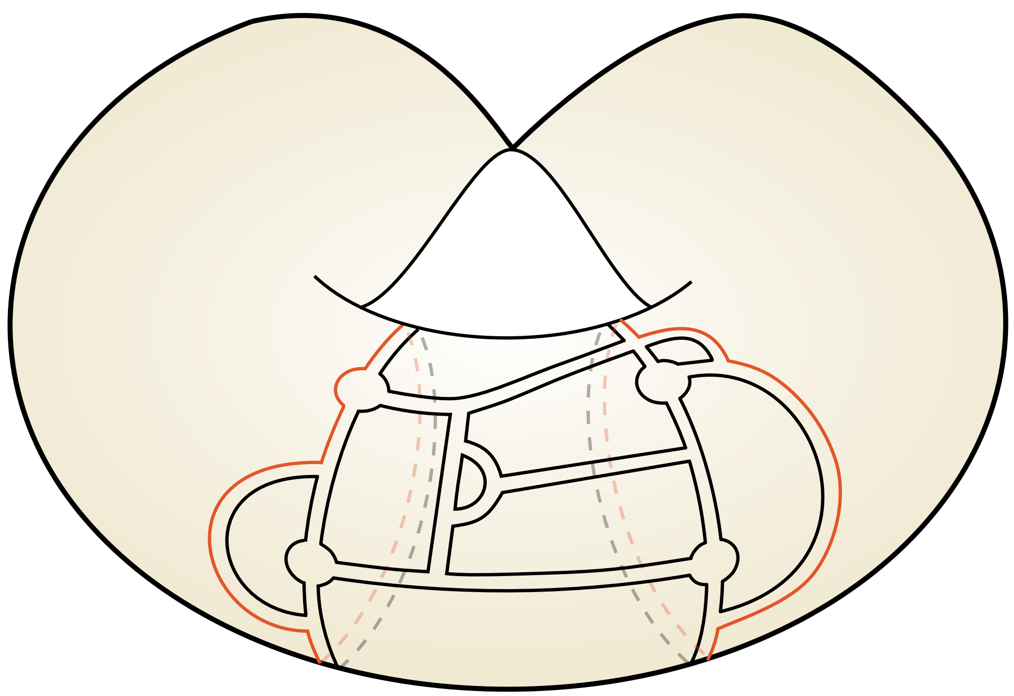
Cut the surface by inserting a weighted complete set of states.

The Octagon



Step 3:

Boundaries of the moduli space; Stratification $\mathcal{S}$



At its boundary, a torus degenerates into a sphere.

Final Proposal:

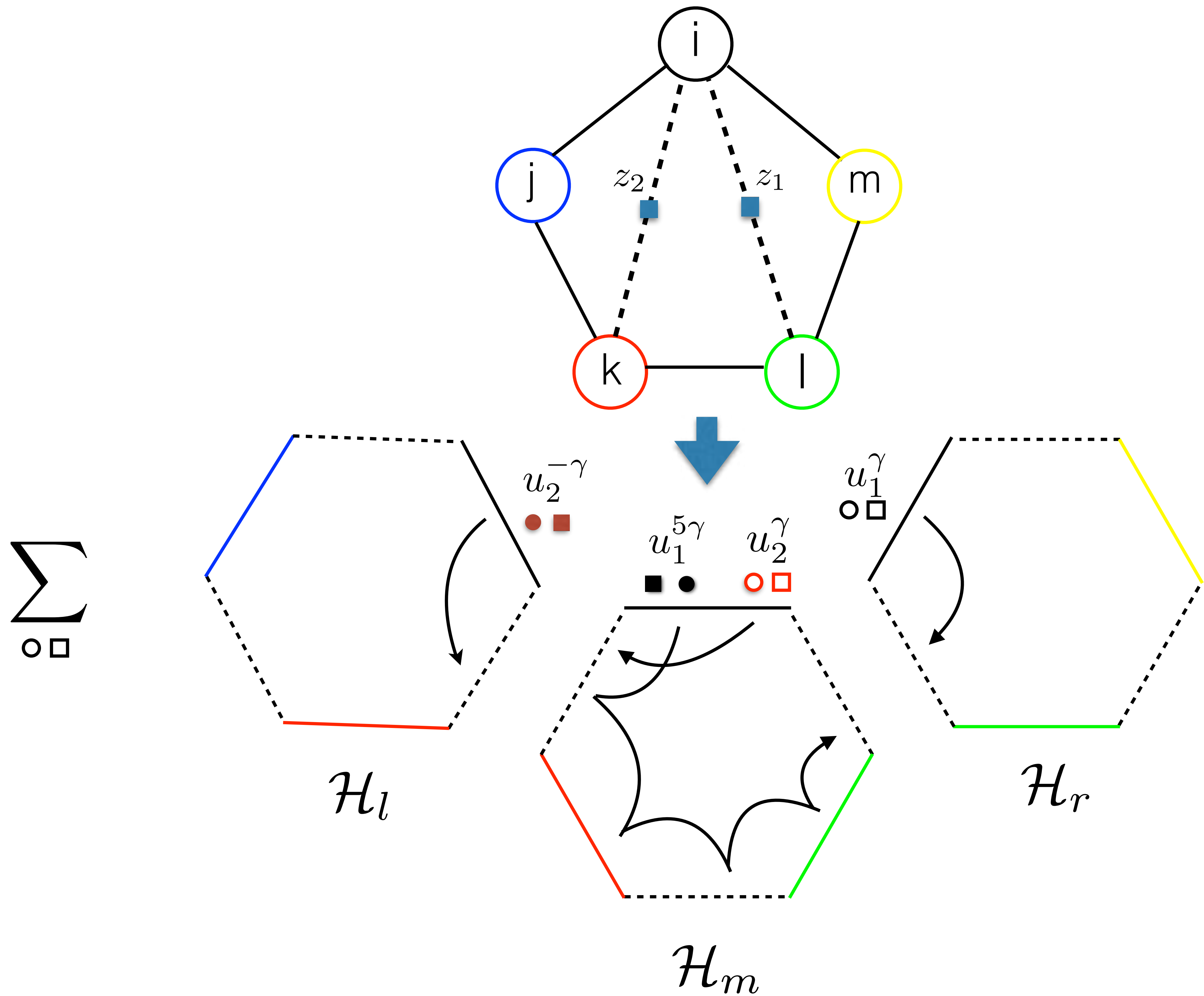
$$\langle \mathcal{O}_{L_1}(x_1)\dots\mathcal{O}_{L_n}(x_n)\rangle =$$

$$\mathcal{S}\cdot\left(\sum_{\substack{\text{tree}-\text{level}\\ \text{graphs}}}\prod_{l_{ij}}(d_{ij})^{l_{ij}}\right)\left(\sum_{i,j,k}\prod^{\text{hexagons}}\mathcal{W}_{i,j,k}\,\mathcal{H}_{\psi_{ij},\psi_{jk},\psi_{ki}}\right)$$

The Two-Particle Contribution

(TF, Komatsu)

(de Leeuw, Eden, le Plat, Meier, Sfondrini)



Technical Remark: Z-markers

$$\text{dressing} + \, : \, \psi_\alpha \to \psi_\alpha \,, \; \psi_{\dot\alpha} \to \psi_{\dot\alpha} \,, \; \{\phi_1 \,, \phi_{\dot 2}\} \to Z^{\frac{1}{2}}\{\phi_1 \,, \phi_{\dot 2}\} \,, \; \{\phi_{\dot 1} \,, \phi_2\} \to Z^{-\frac{1}{2}}\{\phi_{\dot 1} \,, \phi_2\} \,.$$

$\left(\phi_1, \phi_2, \psi_1, \psi_2\right)$ form an $\mathfrak{su}(2|2)$ fundamental multiplet

Comments:

- The dependence on the cross-ratios comes from the weight-factors

$$\mathcal{W}^{\pm}_{\{a,I\}}(u_i^{\gamma}) = e^{-2i\tilde{p}_a(u_i)\log|z_i|}\, e^{iL\phi_i}\, e^{iR(\theta_i \pm \varphi_i)}\,, \qquad e^{i\phi_i} = \sqrt{\frac{z_i}{\bar{z}_i}}\,, \quad e^{i\theta_i} = \sqrt{\frac{\alpha_i}{\bar{\alpha}_i}}\,, \quad e^{i\varphi} = \sqrt{\frac{\alpha_i\bar{\alpha}_i}{z_i\bar{z}_i}}\,.$$

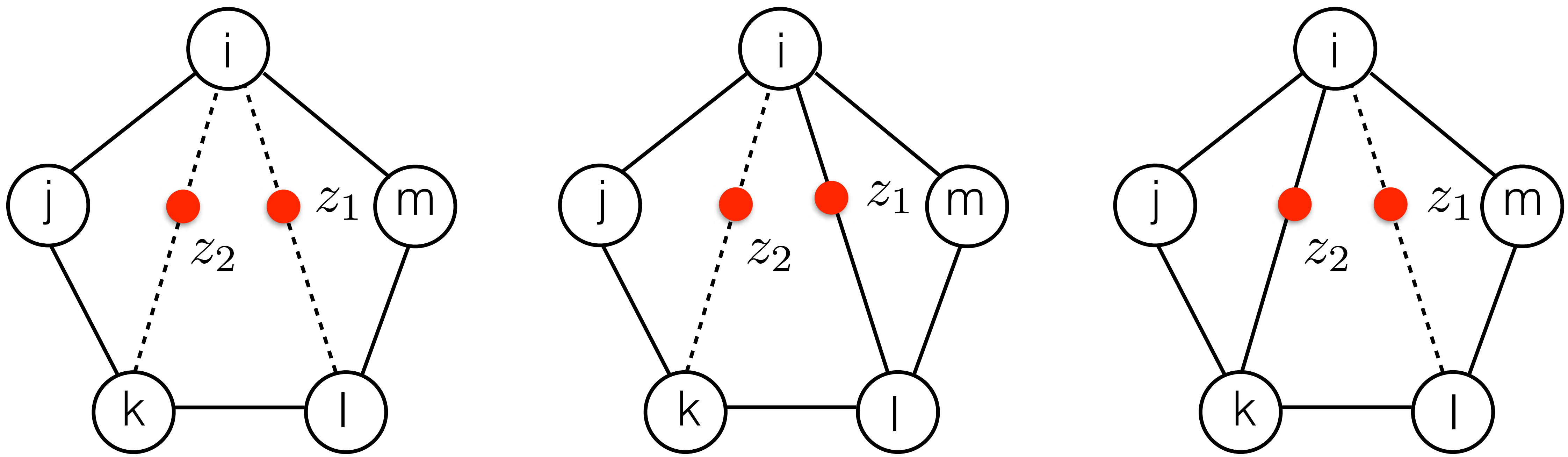
with $L = \frac{1}{2}(L^1{}_1 - L^2{}_2 - L^{\dot{1}}{}_{\dot{1}} + L^{\dot{2}}{}_{\dot{2}})\,, \quad R = \frac{1}{2}(R^1{}_1 - R^2{}_2 - R^{\dot{1}}{}_{\dot{1}} + R^{\dot{2}}{}_{\dot{2}})\,.$

- It is easy to generate power series with integrability. Then one fits against a basis of integrals.

Types of two-particles at two-loops

(TF, Gonçalves)

$$\mathcal{M}^{(2)}_{2,\{0,0\}}(z_1,z_2)\,,\quad \mathcal{M}^{(2)}_{2,\{1,0\}}(z_1,z_2)\,,\quad \mathcal{M}^{(2)}_{2,\{0,1\}}(z_1,z_2)\,.$$



$$z_1\bar{z}_1=\frac{x_{im}^2x_{kl}^2}{x_{ik}^2x_{ml}^2}\,,\quad (1-z_1)(1-\bar{z}_1)=\frac{x_{il}^2x_{km}^2}{x_{ik}^2x_{lm}^2}\,,\quad z_2\bar{z}_2=\frac{x_{il}^2x_{jk}^2}{x_{ij}^2x_{lk}^2}\,,\quad (1-z_2)(1-\bar{z}_2)=\frac{x_{ik}^2x_{jl}^2}{x_{ij}^2x_{kl}^2}\,.$$

$$\alpha_1\bar{\alpha_1}=\frac{y_{im}y_{kl}}{y_{ik}y_{ml}}\,,\quad (1-\alpha_1)(1-\bar{\alpha_1})=\frac{y_{il}y_{km}}{y_{ik}y_{lm}}\,,\quad \alpha_2\bar{\alpha_2}=\frac{y_{il}y_{jk}}{y_{ij}y_{lk}}\,,\quad (1-\alpha_2)(1-\bar{\alpha_2})=\frac{y_{ik}y_{jl}}{y_{ij}y_{kl}}.$$

Results for: $\mathcal{M}^{(2)}_{2,\{1,0\}}(z_1,z_2)\,,\quad \mathcal{M}^{(2)}_{2,\{0,1\}}(z_1,z_2)\,.$

$$\mathcal{M}^{(2)}_{2,\{a,b\}}(z_1,z_2) = -f(z_1)K^1_{\{a,b\}}(z_1,z_2) - f(z_2^{-1})K^2_{\{a,b\}}(z_1,z_2) + f\left(\frac{z_1-1}{z_1 z_2}\right)K^3_{\{a,b\}}(z_1,z_2)$$
$$+ f\left(\frac{1-z_1+z_1 z_2}{z_2}\right)K^4_{\{a,b\}}(z_1,z_2) + f\left(z_1(1-z_2)\right)K^5_{\{a,b\}}(z_1,z_2).$$

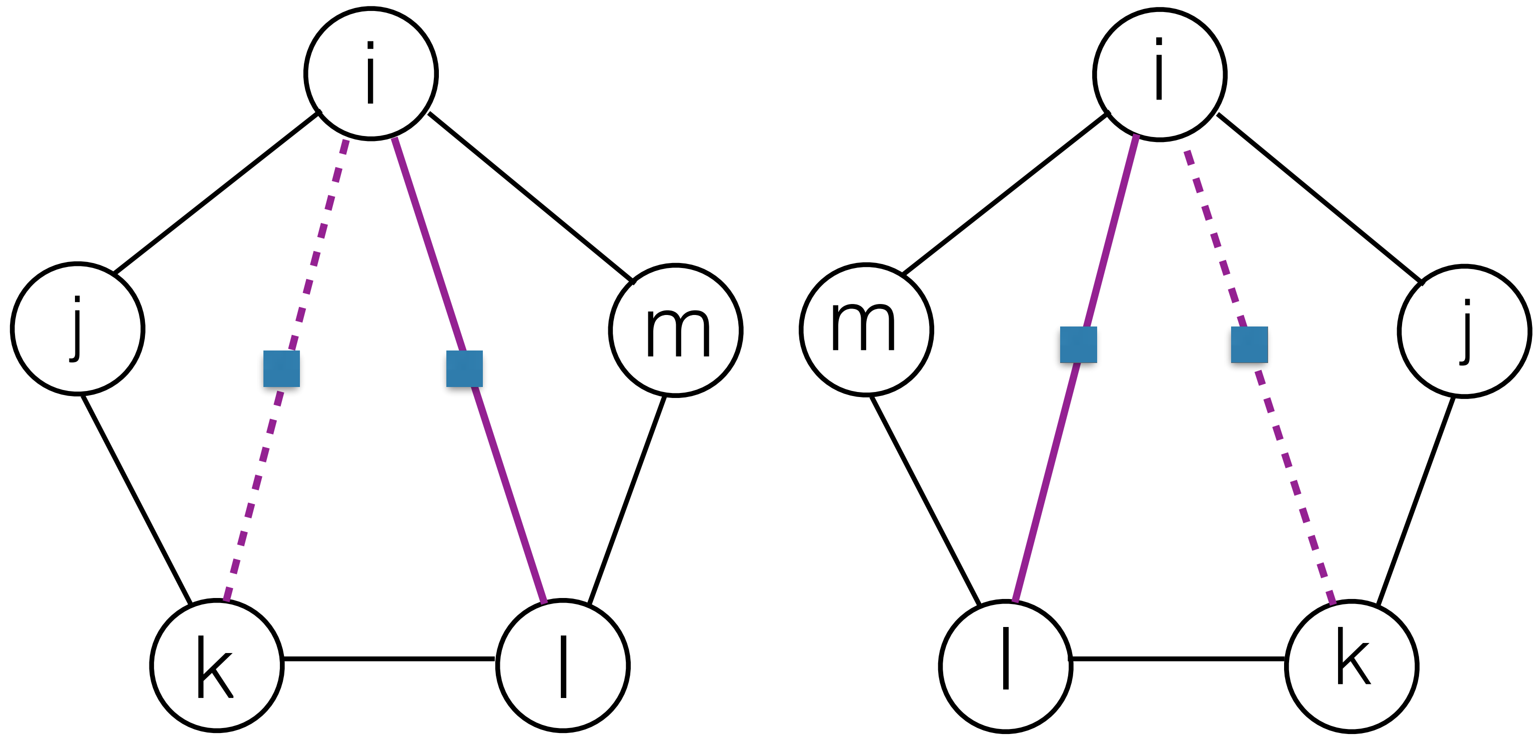
where

$$f(z) = g^4\,\frac{(z+\bar{z})-(\alpha+\bar{\alpha})}{2}$$

and

$$K^1_{\{1,0\}}(z_1,z_2) = F^{(2)}(z_1)\,,\quad K^2_{\{1,0\}}(z_1,z_2) = (x^2_{il})^2 x^2_{kj} P_{m,il,jk}\,,$$
$$K^3_{\{1,0\}}(z_1,z_2) = x^2_{il}x^2_{im}x^2_{jk}L_{i,jk,lm}\,,\quad K^4_{\{1,0\}}(z_1,z_2) = x^2_{il}x^2_{jk}x^2_{lm}L_{l,jk,im}\,,$$
$$K^5_{\{1,0\}}(z_1,z_2) = F^{(2)}(z_1(1-z_2))\,.$$

Parity Invariance



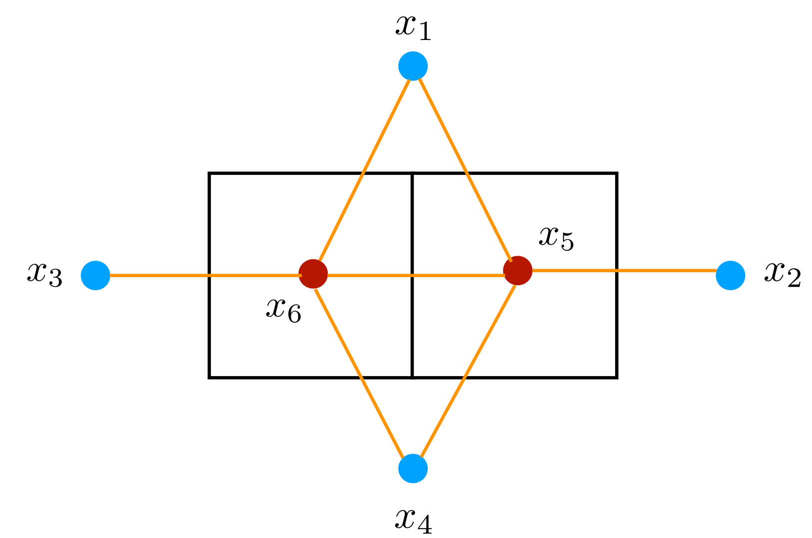
The Figure implies: $$\mathcal{M}^{(L)}_{2;\{1,0\}}(z_1, z_2) = \mathcal{M}^{(L)}_{2;\{0,1\}}(z_2^{-1}, z_1^{-1})\,.$$

Similarly: $$\mathcal{M}^{(L)}_{2,\{0,0\}}(z_1, z_2) = \mathcal{M}^{(L)}_{2,\{0,0\}}(z_2^{-1}, z_1^{-1})\,.$$

The Integrals

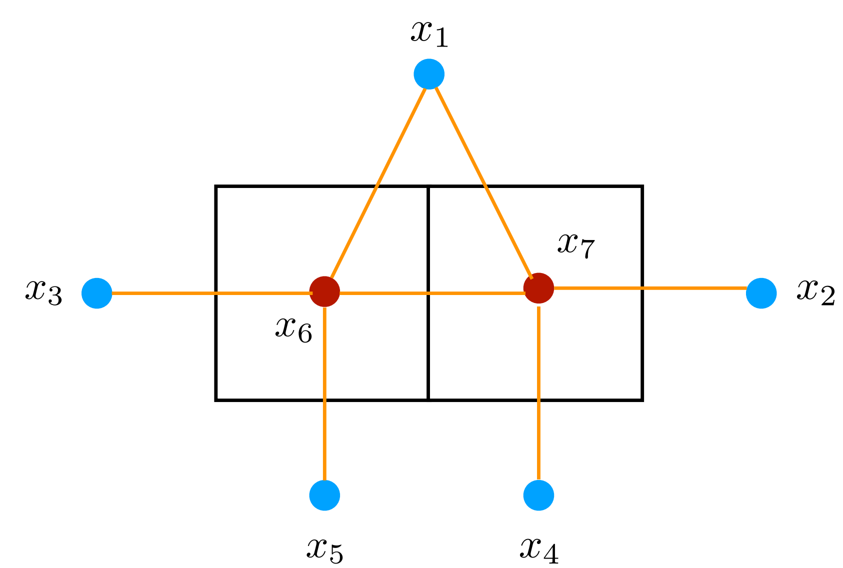
- **Ladder:**

$$F^{(2)}(z,\bar{z}) = \frac{x_{14}^2 x_{13}^2 x_{24}^2}{(\pi^2)^2} \int \frac{d^4x_5 d^4x_6}{x_{15}^2 x_{25}^2 x_{45}^2 x_{56}^2 x_{16}^2 x_{36}^2 x_{46}^2} \,,$$



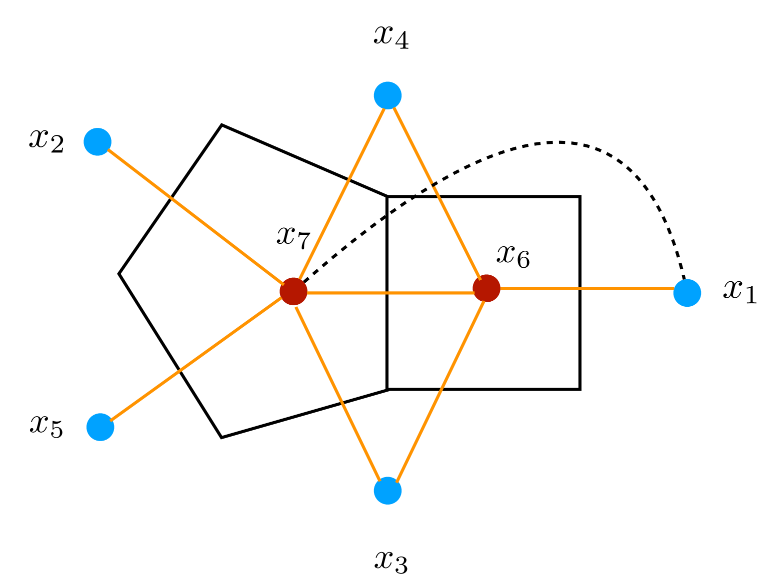
- **Double Box:**

$$L_{1,24,35} = \frac{1}{(\pi^2)^2} \int \frac{d^4x_6 d^4x_7}{x_{16}^2 x_{36}^2 x_{56}^2 x_{67}^2 x_{17}^2 x_{27}^2 x_{47}^2} \,,$$



- **PentaLadder:**

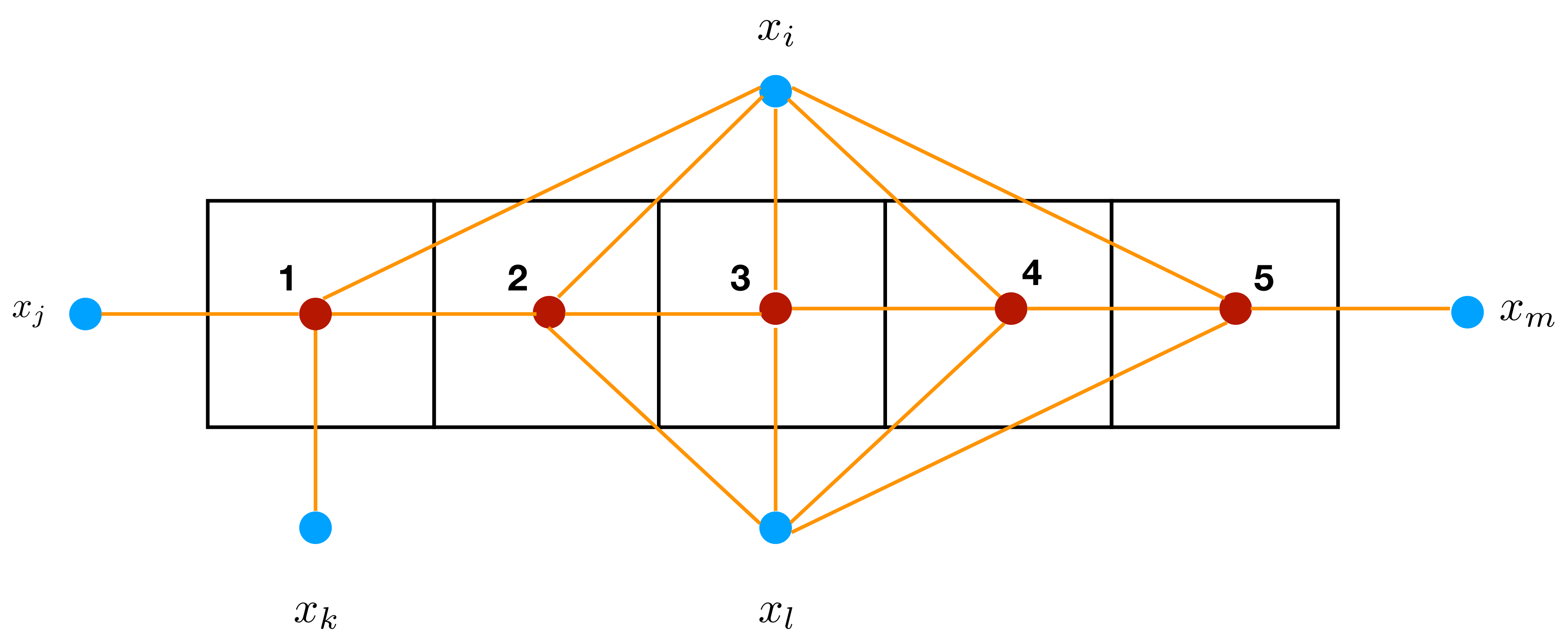
$$P_{1,34,25} = \frac{1}{(\pi^2)^2} \int \frac{d^4x_6 d^4x_7 x_{17}^2}{(x_{16}^2 x_{36}^2 x_{46}^2) x_{67}^2 (x_{27}^2 x_{37}^2 x_{47}^2 x_{57}^2)} \,,$$



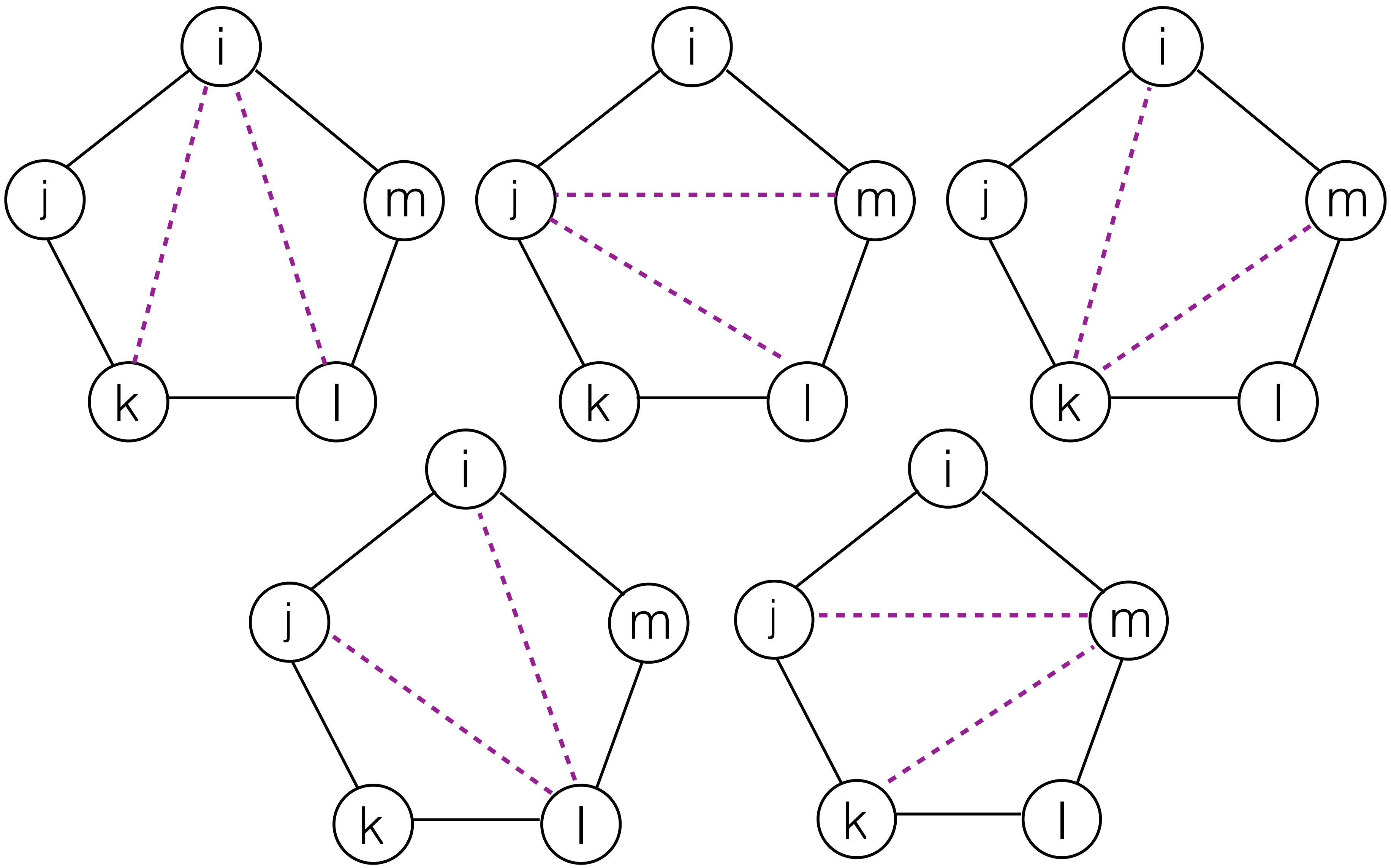
The Components at l-loops $(a+b=l-1)$

Component: $\alpha_1\alpha_2$ Bridges Length: $\{0,l\}$

$$L'_{i,jk,m,l}=\int\prod_{i=1}^{L}d^4x_i\frac{1}{x_{i1}x_{j1}x_{k1}x_{12}x_{iL}x_{mL}x_{lL}}\prod_{p=2}^{L-1}\frac{1}{x_{ip}x_{lp}x_{p(p+1)}}$$



Flip Invariance:



Interesting combination to study:

$$P_{00} \equiv \mathcal{M}^{(2)}_{1,\{0\}}(z_1) + \mathcal{M}^{(2)}_{1,\{0\}}(z_2) + \mathcal{M}^{(2)}_{2;\{0,0\}}(z_1, z_2)$$

Results

$$P_{00} = h^{(2)}(z_1, z_2) + h^{(2)}\left(\frac{z_2}{1+z_1(z_2-1)}, \frac{z_1-1}{z_1 z_2}\right) + h^{(2)}\left(\frac{1}{z_2}, z_1(1-z_2)\right)$$
$$+ h^{(2)}\left(\frac{z_1 z_2}{z_1-1}, \frac{1}{z_1}\right) + h^{(2)}\left(\frac{1}{z_1(1-z_2)}, \frac{1+z_1(z_2-1)}{z_2}\right),$$

One-loop:

$$h^{(1)}(z_1, z_2) = f(z_2) = \frac{g^2}{2}(z_2 + \bar{z}_2 - (\alpha_2 + \bar{\alpha_2}))F^{(1)}(z_2)\,.$$

Two Loop:

$$h^{(2)}(z_1, z_2) = \frac{1}{2}(\alpha_1(1-\alpha_2) - z_1(1-z_2) + \textbf{c.c.})\left[x_{ij}^2 x_{jl}^2 x_{lm}^2 P_{k,lj,m,i}\right.$$
$$\left.-x_{ij}^2 x_{lm}^2(L_{j,mi,lk}x_{jk}^2 + L_{l,mi,jk}x_{lk}^2) + 2F^{(2)}(z_1(1-z_2))\right].$$

Using Z-markers nicely

$$\mathcal{M}^L_{\{1,l_1\},\{1,l_2\}} = \alpha_1\alpha_2\, m^L_{\{l_1,l_2\},\{\alpha_1\,\alpha_2\}} + \frac{1}{\alpha_1\alpha_2}\, m^L_{\{l_1,l_2\},\{1/\alpha_1\,\alpha_2\}} + \alpha_1\, m^L_{\{l_1,l_2\},\{\alpha_1\}}$$
$$+ \frac{1}{\alpha_2}\, m^L_{\{l_1,l_2\},\{1/\alpha_2\}} + \frac{\alpha_1}{\alpha_2}\, m^L_{\{l_1,l_2\},\{\alpha_1/\alpha_2\}} + \frac{1}{\alpha_1}\, m^L_{\{l_1,l_2\},\{1/\alpha_1\}}$$
$$+ \alpha_2\, m^L_{\{l_1,l_2\},\{\alpha_2\}} + \frac{\alpha_2}{\alpha_1}\, m^L_{\{l_1,l_2\},\{\alpha_2/\alpha_1\}} + m^L_{\{l_1,l_2\}}\,,$$

All loop relations:

- $m^L_{\{l_1,l_2\},\{1/\alpha_1\}} = z_1\bar{z}_1 m^L_{\{l_1,l_2+1\},\{\alpha_1\}}$
- $m^L_{\{l_1,l_2\},\{\alpha_2\}} = \frac{1}{z_2\bar{z}_2} m^L_{\{l_1+1,l_2\},\{1/\alpha_2\}}$
- $m^L_{\{l_1+1,l_2\},\{1/\alpha_1\alpha_2\}} = z_1\bar{z}_1 z_2\bar{z}_2 m^L_{\{l_1,l_2+1\},\{\alpha_1\alpha_2\}}$
- $m^L_{\{l_1,l_2\},\{\alpha_2/\alpha_1\}} = \frac{z_1\bar{z}_1}{z_2\bar{z}_2} m^L_{\{l_1+1,l_2+1\},\{\alpha_1/\alpha_2\}}$

Example:

$$m^2_{\{0,0\},\{1/\alpha_1\}} = z_1\bar{z}_1 m^2_{\{0,1\},\{\alpha_1\}}\,.$$

Derivation:

The prescription for the Z-markers is to average between two dressings: + and -

To go from one dressing to the other, one flips the powers of all the Z-markers

This amounts to:

$$\varphi_i \to -\varphi_i \quad , \quad p(u_i^\gamma) \to -p(u_i^\gamma)$$

The two particle integrand: $\cos(\theta_1)\cos(\varphi_1 - \frac{p(u_2^\gamma)}{2})$ .....

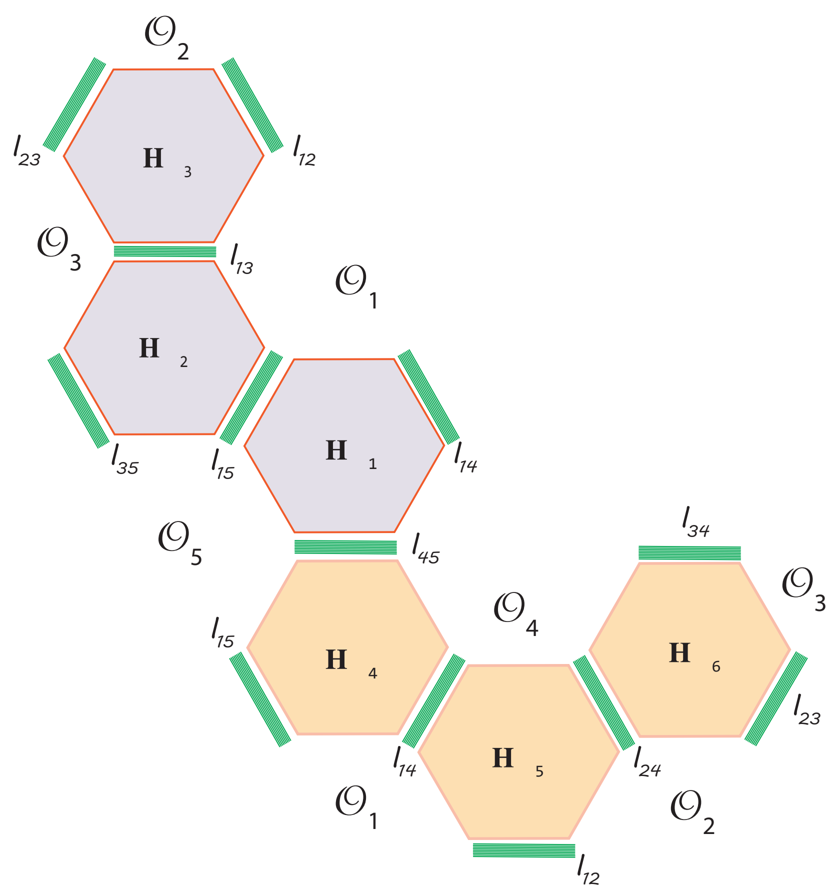
Recall:

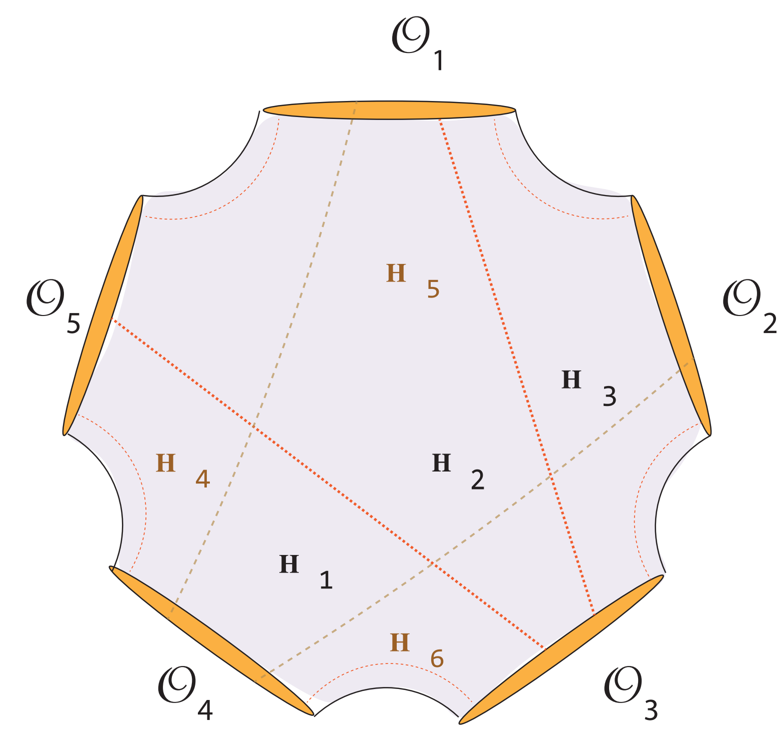
$$e^{-\tilde{E}_a(u)} = e^{ip_a(u^\gamma)}$$

Expanding above:

$$\frac{e^{-\frac{1}{2}ip(u_2^\gamma)}\bar{\alpha}_1}{4\sqrt{\bar{z}_1 z_1}} + \frac{e^{\frac{1}{2}ip(u_2^\gamma)}\sqrt{\bar{z}_1 z_1}}{4\bar{\alpha}_1} + \frac{e^{\frac{1}{2}ip(u_2^\gamma)}\sqrt{\bar{z}_1 z_1}}{4\alpha_1} + \frac{e^{-\frac{1}{2}ip(u_2^\gamma)}\alpha_1}{4\sqrt{\bar{z}_1 z_1}}$$

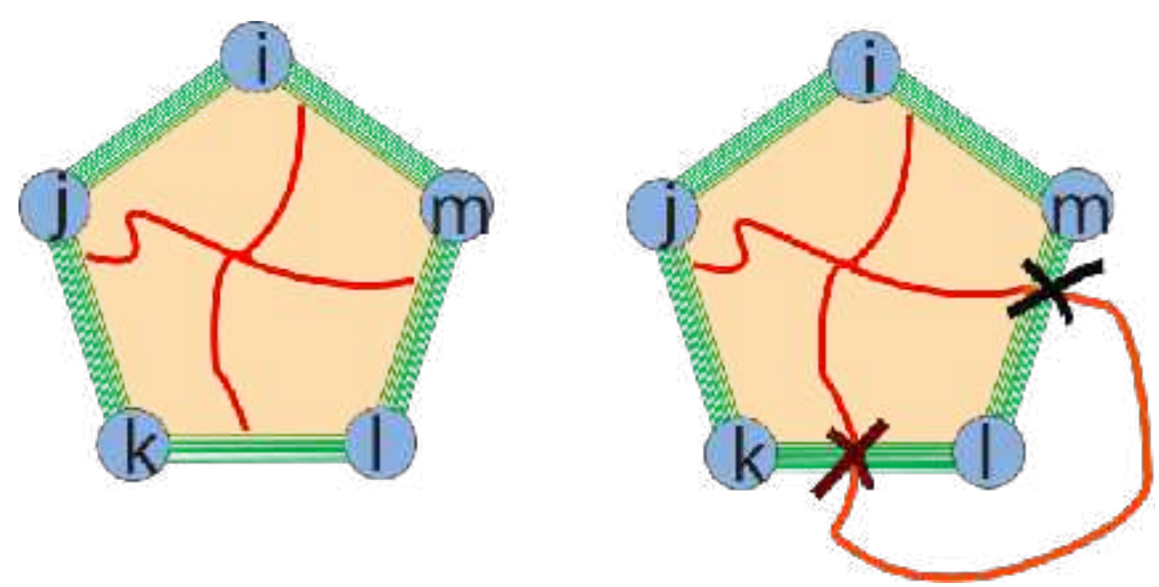
Decagon



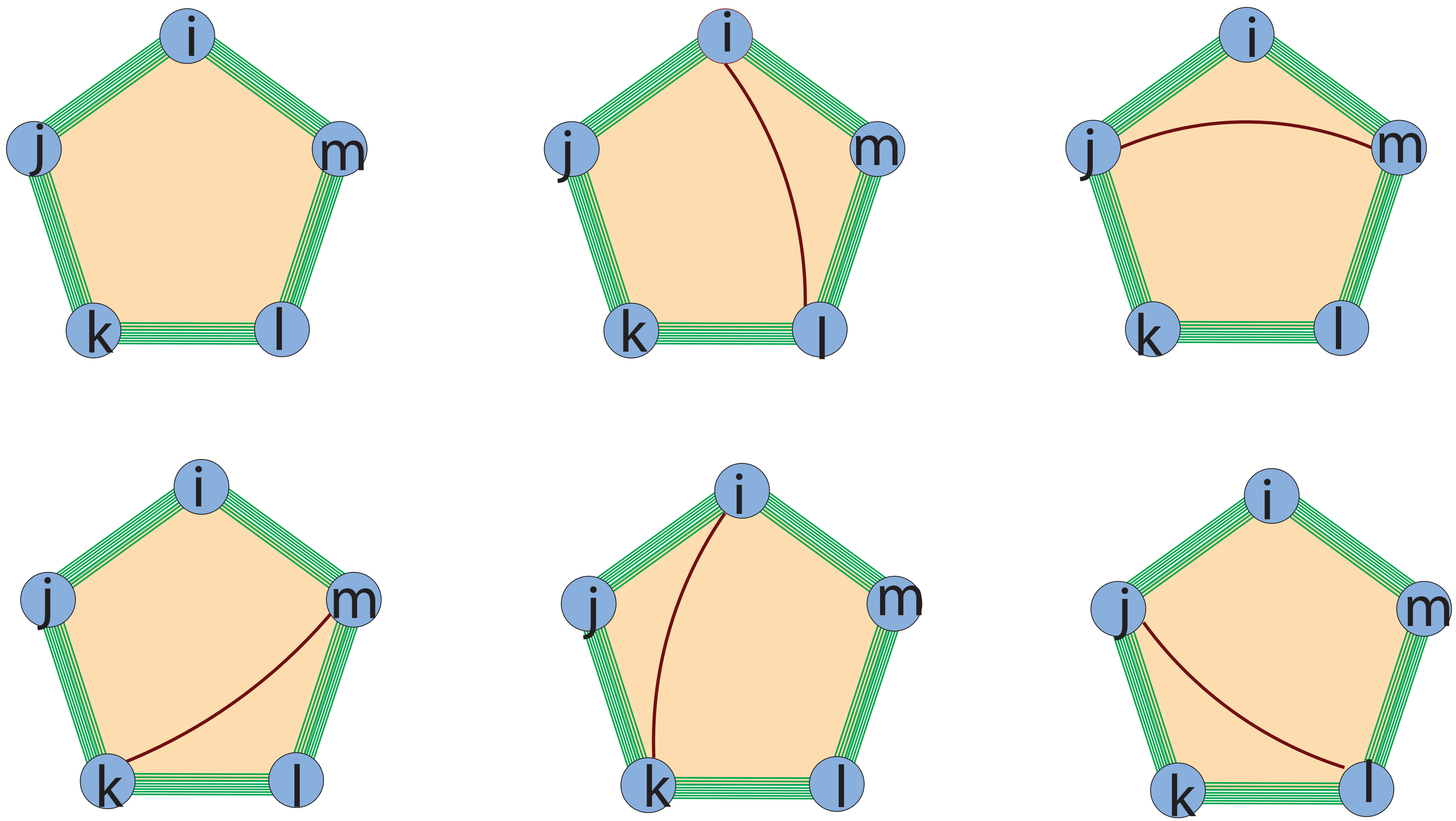


Very long operators:

Length 2k with k infinity



Neighboring graphs:



Result for the decagon:

$$\mathbb{D} = g^2\big[(z_2+\bar z_2-1)\frac{F^{(1)}(z_2)}{2} - z_1\bar z_1 F^{(1)}(z_1)\big]$$

$$+g^4\big[\frac{(1-z_2-\bar z_2)}{2}\left(\frac{P(1-z_1(1-z_2)),1-z_1)}{z_2\bar z_2} - \frac{z_1\bar z_1 L(\frac{1}{1-z_2},z_1)+L(1-z_2,\frac{1}{z_1})}{(1-z_1(1-z_2))(1-\bar z_1(1-\bar z_2))}\right.$$

$$\left.+2F^{(2)}(z_2)\right) + \frac{z_1\bar z_1}{2}P\left(\frac{z_2-1}{z_2},\frac{1-z_1(1-z_2)}{z_1 z_2}\right) + z_1\bar z_1 F^{(2)}(z_1)\big] + \text{cyclic rotations.}$$

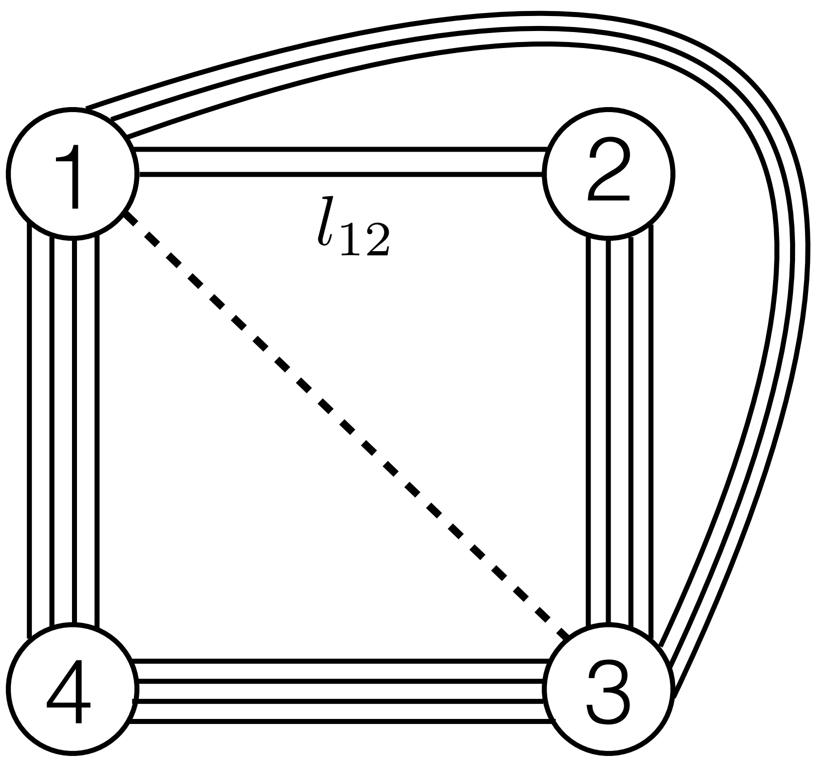
The correlation function:

$$\langle \mathcal{O}_{2K}(x_i,y_i)\mathcal{O}_{2K}(x_m,y_m)\ldots\mathcal{O}_{2K}(x_j,y_j)\rangle = \mathbb{D}^2(d_{im}d_{ml}d_{lk}d_{kj}d_{ji})^K$$

Four-point functions

(Chicherin, Drummond, Heslop, Sokatchev)

(Chicherin, Georgoudis, Gonçalves, Pereira)



$$G'_{1234}\Big|^{g^4}_{l_{12}=0} = (1-z)(1-\bar{z})F^{(2)}[1-z] + F^{(2)}\left[\frac{z}{z-1}\right],$$

$$G'_{1234}\Big|^{g^4}_{l_{12}=1} = (z+\bar{z}-2z\bar{z})F^{(2)}[1-z] + \frac{(1-z\bar{z})}{(1-z)(1-\bar{z})}F^{(2)}\left[\frac{z}{z-1}\right],$$

$$G'_{1234}\Big|^{g^4}_{l_{12}=2} = z\bar{z}\,F^{(2)}[1-z] + \frac{F^{(2)}\left[\frac{z-1}{z}\right]}{z\bar{z}},$$

$$G'_{1234}\Big|^{g^6}_{l_{12}=3} = z\bar{z}\,F^{(3)}[1-z] - 3\frac{F^{(3)}\left[\frac{z-1}{z}\right]}{z\bar{z}},$$

where

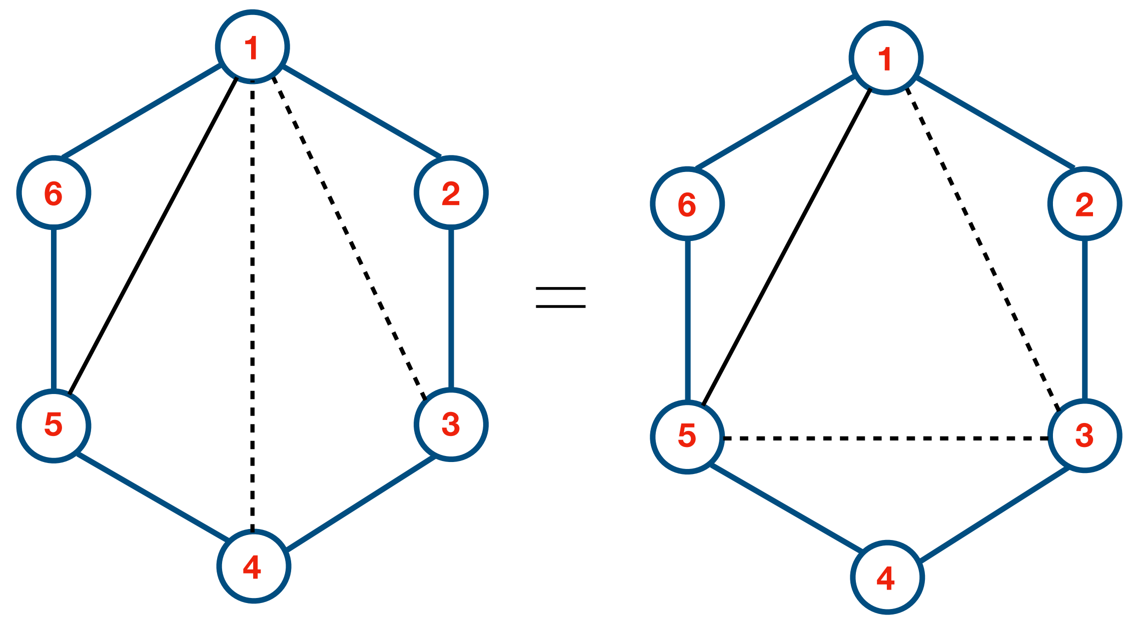
$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad (1-z)(1-\bar{z}) = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}.$$

Can we do higher-points? 6,7, ….

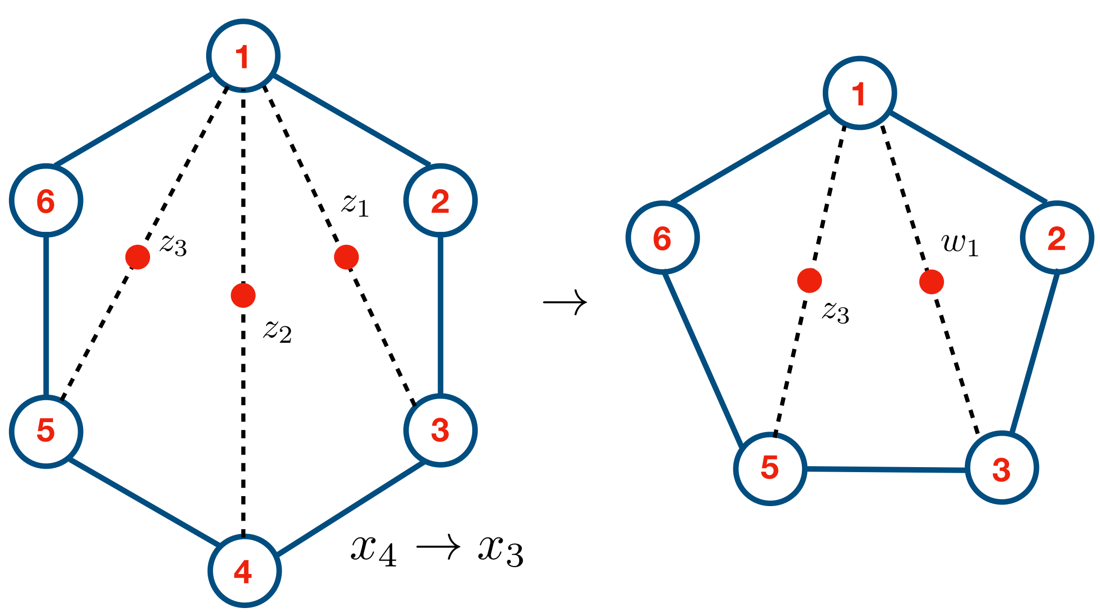
(In progress, with F. Coronado and V. Gonçalves)

It seems so: Z-markers relations, a bit of data, and

Flip invariance:



Decoupling:



$$\mathcal{M}^{(3)}(z_1, z_2, z_3) \to \mathcal{M}^{(2)}(w_1, z_3)\,,$$

as $z_1/z_2 \to 0$ with $z_1 z_2 = -w_1$ fixed

Conclusions:

- Pin down the map between integrability and Feynman integrals.
- Try to get higher-loop information; bootstrap. **(In progress with Coronado, Gonçalves and Vieira)**
- Compute the decagon at strong coupling. **(In progress with Belitsky and Erdogan)**
- Learn how to compute the integrability integrals. **(de Leeuw, Eden, le Plat, Meier, Sfondrini)** **(de Leeuw, Eden, Sfondrini)**
- Connections with other techniques.

Thank you very much!

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