Universality in few-body systems

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2005	PhD in France (Laboratoire Aimé Cotton, Orsay) in ultracold atom theory. Formation of molecules in Bose-Einstein condensates
2005-2008	Postdoctoral researcher at NIST (National Institute of Standards and Technology) Properties of Alkaline-earth atoms for atomic clocks
2008-2012	Postdoctoral researcher at the ERATO project of Masahito Ueda (The University of Tokyo). Efimov states in ultracold-atom experiments
2012	Research Scientist at RIKEN Universal few-body and many-body physics

Aim

Look at a particular aspect of quantum physics: the **universal physics** that arises for **nearly-resonant short-range interactions**.

> nuclear physics

> atomic physics (cold atoms)

> condensed matter, etc.

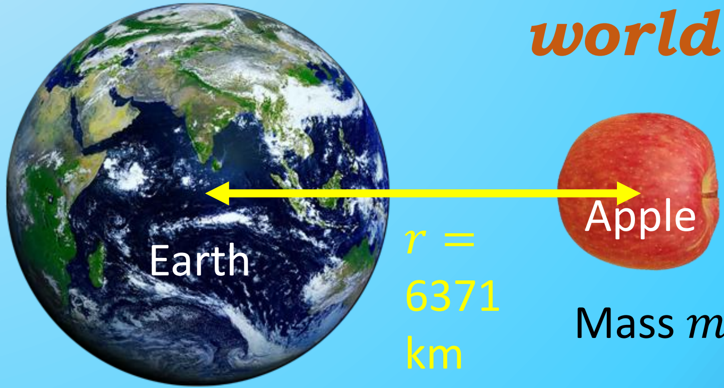
Look in detail at the case of two-body and three-body systems, and in particular the **Efimov effect**.

What is universality?

A phenomenon is ***universal*** when it applies to *many different physical systems* with a simple law that depends upon just *a few parameters*.

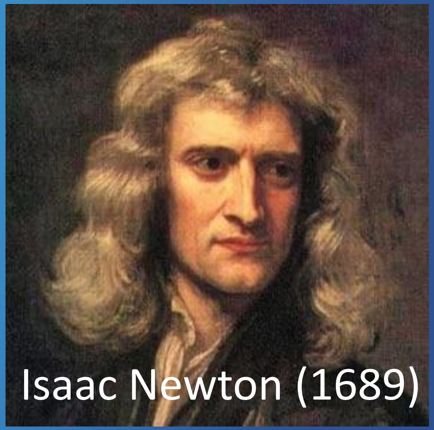
Gravitation is universal

Terrestrial world



Mass $m = 0.1$ kg

Mass $M = 6 \times 10^{24}$ kg


Isaac Newton (1689)

Universal law: long-range force

$$V(r) = -G\frac{Mm}{r}$$

Celestial world

$$V(R) = -G\frac{MM'}{R}$$

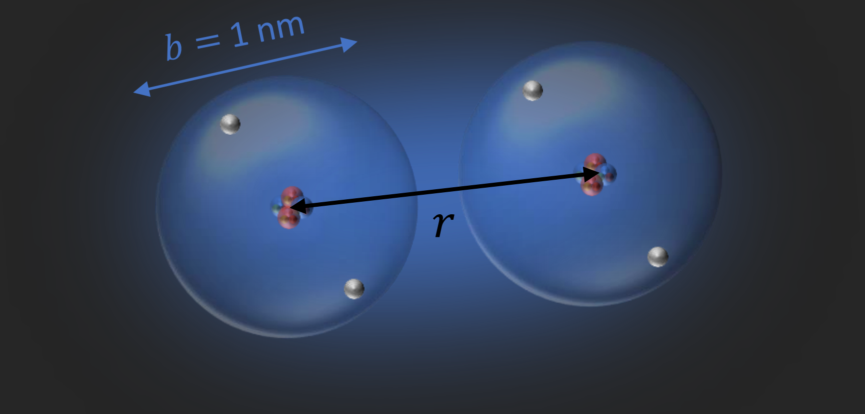
Mass $M' = 7 \times 10^{22}$ kg

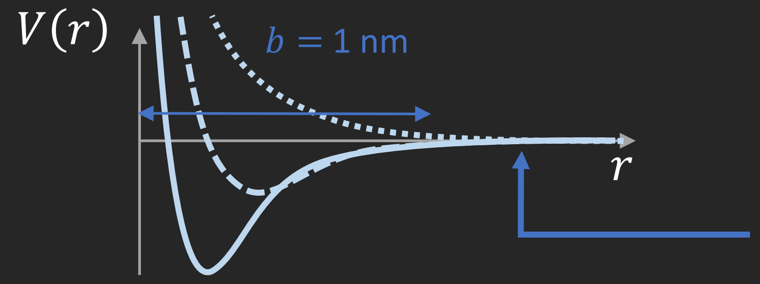
No apparent universality in the microscopic world

- Short-range forces
- Many different interaction potentials that depend on the nature and states of the particles

Atomic world

Electromagnetic force

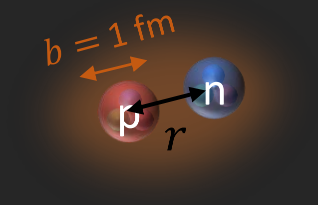


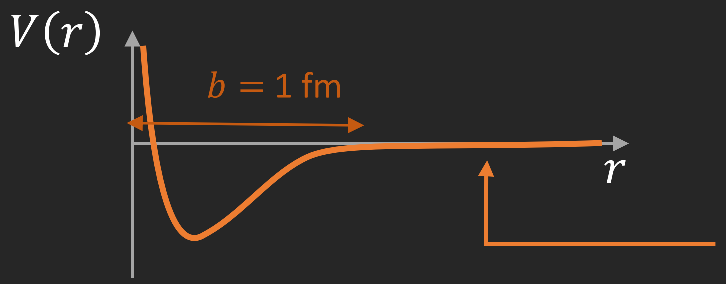


$$-\frac{C_6}{r^6}$$ (van der Waals)

Subatomic world

Strong force





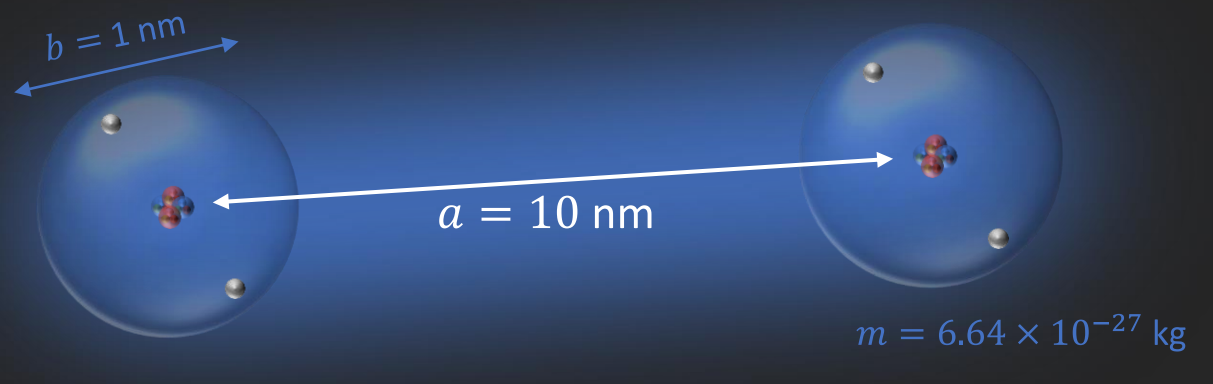
$$-\frac{e^{-\alpha r}}{r}$$ (Yukawa)

“Zero-range” universality

Approximate universal law

Atomic world Electromagnetic force

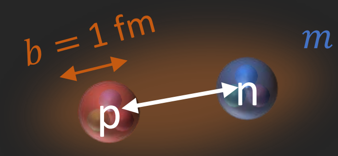
Helium-4 dimer



$$E \approx -\frac{\hbar^2}{ma^2} = 1.0 \times 10^{-7}\ \text{eV}$$

Subatomic world Strong force

Deuteron



$m = 1.66 \times 10^{-27}$ kg

$a = 5.4$ fm

$$E \approx -\frac{\hbar^2}{ma^2} = 1.4 \times 10^{6}\ \text{eV}$$

“Zero-range” universality

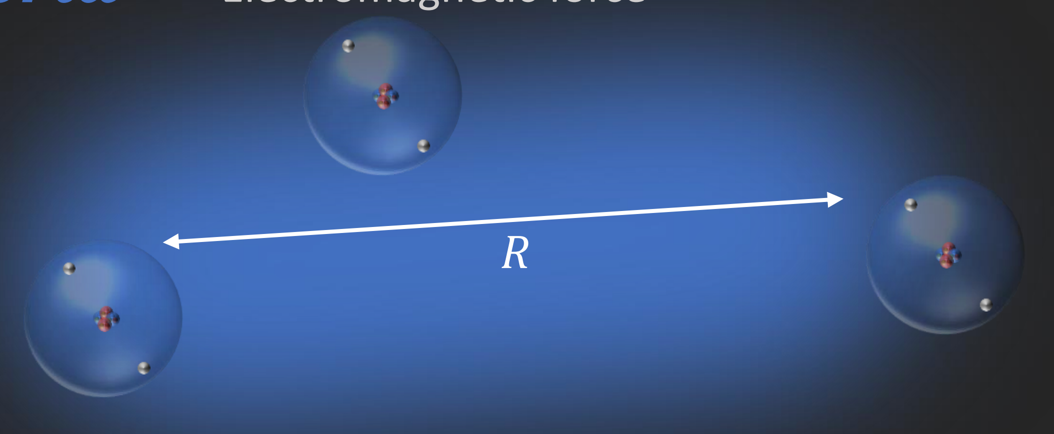
Approximate universal law: effective long-range force

Efimov attraction

Atomic world

Electromagnetic force

Helium-4 trimer

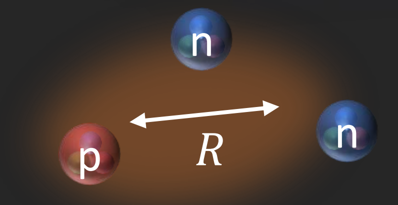


$$V(R) \approx -\frac{\hbar^2}{mR^2}$$

Subatomic world

Strong force

Triton



$$V(R) \approx -\frac{\hbar^2}{mR^2}$$

Plan

1 TWO-BODY PHYSICS

- Basic theory (resonances)
- Zero-range universality
- Feshbach resonances
- Zero-range theory
- Extras (Separable theory)

2 THREE-BODY PHYSICS

- History
  - Thomas collapse
  - STM’s zero-range theory
  - Efimov’s breakthrough
- Efimov effect and Efimov states

3 UNIVERSAL CLUSTERS

- Experimental observations in nuclear and atomic systems
- Mixtures of particles

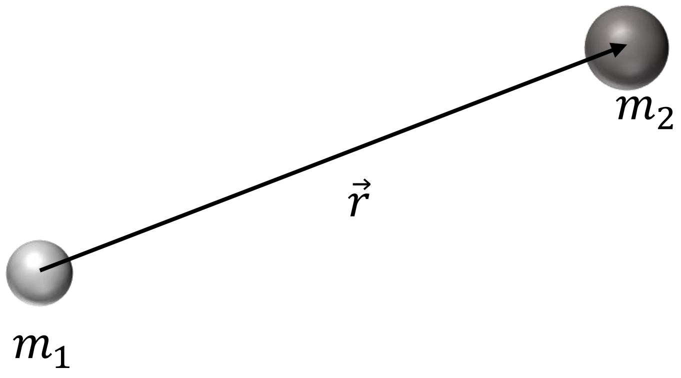
4 VAN DER WAALS UNIVERSALITY

- Two-body systems
- Three-body systems

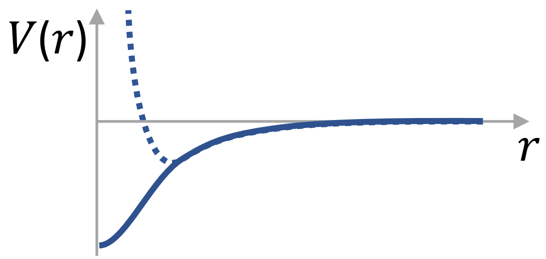
Two-body physics

Basic theory

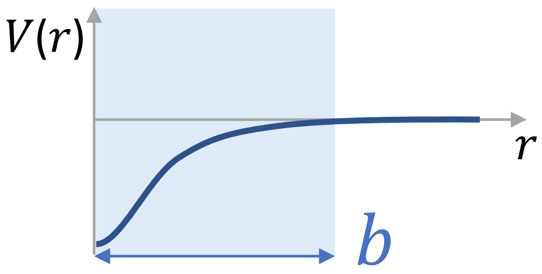
We consider two particles of masses m_1 and m_2 with no internal degree of freedom.
In their centre of mass, they are described by a relative vector $\vec{r}$.



The particles interact via an isotropic (central) interaction potential $V(r)$, with $r = |\vec{r}|$



Attractive
⟵ Necessary for binding



Short range
⟷ $V(r)$ decays strictly faster than r^{-3}

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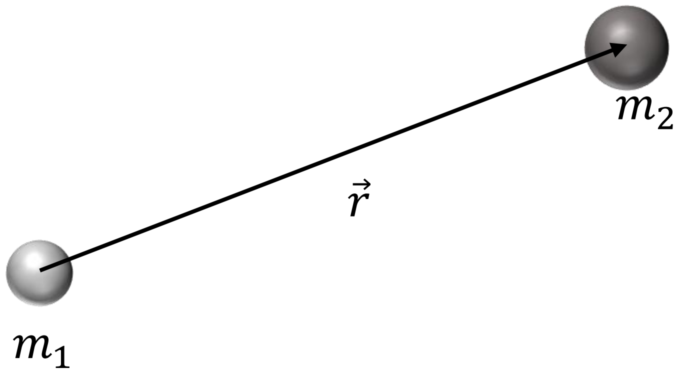

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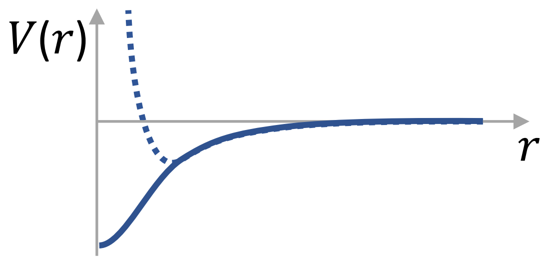

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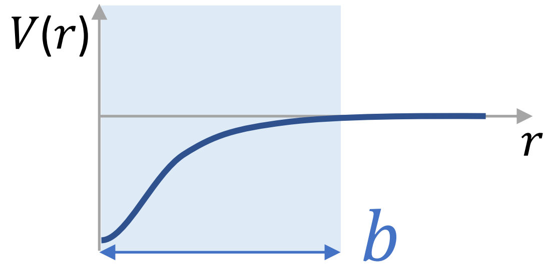

Short range

⟺ $V(r)$ decays strictly faster than r^{-3}

Classically:

$$\underbrace{\frac{1}{2}\mu\left(\frac{d\vec{r}}{dt}\right)^2}_{\text{Kinetic energy}} + \underbrace{V(r)}_{\text{Interaction energy}} = \underbrace{E}_{\text{Total energy}}$$

Reduced mass: $\mu = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1}$

Basic theory

We consider two particles of masses m_1 and m_2 with no internal degree of freedom.
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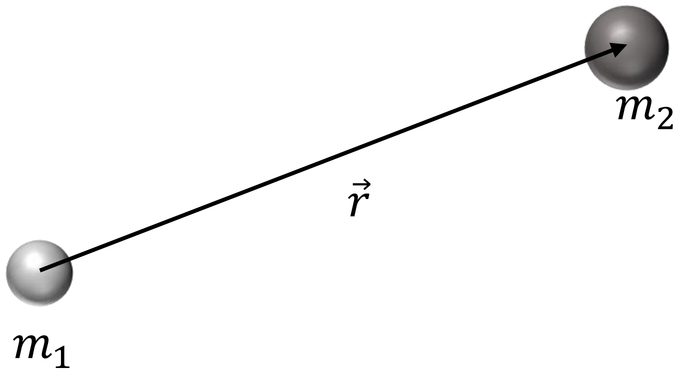

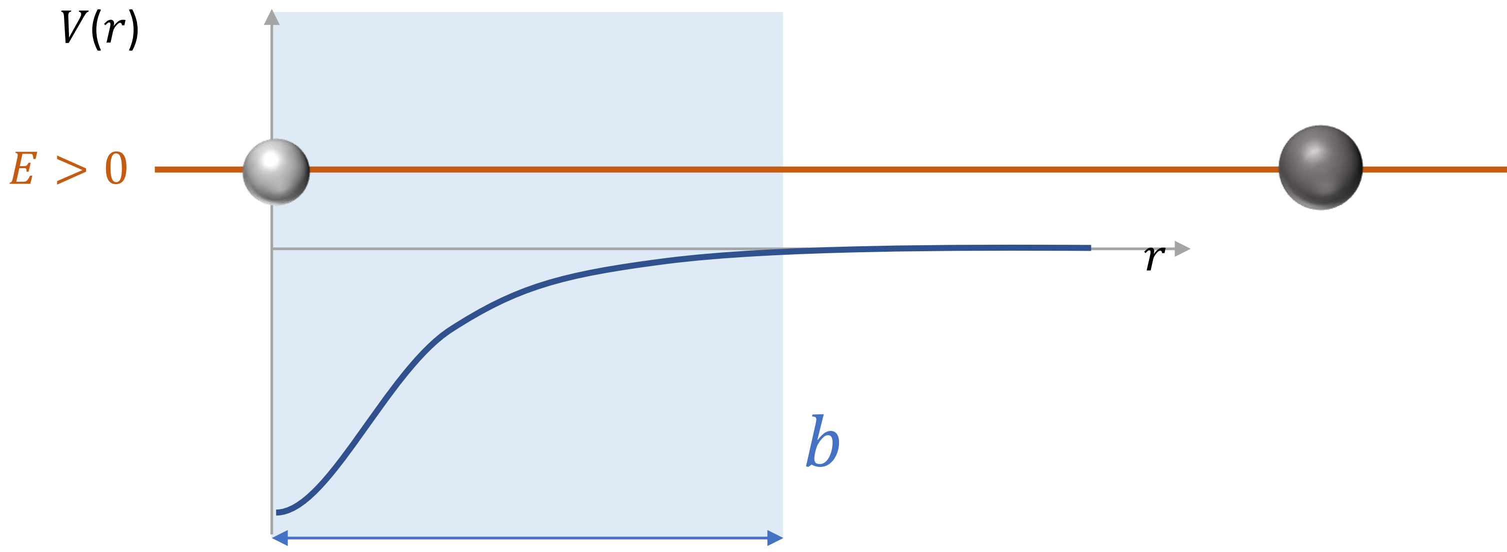

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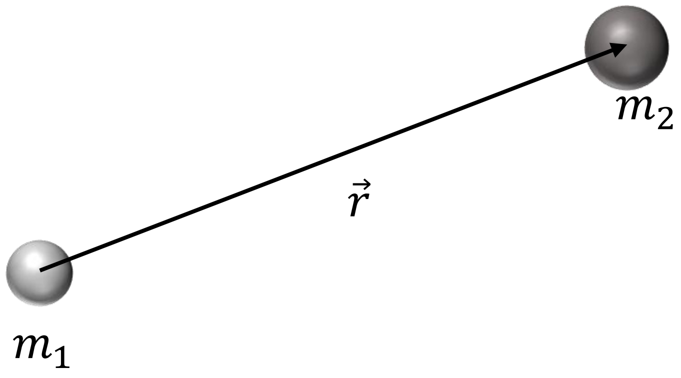

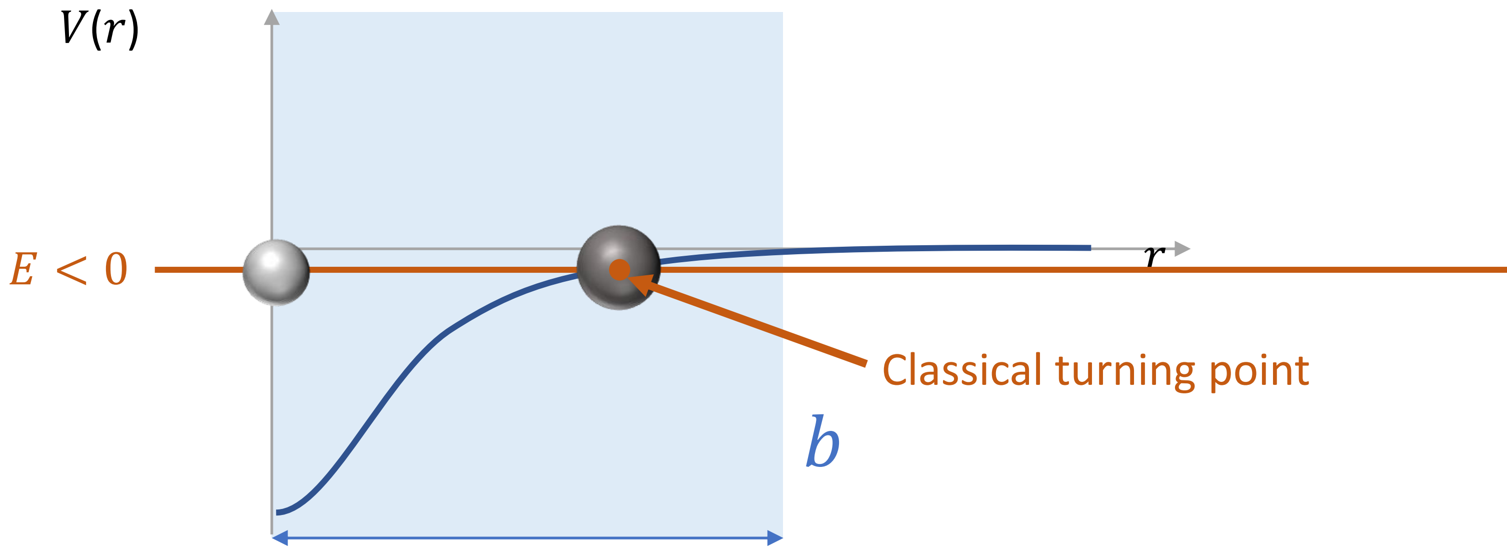

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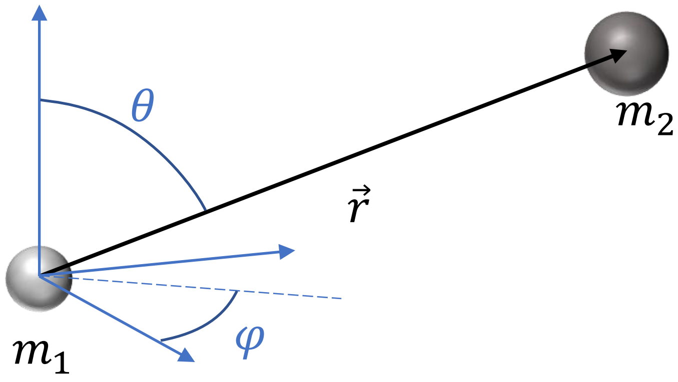
Basic theory



Quantum mechanically, the vector $\vec{r}$ is described by a wave function $\psi(\vec{r})$.

Schrödinger equation at energy E: $$\left(-\frac{\hbar^2}{2\mu}\nabla_{\boldsymbol{r}}^2 + V(r) - E\right)\psi(\vec{r}) = 0$$

$$\left(-\nabla_{\boldsymbol{r}}^2 + \frac{2\mu}{\hbar^2}V(r) - \frac{2\mu}{\hbar^2}E\right)\psi(\vec{r}) = 0$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\,\cdot) + \frac{1}{r^2}\left(\underbrace{\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}}_{\ell(\ell+1)} + \cancel{\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}}\right)\underbrace{P_\ell(\cos\theta)}_{\text{Legendre polynomial}}$$

Spherical coordinates: $\psi(\vec{r}) = \psi(r, \theta, \cancel{\varphi})$

Partial wave expansion: $= \sum_{\ell=0}^{\infty} \frac{u_\ell(r)}{r} P_\ell(\cos\theta)$

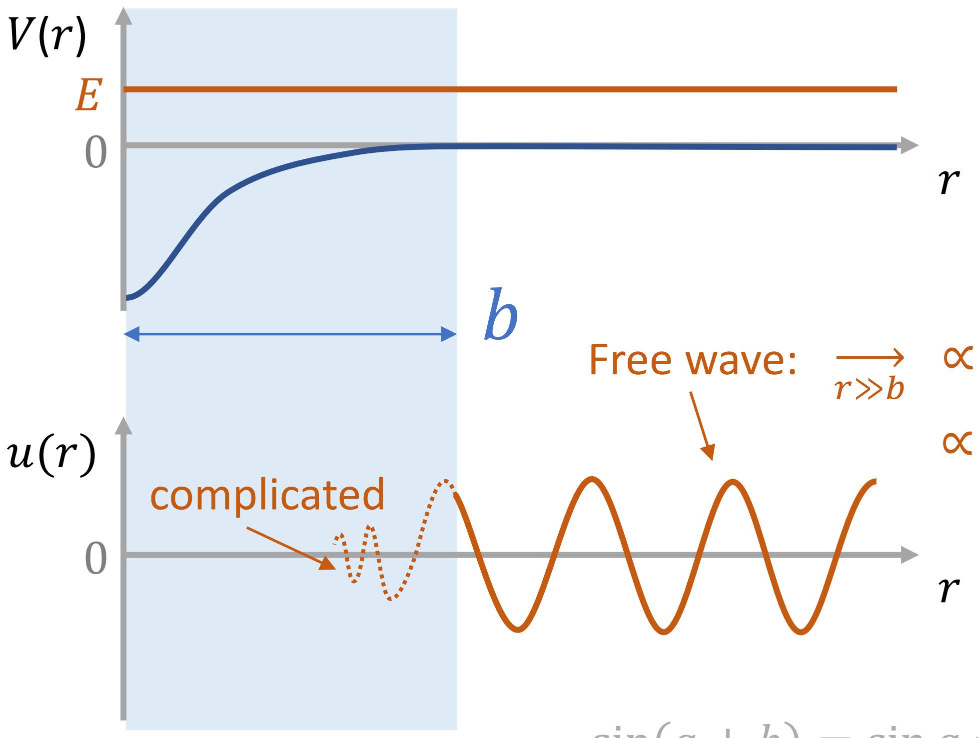
$\ell = 0$: s wave
$\ell = 1$: p wave
$\ell = 2$: d wave

Radial Schrödinger equations: $$\left(\underbrace{-\frac{d^2}{dr^2}}_{\text{Radial kinetic energy}} + \underbrace{\frac{\ell(\ell+1)}{r^2}}_{\text{Centrifugal repulsion}} + \frac{2\mu}{\hbar^2}V(r) - \frac{2\mu}{\hbar^2}E\right)u_\ell(r) = 0$$

Basic theory: s wave

Radial Schrödinger equations: $\left(-\frac{d^2}{dr^2}+\frac{2\mu}{\hbar^2}V(r)\underbrace{-\frac{2\mu}{\hbar^2}E}_{-k^2}\right)u(r)=0$ k: wave number

Positive energy $\boldsymbol{E=\frac{\hbar^2 k^2}{2\mu}\geq 0}$



$\xrightarrow[r\gg b]{} \propto \sin kr + \tan\delta\cos kr$

$\propto \frac{1}{\cos\delta}\sin(kr+\delta)$

$\sin(a+b)=\sin a\cos b+\sin b\cos a$

Phase shift:

$\delta=0$ modulo π : effectively non-interacting

$\delta\approx\frac{\pi}{2}$ modulo π : **resonant interaction**

At low scattering energy E, in the s wave:

For $k\ll b^{-1}$, $\boxed{\tan\delta\approx -ka}$

"Scattering length"
(positive or negative)

Justification:

With potential v: $\left(-\frac{d^2}{dr^2} + v(r) - k^2\right) u(r) = 0$

With potential $\bar{v}$: $\left(-\frac{d^2}{dr^2} + \bar{v}(r) - k^2\right) \bar{u}(r) = 0$

$\times\, \bar{u}_\ell$

$$-\bar{u}u'' + (v - k^2)\bar{u}u = 0$$

$\times\, u_\ell$

$$-u\bar{u}'' + (\bar{v} - k^2)u\bar{u} = 0$$

Subtract: $\underbrace{u\bar{u}'' - \bar{u}u''}_{} + (v - \bar{v})\bar{u}u = 0$

$$(u\bar{u}' - \bar{u}u')' + (v - \bar{v})\bar{u}u = 0$$

$$(u\bar{u}' - \bar{u}u')' = -(v - \bar{v})\bar{u}u$$

At low scattering energy E, in the s wave:

For $k \ll b^{-1}$, $\boxed{\tan\delta \approx -ka}$

"Scattering length"
(positive or negative)

Condition for $r = 0$:
$u(0) = 0$
$\bar{u}(0) = 0$

For $r \gg b$:

$$u(r) \to \frac{1}{k\cos\delta}\sin(kr + \delta)$$

$$\bar{u}(r) \to \frac{1}{k\cos\bar{\delta}}\sin(kr + \bar{\delta})$$

Justification:

With potential v: $\left(-\frac{d^2}{dr^2}+v(r)-k^2\right)u(r)=0$

With potential $\bar{v}$: $\left(-\frac{d^2}{dr^2}+\bar{v}(r)-k^2\right)\bar{u}(r)=0$

$$(u\bar{u}'-\bar{u}u')'=-(v-\bar{v})\bar{u}u$$

$$[u\bar{u}'-\bar{u}u']_0^\infty=-\int_0^\infty(v-\bar{v})\bar{u}u dr$$

$$\left[\frac{\sin(kr+\delta)\cos(kr+\bar{\delta})-\sin(kr+\bar{\delta})\cos(kr+\delta)}{k\cos\delta\cos\bar{\delta}}-0\right]=-\int_0^\infty(v-\bar{v})\bar{u}u dr$$

$$\frac{\sin(\delta-\bar{\delta})}{k\cos\delta\cos\bar{\delta}}=-\int_0^\infty(v-\bar{v})\bar{u}u dr$$

$\bar{v}\to 0\ (\bar{\delta}\to 0)$

$$\frac{\tan\delta}{k}=-\int_0^\infty v(r)\frac{\sin(kr)}{k}u(r)dr$$

$\sin(a+b)=\sin a\cos b+\sin b\cos a$

At low scattering energy E, in the s wave:

For $k\ll b^{-1}$, $\boxed{\tan\delta\approx -ka}$

"Scattering length"
(positive or negative)

Condition for $r=0$:
$u(0)=0$
$\bar{u}(0)=0$

For $r\gg b$:

$$u(r)\to\frac{1}{k\cos\delta}\sin(kr+\delta)$$

$$\bar{u}(r)\to\frac{1}{k\cos\bar{\delta}}\sin(kr+\bar{\delta})$$

$$u'(r)\xrightarrow[r\gg b]{}\frac{1}{\cos\delta}\cos(kr+\delta)$$

$$\bar{u}'(r)\xrightarrow[r\gg b]{}\frac{1}{\cos\bar{\delta}}\cos(kr+\bar{\delta})$$

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With potential v: $\left(-\frac{d^2}{dr^2}+v(r)-k^2\right)u(r)=0$

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$\bar{v}\to 0\ (\bar{\delta}\to 0)$

$$\frac{\tan\delta}{k}=-\int_0^\infty v(r)\frac{\sin(kr)}{k}u(r)dr$$

$$\approx -\underbrace{\int_0^b v(r)r\frac{\sin(kr)}{kr}u(r)dr}$$

For $kb \ll 1$

$$\underbrace{\int_0^b v(r)ru^{(0)}(r)dr}_{a}$$

$\frac{\sin kr}{kr}\xrightarrow[kb\ll 1]{}1$

$u(r)\xrightarrow[kb\ll 1]{}u^{(0)}(r)$

The scattering length a is well defined if the potential $v(r)$ decays faster than $1/r^3$.

At low scattering energy E, in the s wave:

For $k \ll b^{-1}$, $\boxed{\tan\delta\approx -ka}$

"Scattering length"
(positive or negative)

Condition for $r=0$:
$u(0)=0$
$\bar{u}(0)=0$

$$u(r)\xrightarrow[r\gg b]{}\frac{\sin kr+\tan\delta\cos kr}{k}\xrightarrow[kb\to o]{}r+\frac{\tan\delta}{k}$$

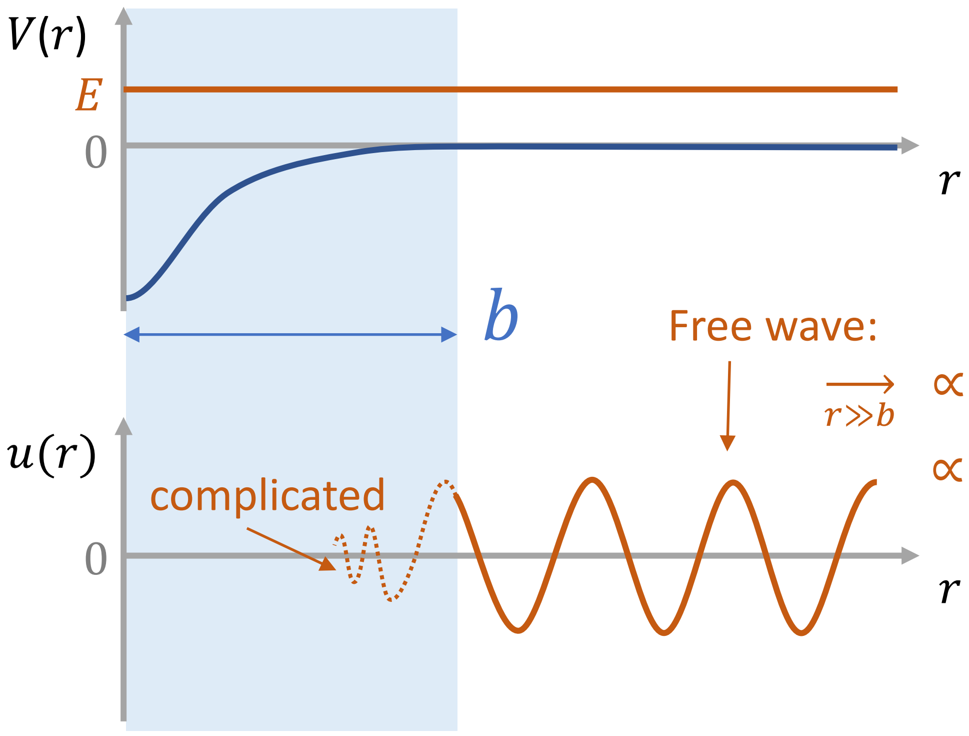
$$\bar{u}(r)=\frac{\sin kr}{k}$$

Basic theory: s wave

Radial Schrödinger equations: $\left(-\frac{d^2}{dr^2}+\frac{2\mu}{\hbar^2}V(r)-\frac{2\mu}{\hbar^2}E\right)\underbrace{u(r)=}_{-k^2}0$

k: wave number

Positive energy $\boldsymbol{E=\frac{\hbar^2k^2}{2\mu}\geq 0}$



$\xrightarrow[r\gg b]{} \propto \sin kr+\tan\delta\cos kr$

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$\delta=0$ modulo π : effectively non-interacting

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At low scattering energy E, in the s wave:

For $k\ll b^{-1}$, $\boxed{\tan\delta\approx -ka}$

"Scattering length"
(positive or negative)

Low-energy s-wave **resonance**: $|a|\gg b$

Unitarity (or ***unitary limit***): $a\to\pm\infty$

Cross section: $\sigma=\frac{4\pi}{k^2}\sin^2\delta\leq\frac{4\pi}{k^2}$

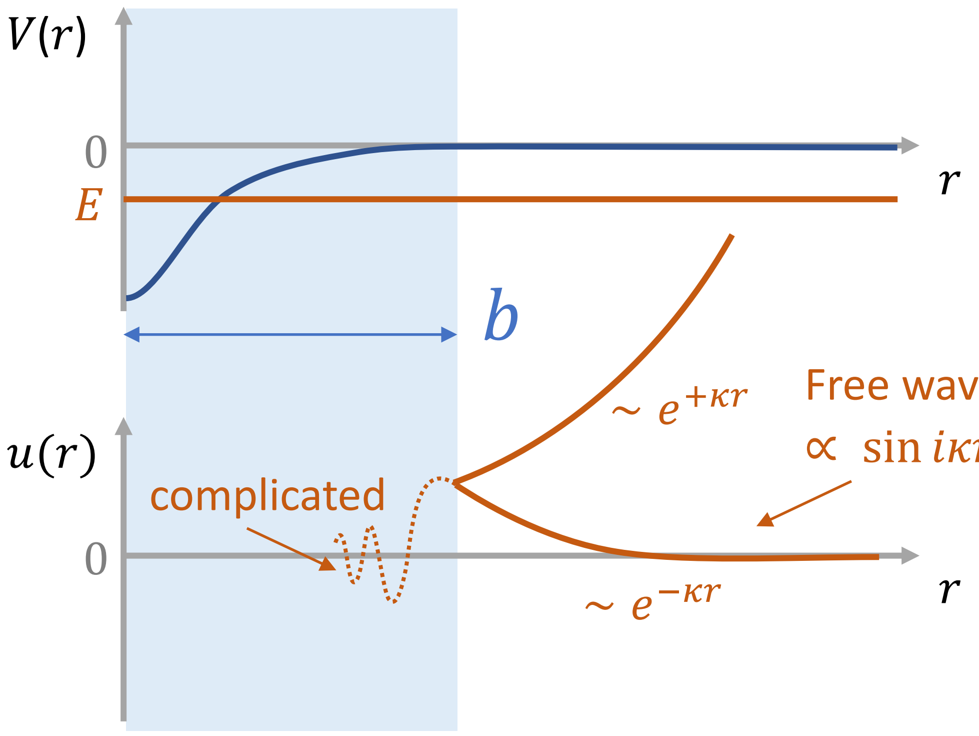
Basic theory

Radial Schrödinger equations: $$\left(-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+\frac{2\mu}{\hbar^2}V(r)\underbrace{-\frac{2\mu}{\hbar^2}E}_{+\kappa^2}\right)u_\ell(r)=0$$

$\boxed{k\rightarrow i\kappa}$

$\kappa\geq 0$: binding wave number

Negative energy $E=-\frac{\hbar^2\kappa^2}{2\mu}\leq 0$



Free wave:
$\propto\ \sin i\kappa r+\tan\delta\cos i\kappa r$

$$\sin i\kappa r+\tan\delta\cos i\kappa r$$
$$\propto\frac{e^{-\kappa r}-e^{\kappa r}}{2i}+\tan\delta\frac{e^{-\kappa r}+e^{\kappa r}}{2}$$
$$\propto\left(\tan\delta+\frac{1}{i}\right)\frac{e^{-\kappa r}}{2}+\underbrace{\left(\tan\delta-\frac{1}{i}\right)}_{0}\frac{e^{+\kappa r}}{2}$$

Physical state (bound state) : $\tan\delta=\frac{1}{i}$
(quantisation of the energy: discrete levels)

For small energy E, in the s wave:

For $k\ll b^{-1}$, $\boxed{\tan\delta\approx -ka=-i\kappa a}$

➡ $\kappa\approx 1/a$ $\quad\boxed{E\approx-\frac{\hbar^2}{2\mu a^2}}$ (For $|a|\gg b$)

Basic theory

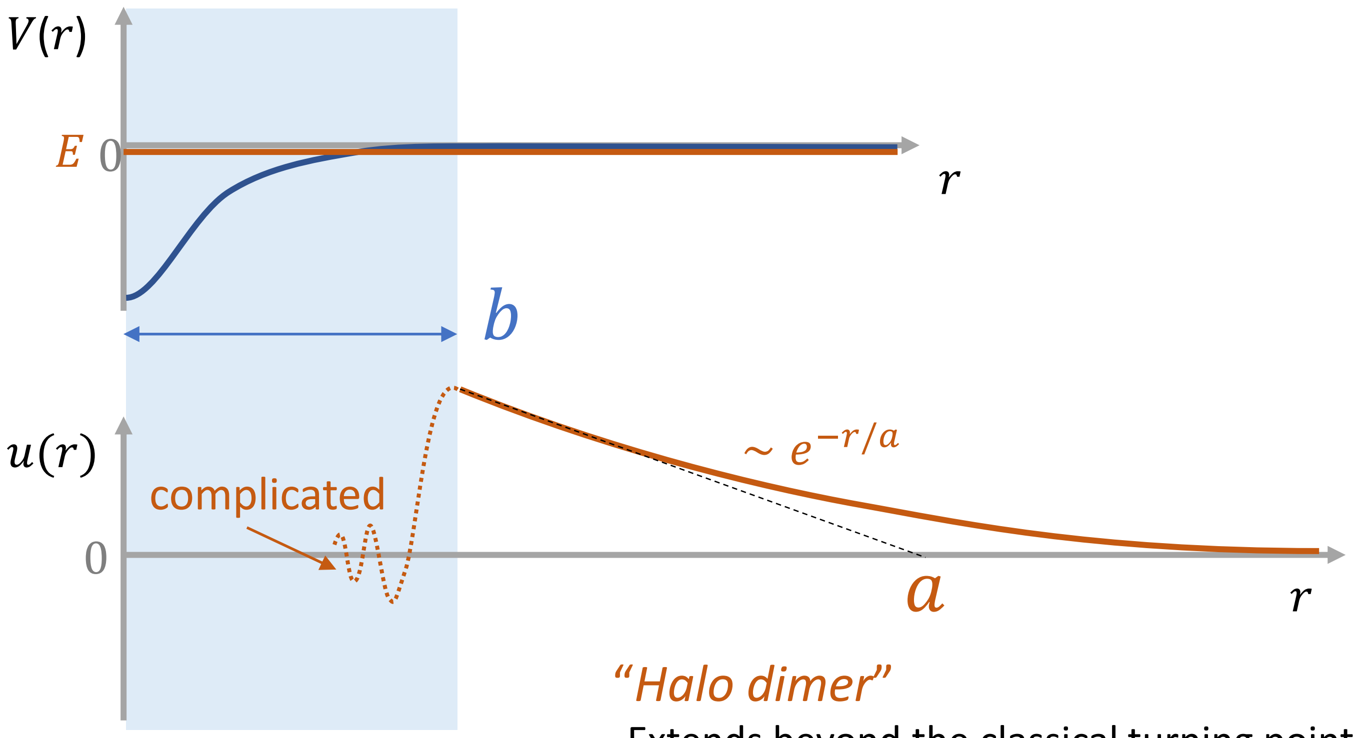
Radial Schrödinger equations: $\left(-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+\frac{2\mu}{\hbar^2}V(r)\underbrace{-\frac{2\mu}{\hbar^2}E}_{+\kappa^2}\right)u_\ell(r)=0$

$\boxed{k \to i\kappa}$

$\kappa \geq 0$: binding wave number

Negative energy $E=-\frac{\hbar^2\kappa^2}{2\mu}\leq 0$

$$\sin i\kappa r+\tan\delta\cos i\kappa r$$
$$\propto \frac{e^{-\kappa r}-e^{\kappa r}}{2i}+\tan\delta\,\frac{e^{-\kappa r}+e^{\kappa r}}{2}$$
$$\propto \left(\tan\delta+\frac{1}{i}\right)\frac{e^{-\kappa r}}{2}+\underbrace{\left(\tan\delta-\frac{1}{i}\right)}_{0}\frac{e^{+\kappa r}}{2}$$



"Halo dimer"

Extends beyond the classical turning point

Physical state (bound state) : $\tan\delta=\frac{1}{i}$

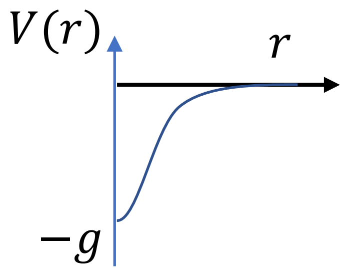
(quantisation of the energy: discrete levels)

For small energy E, in the s wave:

For $k \ll b^{-1}$, $\boxed{\tan\delta \approx -ka=-i\kappa a}$

➡ $\kappa \approx 1/a$ $\boxed{E \approx -\frac{\hbar^2}{2\mu a^2}}$ (For $|a| \gg b$)

Short-range attractive interaction potential



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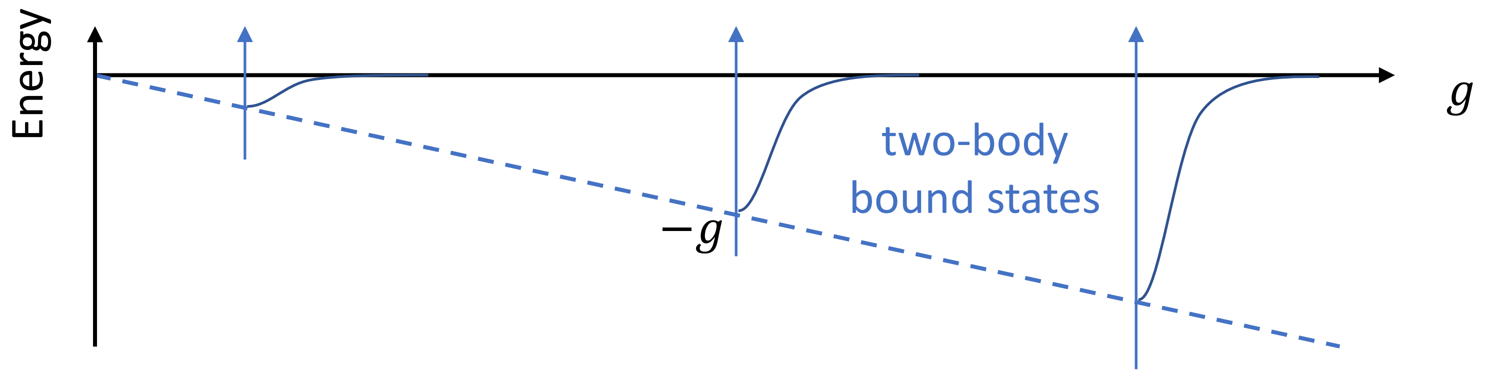
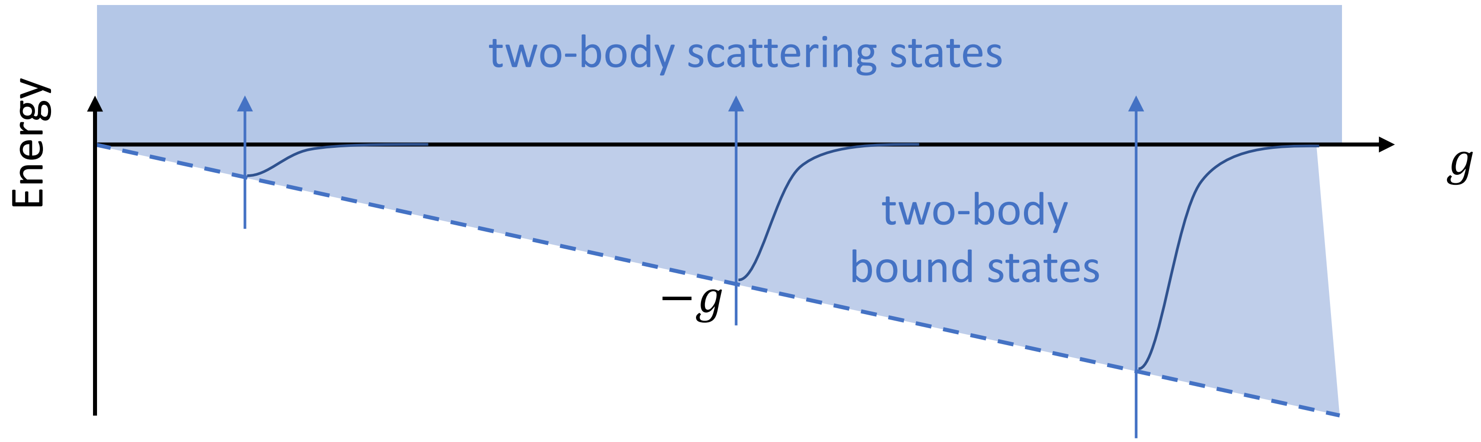
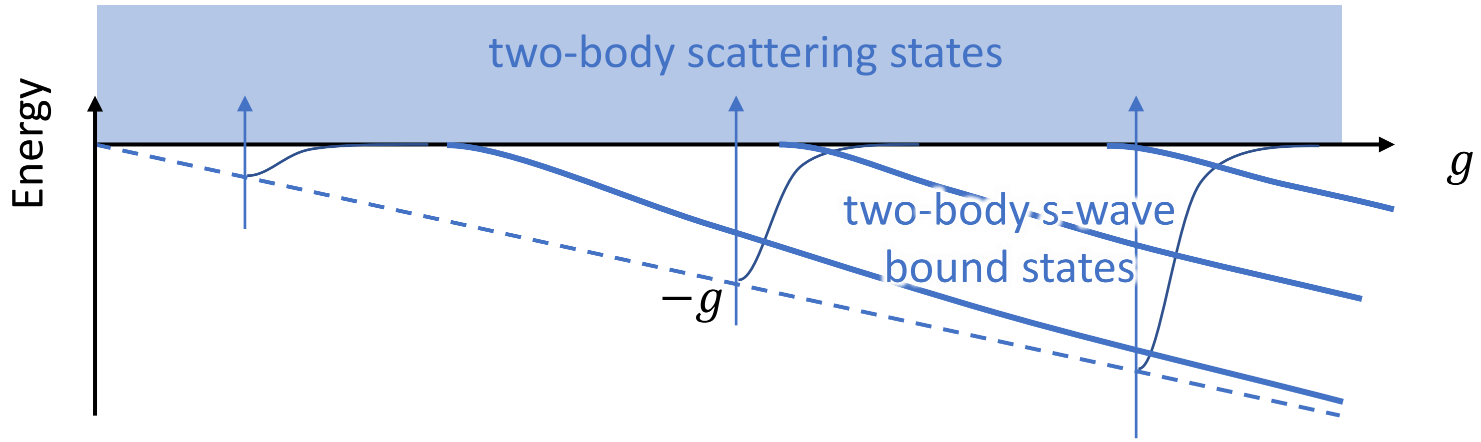


Binding in classical systems



Binding in quantum systems

Critical strength g (zero-point) and quantisation



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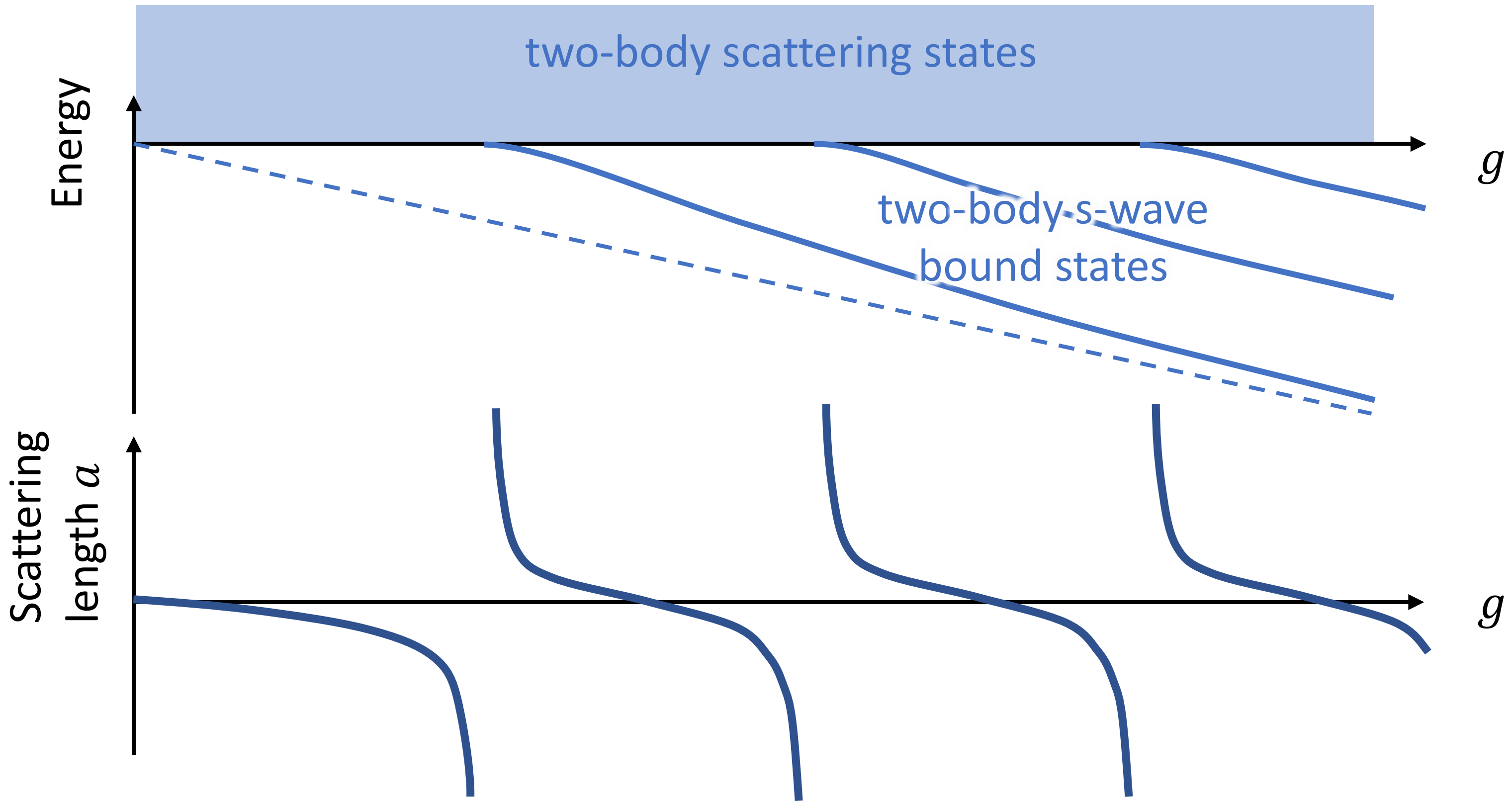
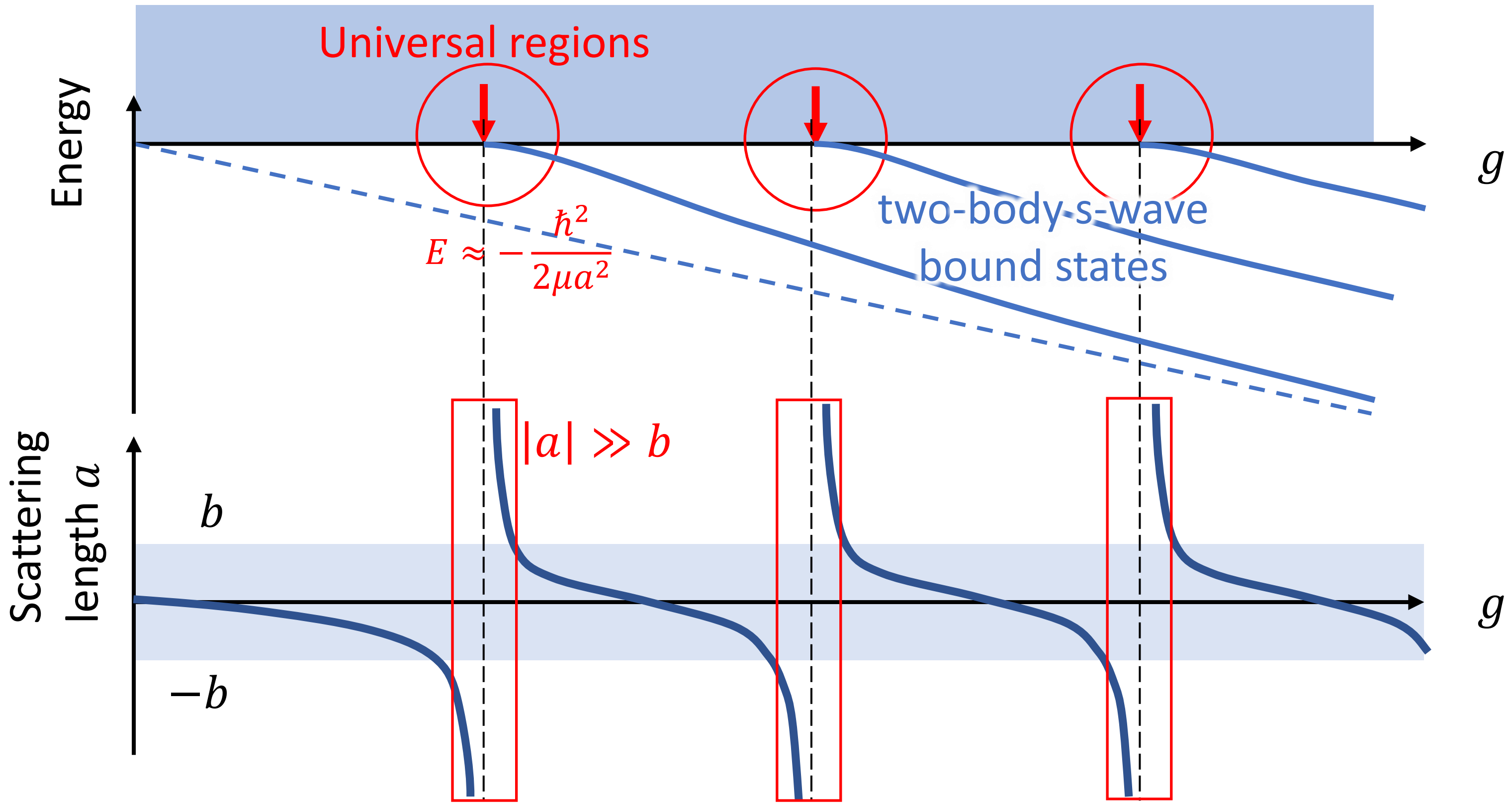


Binding in quantum systems

Two-body resonances $\Longrightarrow$ unitarity points, universality and scale invariance



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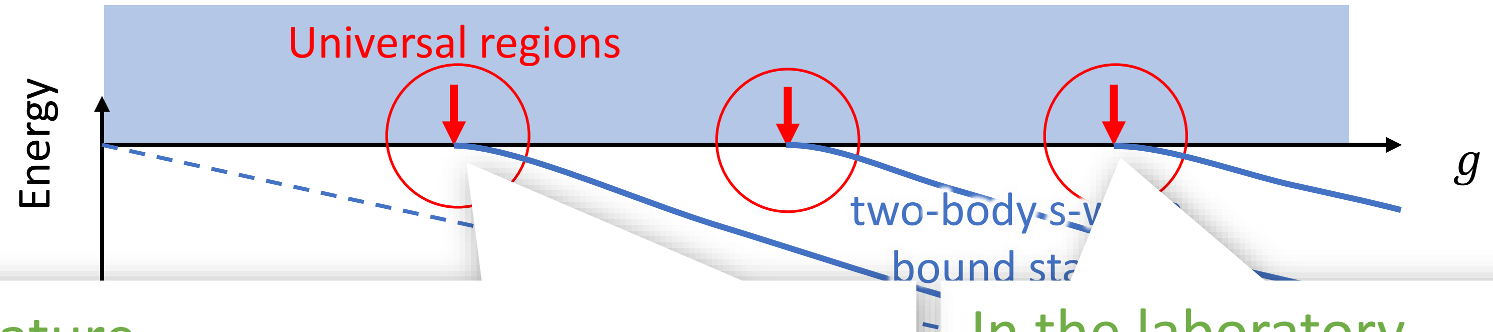
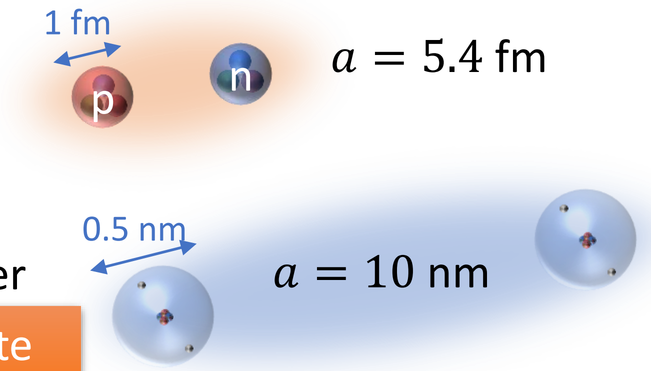


In nature

Deuteron



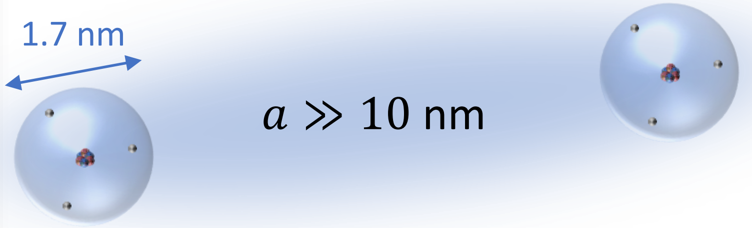
➡ Prof Alejandro Kievsky's lecture.

Helium-4 dimer

➡ Prof Doerte Blume's lecture.

In the laboratory

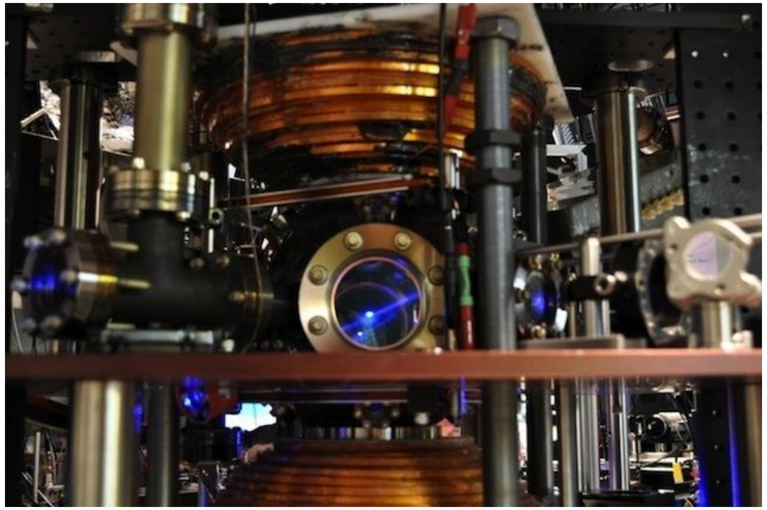
Feshbach dimers from cold atoms

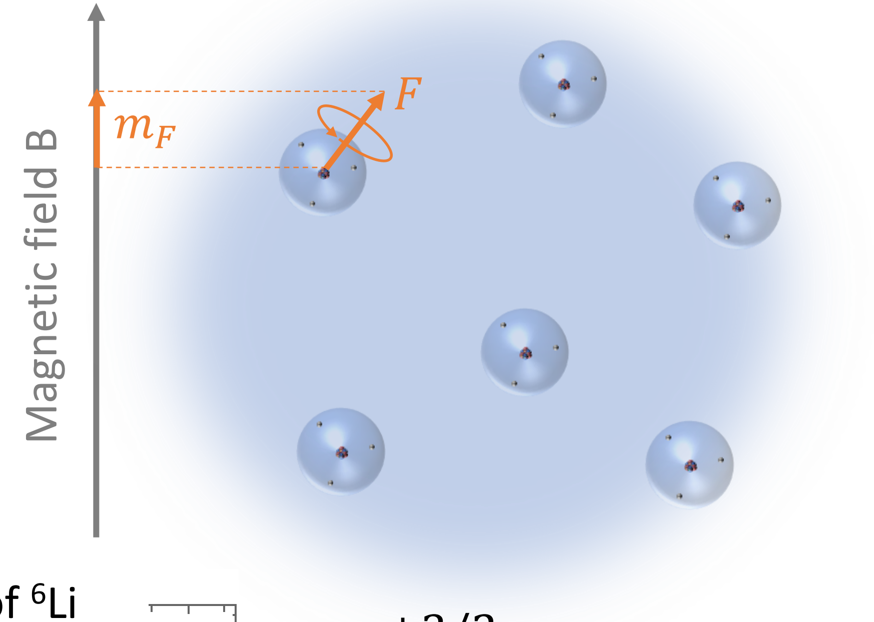


"Halo dimers"

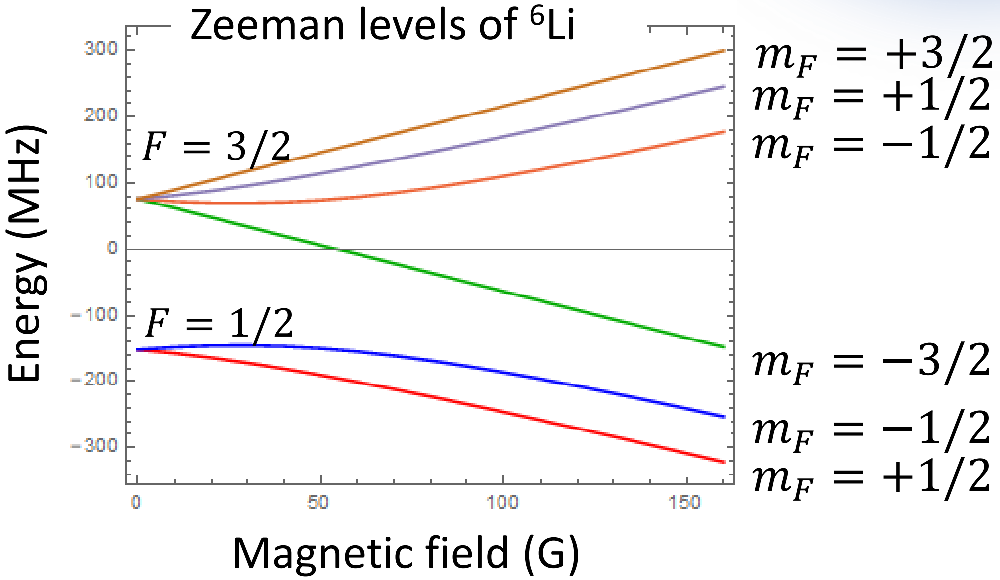
Observations in ultra-cold atomic gases

Cloud of atoms cooled to T < 1 μK in a vacuum chamber





Zeeman effect:
Different internal atomic states shift differently with magnetic field



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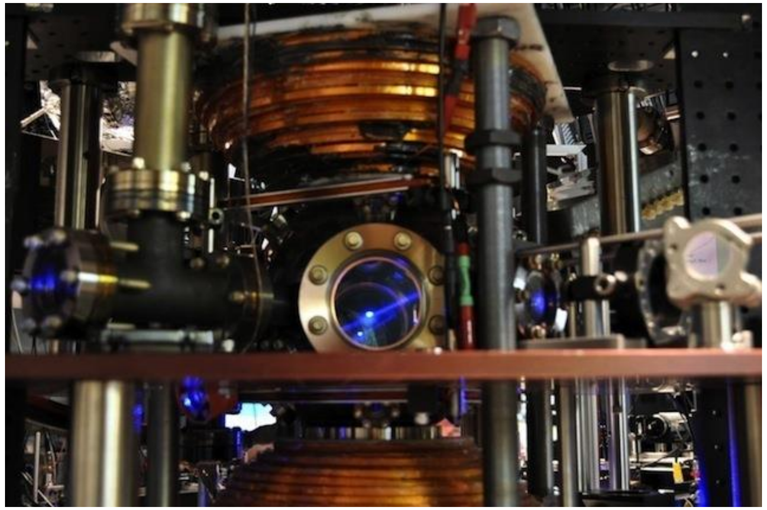

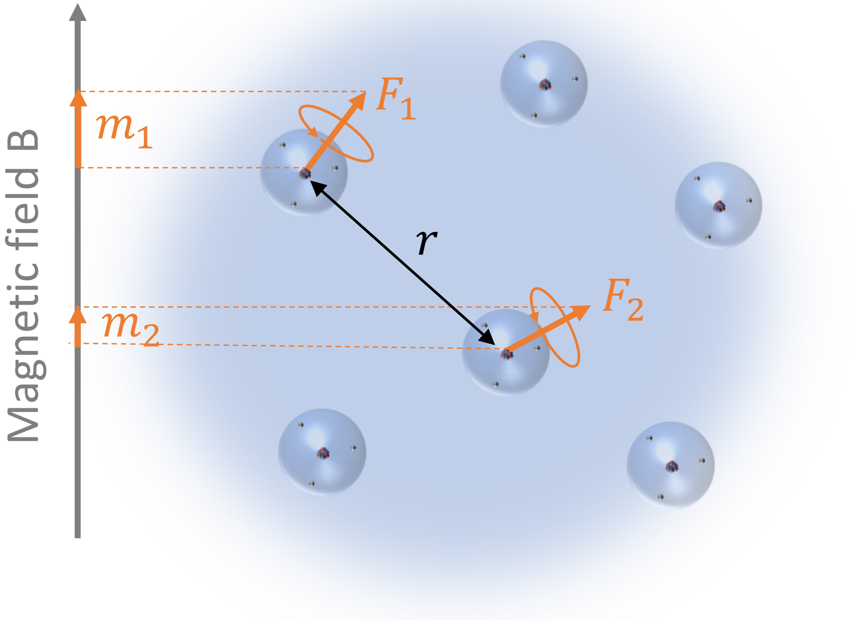

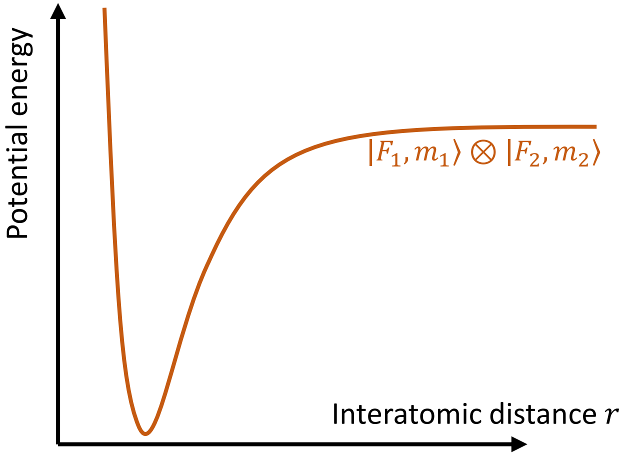

Observations in ultra-cold atomic gases

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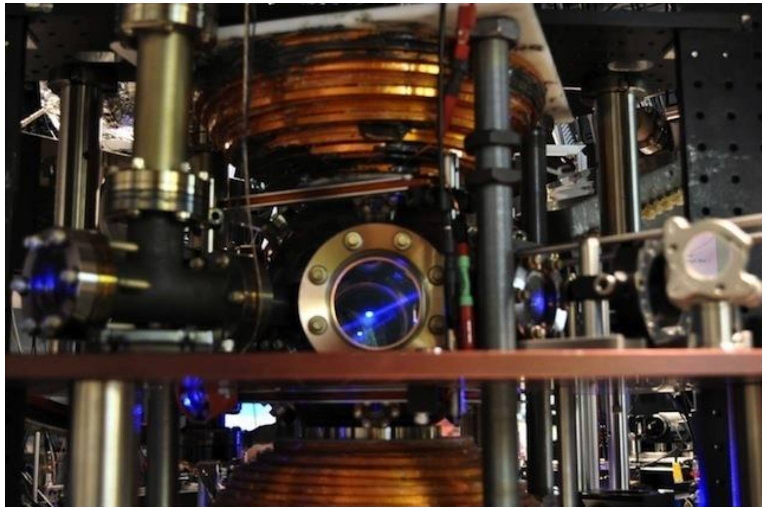

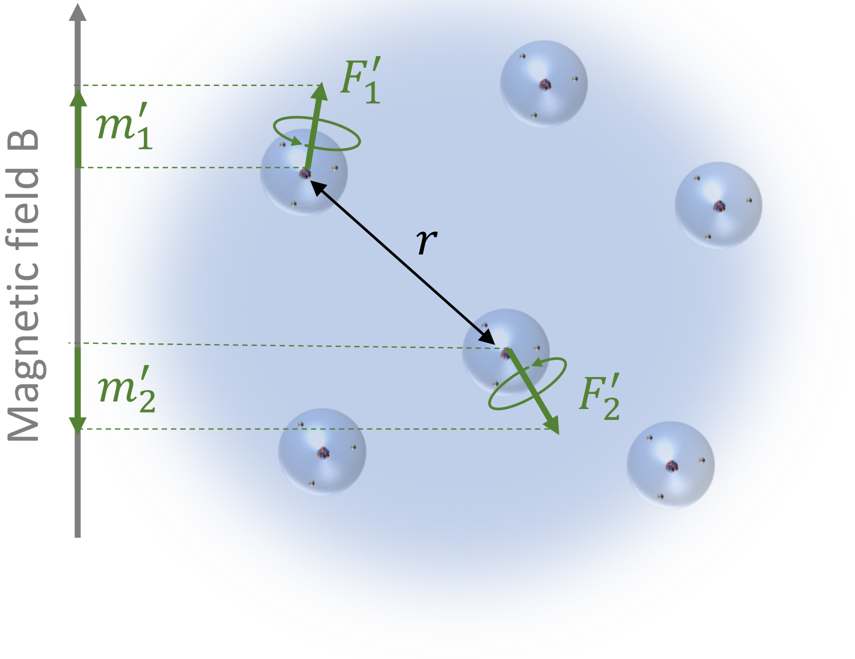

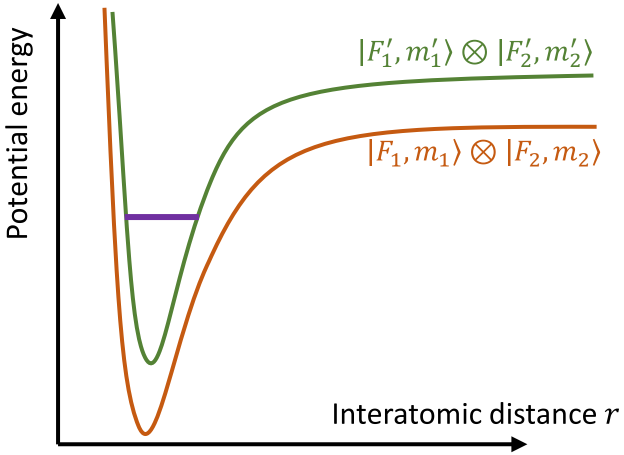

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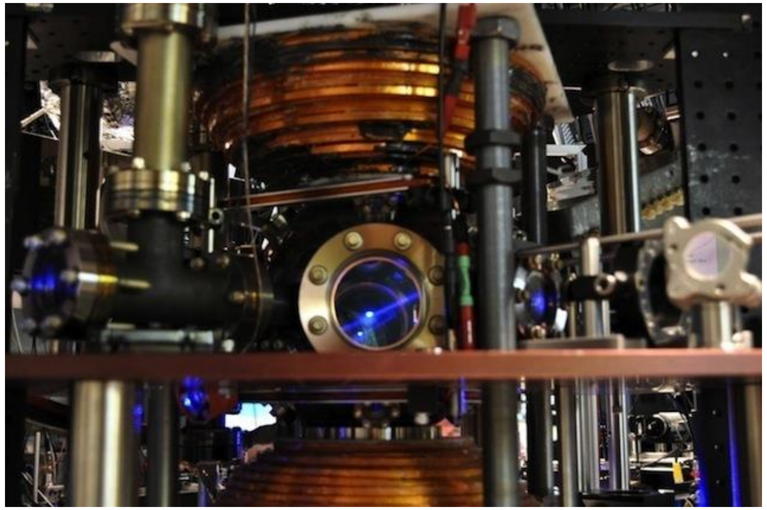
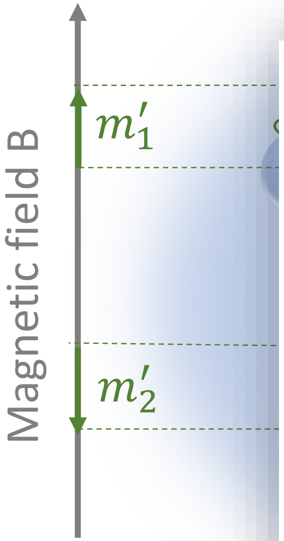
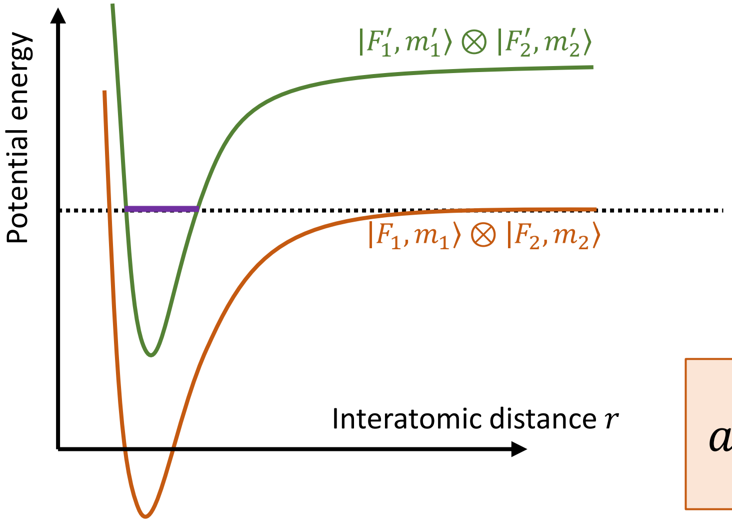
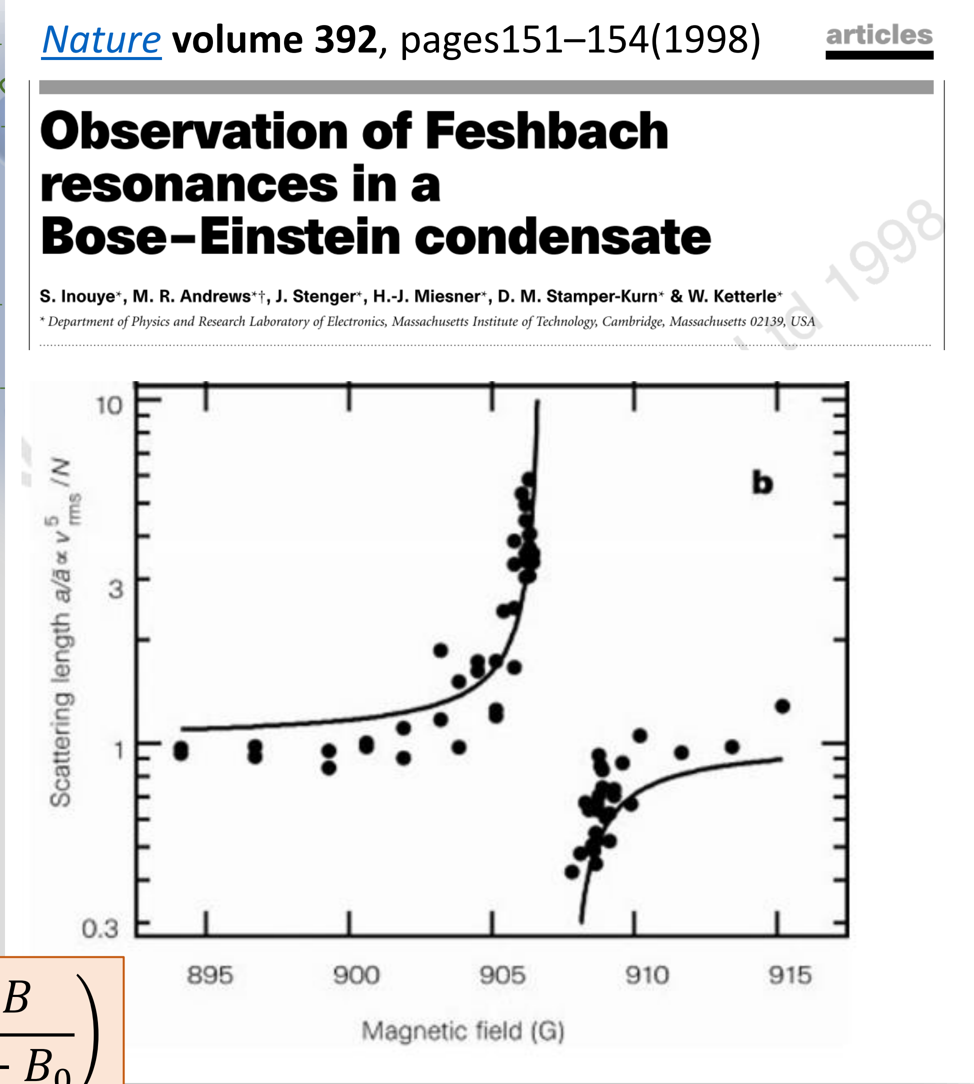






Feshbach resonance
$a \to \infty$

$$a = a_{\mathrm{bg}}\left(1 - \frac{\Delta B}{B - B_0}\right)$$



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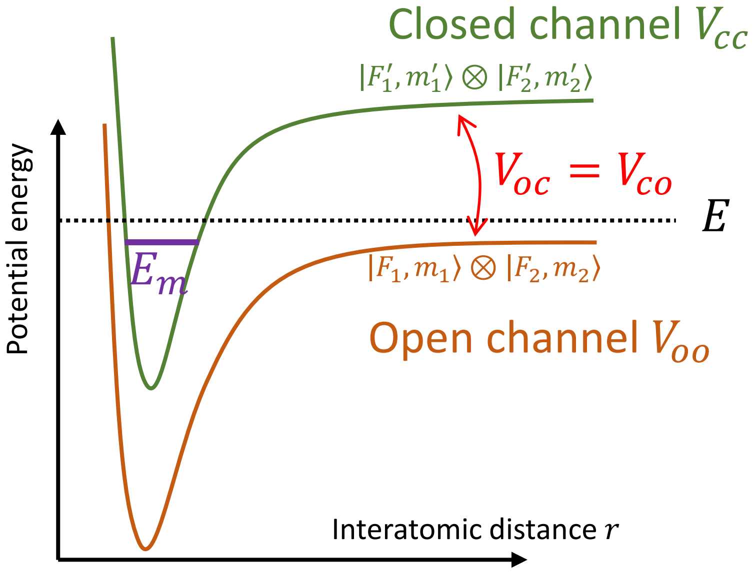
articles

Observation of Feshbach resonances in a Bose–Einstein condensate

S. Inouye*, M. R. Andrews*†, J. Stenger*, H.-J. Miesner*, D. M. Stamper-Kurn* & W. Ketterle*

* Department of Physics and Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

Two-channel model



Radial wave function (for $\ell = 0$):

$$u(r) = u_o(r)|F_1, m_1\rangle \otimes |F_2, m_2\rangle + u_c(r)|F_1', m_1'\rangle \otimes |F_2', m_2'\rangle$$

$$(K + V_{oo} - E)|u_o\rangle + V_{oc}|u_c\rangle = 0$$
$$(K + V_{cc} - E)|u_c\rangle + V_{co}|u_o\rangle = 0$$

With $K = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}$

$$|u_o\rangle = |\bar{u}_o\rangle + G_o V_{oc}|u_c\rangle$$
$$|u_c\rangle = 0 + G_c V_{co}|u_o\rangle$$

With the resolvents:

$G_o = (E - K - V_{oo})^{-1}$

$G_c = (E - K - V_{cc})^{-1}$

And the open-channel eigenstate:

$(K + V_{oo})\,|\bar{u}_o\rangle = E\,|\bar{u}_o\rangle$

$$|u_o\rangle = |\bar{u}_o\rangle + G_o V_{oc} G_c V_{co}|u_o\rangle$$

Closed-channel eigenstates:

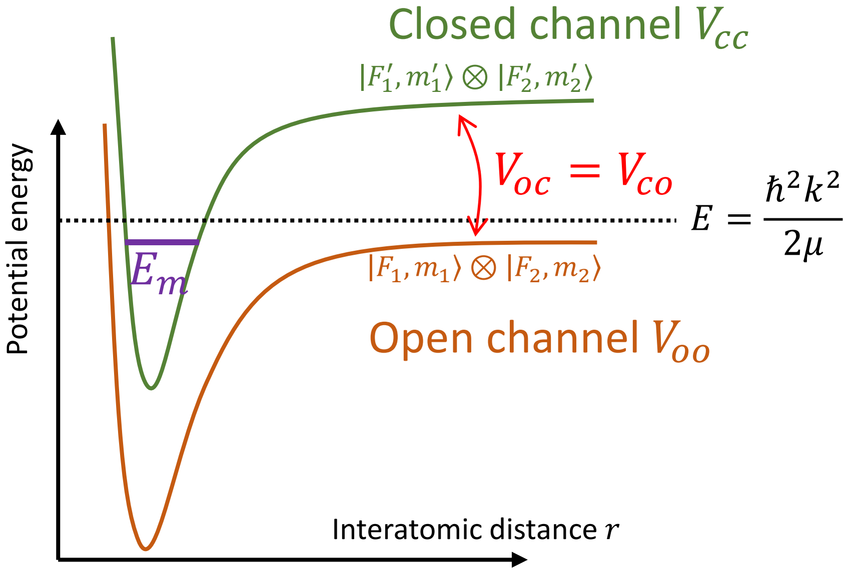
$$(K + V_{cc})\left|\bar{u}_c^{(n)}\right\rangle = E_c^{(n)}\left|\bar{u}_c^{(n)}\right\rangle$$

Isolated resonance approximation:

$$G_c = \sum_n \frac{|\bar{u}_c^{(n)}\rangle\langle\bar{u}_c^{(n)}|}{E - E_c^{(n)}} \approx \frac{|\bar{u}_m\rangle\langle\bar{u}_m|}{E - E_m}$$

$$\boxed{|u_o\rangle = |\bar{u}_o\rangle + G_o V_{oc}|\bar{u}_m\rangle \frac{\langle\bar{u}_m|V_{co}|u_o\rangle}{E - E_m}}$$

Two-channel model



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$$\boxed{|u_o\rangle = |\bar{u}_o\rangle + G_o V_{oc}|\bar{u}_m\rangle \frac{\langle \bar{u}_m|V_{co}|u_o\rangle}{E - E_m}}$$

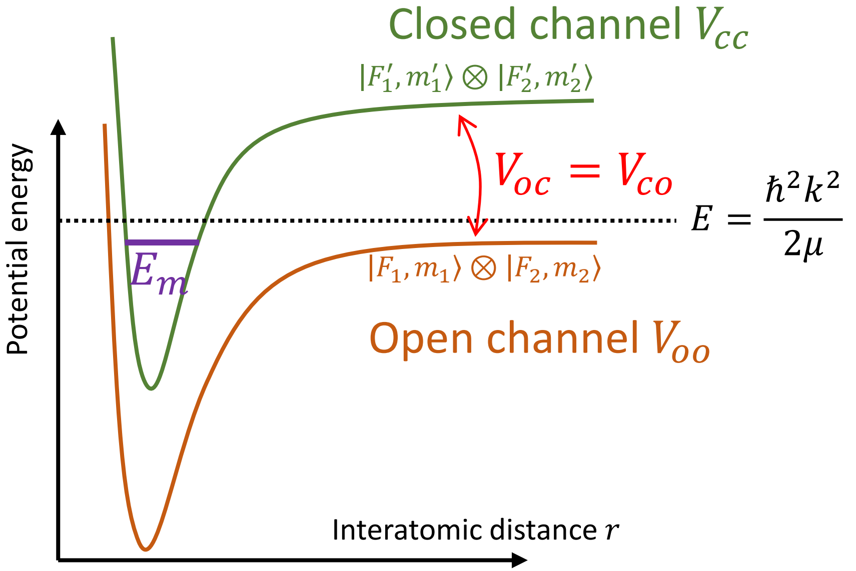
Multiply on the left by $\langle \bar{u}_m|V_{co}$

$$\langle \bar{u}_m|V_{co}|u_o\rangle = \langle \bar{u}_m|V_{co}|\bar{u}_o\rangle + \underbrace{\langle \bar{u}_m|V_{co}G_o V_{oc}|\bar{u}_m\rangle}_{\Delta E_m\text{ : resonance shift}} \frac{\langle \bar{u}_m|V_{co}|u_o\rangle}{E - E_m}$$

$$\langle \bar{u}_m|V_{co}|u_o\rangle = \frac{\langle \bar{u}_m|V_{co}|\bar{u}_o\rangle}{1 - \frac{\Delta E_m}{E - E_m}}$$

$$\boxed{|u_o\rangle = |\bar{u}_o\rangle + G_o V_{oc}|\bar{u}_m\rangle \frac{\langle \bar{u}_m|V_{co}|\bar{u}_o\rangle}{E - E_m - \Delta E_m}}$$

Two-channel model



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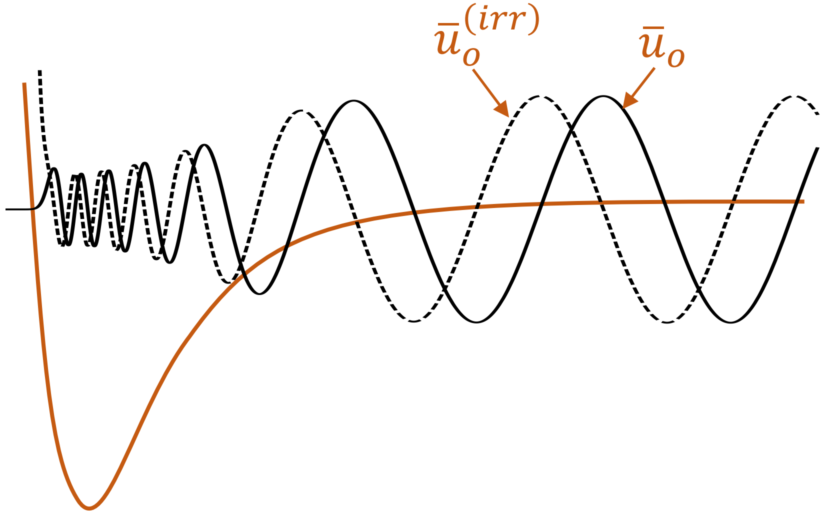
$$\boxed{|u_o\rangle = |\bar{u}_o\rangle + G_o V_{oc}|\bar{u}_m\rangle \frac{\langle \bar{u}_m|V_{co}|\bar{u}_o\rangle}{E - E_m - \Delta E_m}}$$

$$\bar{u}_o(r) \xrightarrow[r \gg b]{} k^{-1}\sin(kr + \bar{\delta})$$

$$\bar{u}_o^{(irr)}(r) \xrightarrow[r \gg b]{} k^{-1}\cos(kr + \bar{\delta})$$

(irregular solution)

$$G_o = -k|\bar{u}_o^{(irr)}\rangle\langle \bar{u}_o|$$



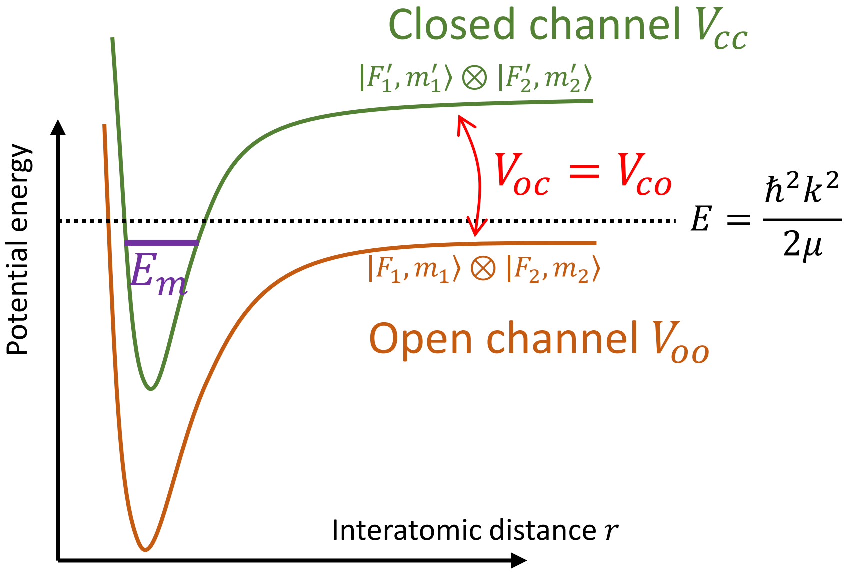
α : resonance strength

$$|u_o\rangle = |\bar{u}_o\rangle - |\bar{u}_o^{(irr)}\rangle \frac{\overbrace{k|\langle \bar{u}_m|V_{co}|\bar{u}_o\rangle|^2}^{\alpha}}{E - E_m - \Delta E_m}$$

$$u_o(r) \xrightarrow[r \gg b]{} k^{-1}\left(\sin(kr + \bar{\delta}) - \cos(kr + \bar{\delta})\frac{k\alpha}{E - E_m - \Delta E_m}\right) \propto \sin\left(kr + \underbrace{\bar{\delta} - \arctan\frac{k\alpha}{E - E_m - \Delta E_m}}\right)$$

$$\boxed{\delta = \bar{\delta} - \arctan\frac{k\alpha}{E - E_m - \Delta E_m}}$$

Two-channel model



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$$\boxed{\delta = \bar{\delta} - \arctan \frac{k\alpha}{E - E_m - \Delta E_m}}$$

Background phase shift

Resonant phase shift

For $E \to 0$, $\quad -ka \quad -ka_{\mathrm{bg}}$ Background scattering length

$$a = a_{\mathrm{bg}} - \frac{\alpha}{E_m + \Delta E_m}$$

The scattering length diverges for $E_m + \Delta E_m = 0$

The molecular state energy varies with the magnetic field (approximately linearly)

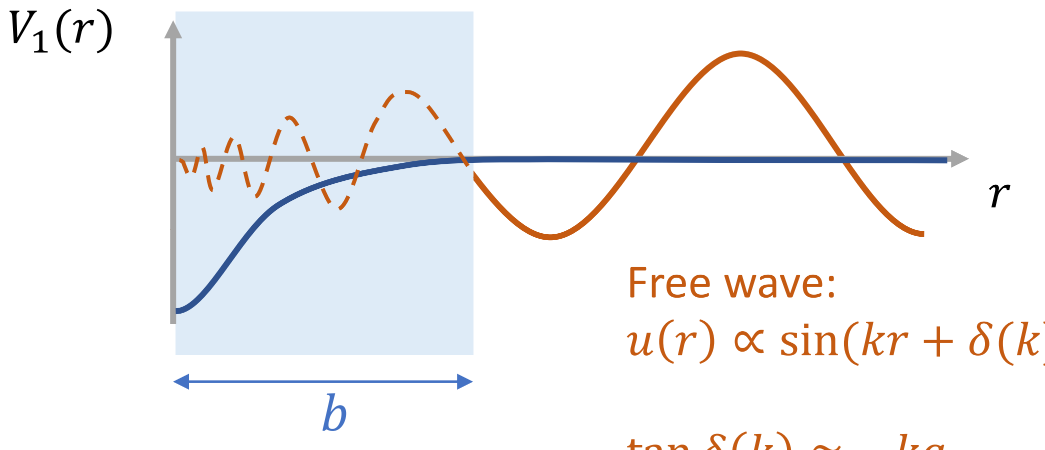
$$E_m = \mu_m (B - B_1)$$

$$\boxed{a = a_{\mathrm{bg}} \left(1 - \frac{\Delta B}{B - B_0}\right)}$$

The scattering length diverges for $B = B_0 \equiv B_1 - \Delta E_m / \mu_m$

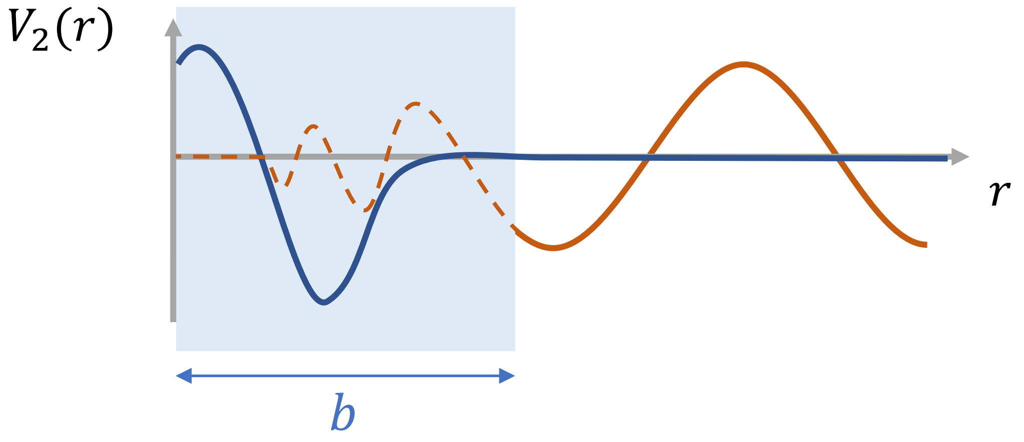
$\Delta B = \frac{\alpha}{a_{\mathrm{bg}} \mu_m}$ magnetic width of the resonance

Zero-range theory



Free wave:
$u(r) \propto \sin(kr + \delta(k))$

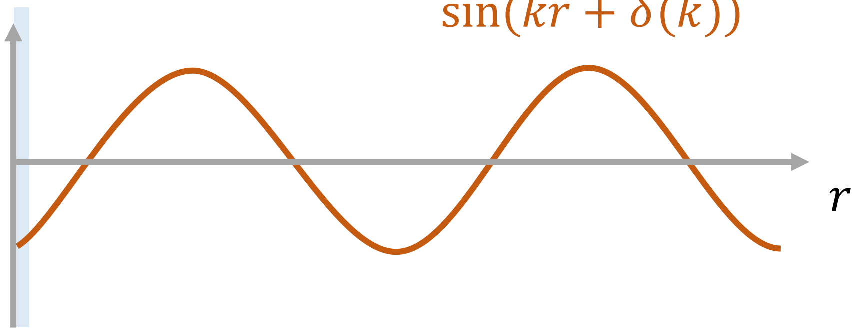
$\tan \delta(k) \approx -ka$



Free wave: $-\frac{\hbar^2}{2\mu}\nabla^2\psi = E\psi$

Zero-range Boundary condition $\begin{cases} u(0) = ? \\ u'(0) = ? \end{cases}$

Free wave:
$\sin(kr + \delta(k))$



Zero-range theory

Free wave: $-\frac{\hbar^2}{2\mu}\nabla^2\psi = E\psi$

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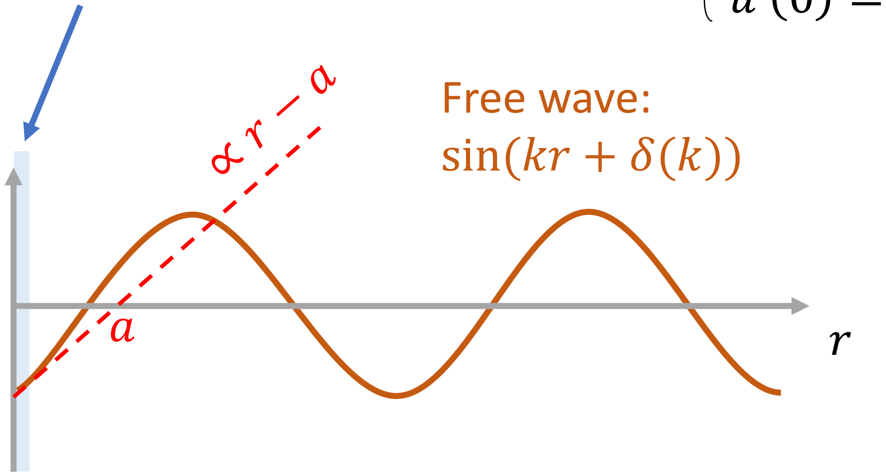


$\tan\delta(k) \approx -ka$

Free wave:
$u(r) \propto \sin(kr + \delta(k))$

$u(r) \propto \sin kr + \tan\delta(k)\cos kr$

$u(r) \propto \sin kr - ka\cos kr$

$u'(r) \propto k(\cos kr + ka\sin kr)$

$\begin{cases} u(0) \propto -ka \\ u'(0) \propto k \end{cases}$ ➡ $\boxed{\frac{u'(0)}{u(0)} = -\frac{1}{a}}$

$u(r) \propto kr - ka + O(r^2)$ ➡ $\boxed{u(r) \propto r - a}$

For $r \to 0$

Bethe-Peierls boundary condition:

$$\boxed{\frac{1}{r\psi}\frac{d}{dr}(r\psi) \xrightarrow[r\to 0]{} -\frac{1}{a}}$$

(this condition being isotropic, it only affects the s wave)

$\psi(\boldsymbol{r}) = \frac{u(r)}{r} + \cdots$

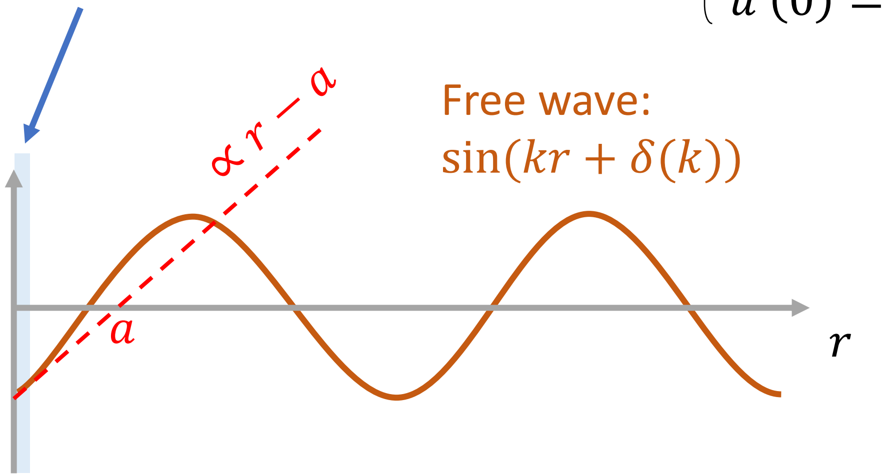
$$\boxed{\psi \xrightarrow[r\to 0]{} F \times \left(\frac{1}{r} - \frac{1}{a}\right)}$$

➡ **"universal theory" (in terms of a)**

Zero-range theory

Free wave: $-\frac{\hbar^2}{2\mu}\nabla^2\psi = E\psi$

Zero-range Boundary condition $\begin{cases} u(0) = ? \\ u'(0) = ? \end{cases}$



Bethe-Peierls boundary condition:

$$\boxed{\frac{1}{r\psi}\frac{d}{dr}(r\psi) \underset{r\to 0}{\longrightarrow} -\frac{1}{a}}$$

$$\Updownarrow$$

$$\boxed{\psi \underset{r\to 0}{\longrightarrow} F \times \left(\frac{1}{r} - \frac{1}{a}\right)}$$

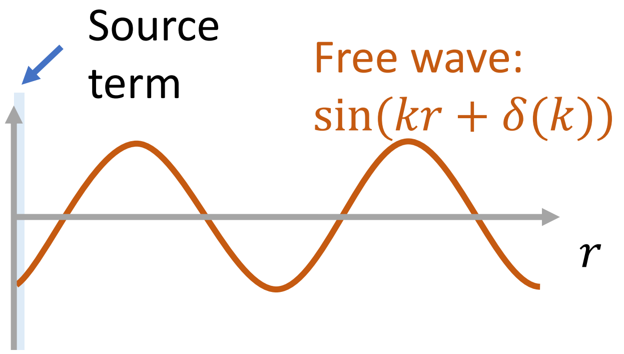
(this condition being isotropic, it only affects the s wave)

➡ **"universal theory" (in terms of a)**

Alternative formulations:

(1) "source term"

Including the boundary condition by a source term inside the Schrödinger equation:



$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + \underbrace{V(r)\psi}_{-\frac{4\pi\hbar^2}{2\mu}F\delta^3(\vec{r})} = E\psi$$

Obtained from the relation: $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r})$

(2) "Pseudopotential" (Huang-Yang, 1957)

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + \underbrace{\frac{4\pi\hbar^2 a}{2\mu}\delta^3(\vec{r})\frac{d}{dr}(r\psi)}_{\text{pseudopotential}} = E\psi$$

(3) "zero-range" or "contact" interaction

Take any potential, set the range to zero, while increasing the depth to keep a fixed scattering length a.

(3) "zero-range" or "contact" interaction

Take any potential, set the range to zero, while increasing the depth to keep a fixed scattering length a.

Example in momentum space: $\tilde{V}(p) = \begin{cases} -g & \text{for } p \leq b^{-1} \\ 0 & \text{for } p > b^{-1} \end{cases}$

We can set $b \to 0$ only at the end of calculations

$$\frac{\hbar^2 p^2}{2\mu}\tilde{\psi}(\vec{p}) + \int \frac{d^3q}{(2\pi)^3}\tilde{V}(q)\tilde{\psi}(\vec{p}-\vec{q}) = E\tilde{\psi}(\vec{p})$$

At zero energy $E = 0$:

$$p^2\tilde{\psi}(\vec{p}) - \underbrace{\frac{2\mu}{\hbar^2} g \int_{q<b^{-1}} \frac{d^3q}{(2\pi)^3}\tilde{\psi}(\vec{p}-\vec{q})}_{f \approx \frac{2\mu}{\hbar^2} g \int_{q<b^{-1}} \frac{d^3q}{(2\pi)^3}\tilde{\psi}(\vec{q})} = 0$$

$$f = \frac{2\mu}{\hbar^2} g \left(1 + f \int_{q<b^{-1}} \frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{q^2}\right)$$

$$\frac{\hbar^2}{2\mu}\frac{1}{g} = \frac{1}{f} + \underbrace{\int_{q<b^{-1}} \frac{d^3q}{(2\pi)^3}\frac{1}{q^2}}_{\frac{1}{2\pi^2}b^{-1}}$$

F.T.

$$\tilde{\psi}(\vec{p}) = (2\pi)^3\delta^3(\vec{p}) + \frac{f}{p^2}$$

$$\tilde{\psi}(\vec{r}) = 1 + \frac{f}{4\pi r} \equiv 1 - \frac{a}{r} \quad \Rightarrow \quad f = -4\pi a$$

$$\boxed{\frac{4\pi\hbar^2}{2\mu}\frac{1}{g} = -\frac{1}{a} + \frac{2}{\pi}\frac{1}{b}}$$

"Renormalisation relation"

Bethe-Peierls condition in momentum space

Schrödinger equation in coordinate space

$$\boxed{-\frac{\hbar^2}{2\mu}\nabla^2\psi-\frac{4\pi\hbar^2}{2\mu}F\delta^3(\vec{r})=E\psi}$$

F.T.

Schrödinger equation in momentum space

$$\boxed{\frac{\hbar^2p^2}{2\mu}\tilde{\psi}(\vec{p})-\frac{4\pi\hbar^2}{2\mu}F=E\tilde{\psi}(\vec{p})}$$

$$p^2\tilde{\psi}(\vec{p})-4\pi F=\frac{2\mu}{\hbar^2}E\tilde{\psi}(\vec{p})$$

$$\lim_{p\to\infty}\left(p^2\tilde{\psi}(\vec{p})-4\pi F\right)=\frac{2\mu}{\hbar^2}E\lim_{p\to\infty}\tilde{\psi}(\vec{p})=0$$

$$4\pi F=\lim_{p\to\infty}p^2\tilde{\psi}(\vec{p})$$

Condition for the source in coordinate space

$$\boxed{\psi\xrightarrow[r\to 0]{}F\times\left(\frac{1}{r}-\frac{1}{a}\right)}$$

$$\psi(r)-\frac{F}{r}\xrightarrow[r\to 0]{}-F/a$$

$$\int\frac{d^3p}{(2\pi)^3}\left(\tilde{\psi}(\vec{p})-\frac{4\pi F}{p^2}\right)e^{i\vec{p}\cdot\vec{r}}\xrightarrow[r\to 0]{}-F/a$$

$$\int\frac{d^3p}{(2\pi)^3}\left(\tilde{\psi}(\vec{p})-\frac{4\pi F}{p^2}\right)=-F/a$$

Condition for the source in momentum space

$$\boxed{F=-a\int\frac{d^3p}{(2\pi)^3}\left(\tilde{\psi}(\vec{p})-\frac{\lim_{q\to\infty}q^2\tilde{\psi}(\vec{q})}{p^2}\right)}$$

Bound state in the Zero-Range Theory

There is only one bound state in the zero-range model. It has zero angular momentum.

We look for eigenstates of energy $E = -\hbar^2\kappa^2/2\mu$

Coordinate space

Equation: $-\frac{d^2u(r)}{dr^2} = -\kappa^2 u(r)$

$$u(r) = Ae^{-\kappa r} + \cancel{Be^{+\kappa r}}$$

Condition: $\frac{du(r)}{dr} = -\frac{1}{a}u(r)$

$$-\kappa u(r) = -\frac{1}{a}u(r)$$

$$\boxed{\kappa = \frac{1}{a}}$$

$$\psi(\vec{r}) = \frac{e^{-r/a}}{\sqrt{2\pi a r}}$$

Momentum space

Equation: $p^2\,\tilde{\psi}(\vec{p}) - 4\pi F = -\kappa^2\tilde{\psi}(\vec{p})$

$$\tilde{\psi}(\vec{p}) = \frac{4\pi F}{p^2+\kappa^2}$$

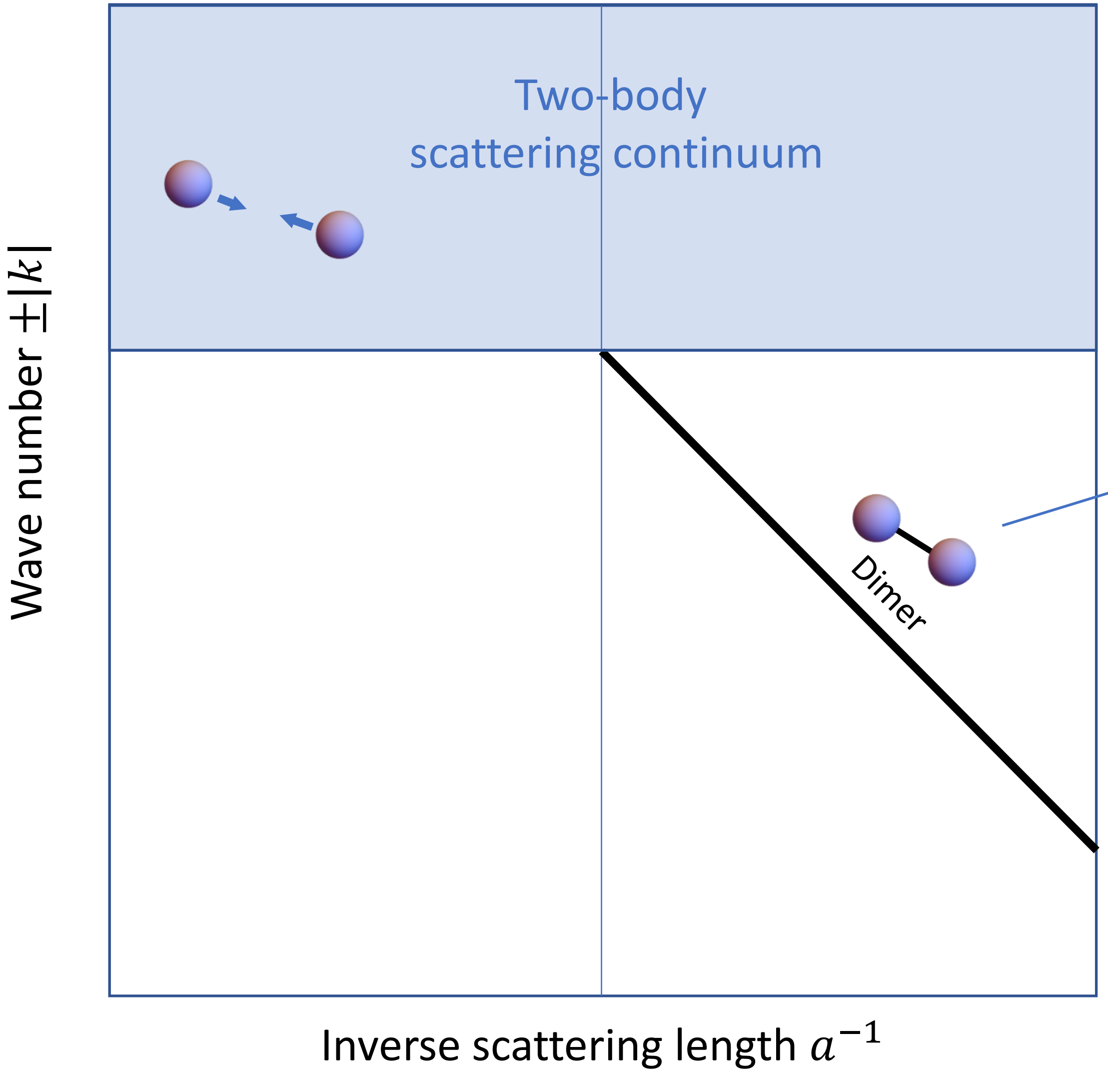
Condition: $\int \frac{d^3p}{(2\pi)^3}\left(\tilde{\psi}(\vec{p}) - \frac{4\pi F}{p^2}\right) = -F/a$

$$F\, \underbrace{4\pi \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{p^2+\kappa^2} - \frac{1}{p^2}\right)}_{-\kappa} = -F/a$$

$$\boxed{\kappa = \frac{1}{a}}$$

$$\tilde{\psi}(\vec{p}) = \frac{1}{\pi\sqrt{a}}\frac{1}{p^2+a^{-2}}$$

Two-body spectrum in the Zero-Range Theory



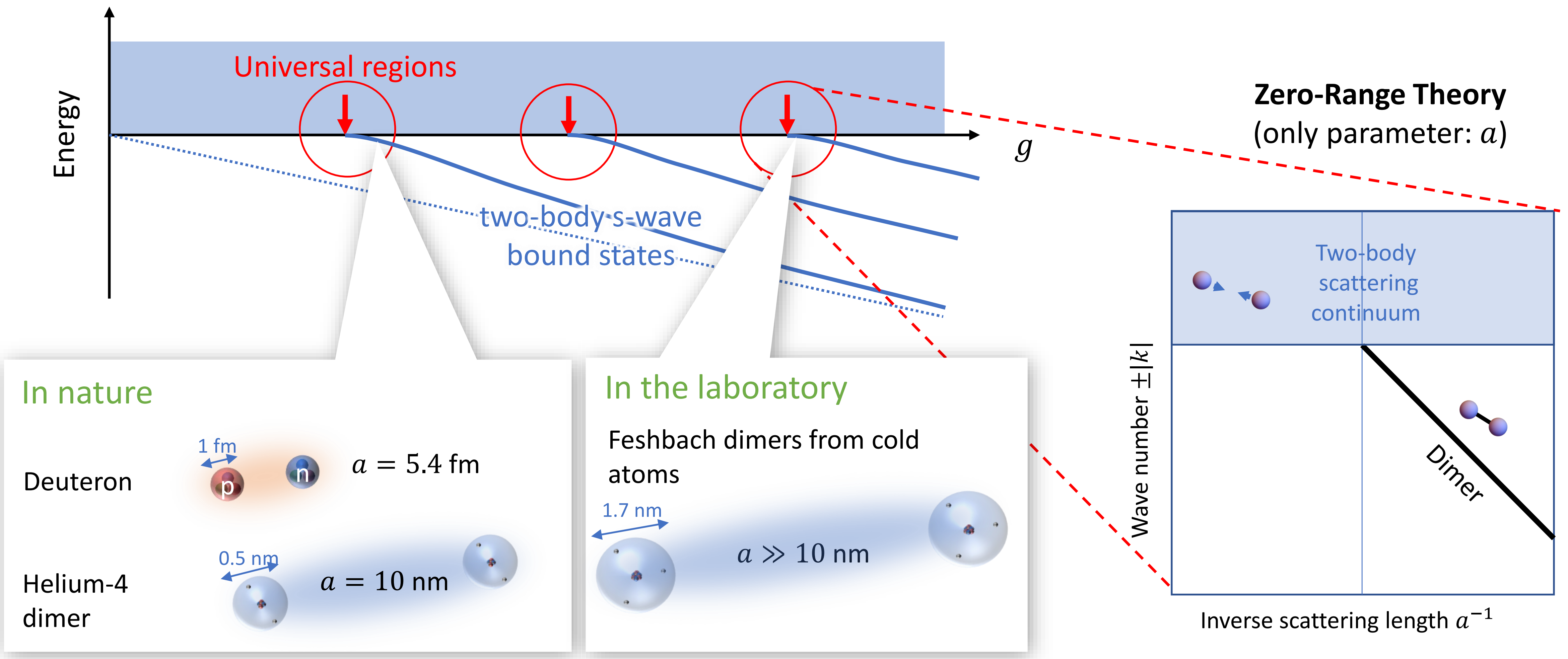
$$\psi(\vec{r}) = \frac{1}{\sqrt{2\pi a r}} e^{-r/a}$$

$$E = -\frac{\hbar^2}{2\mu}\kappa^2$$

With

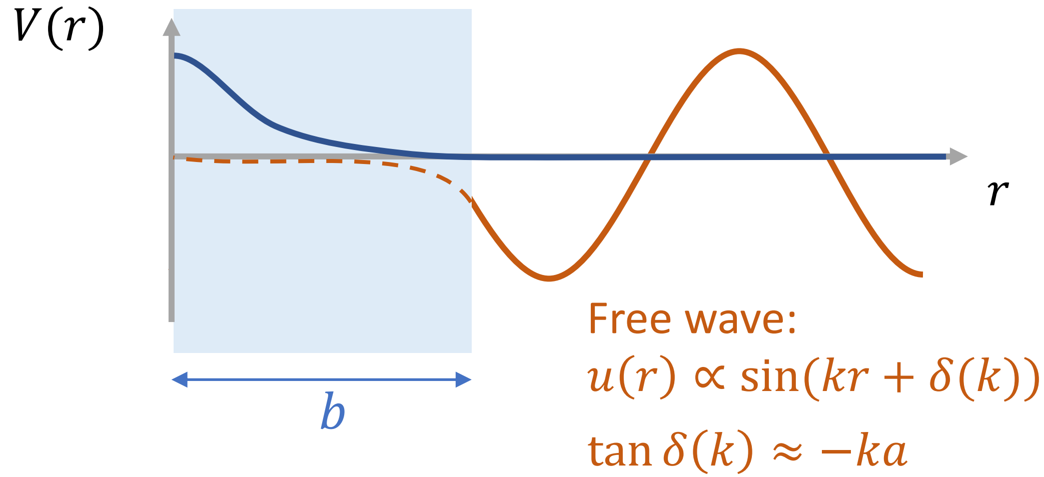
$\kappa = a^{-1}$

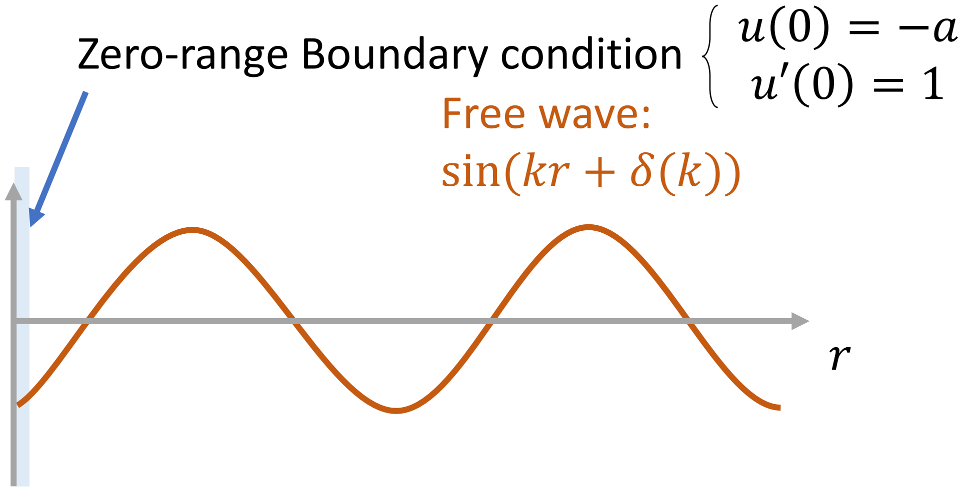
Summary



Universal "Halo dimers"

Zero-range theory for repulsive forces?





The scattering length cannot be larger than b (hard sphere limit $a = b$).

Therefore, in the limit of small b, the scattering length is always zero.

If we simply reproduce the scattering length, we retrieve the previous zero-range theory. Therefore there is a *bound state*!

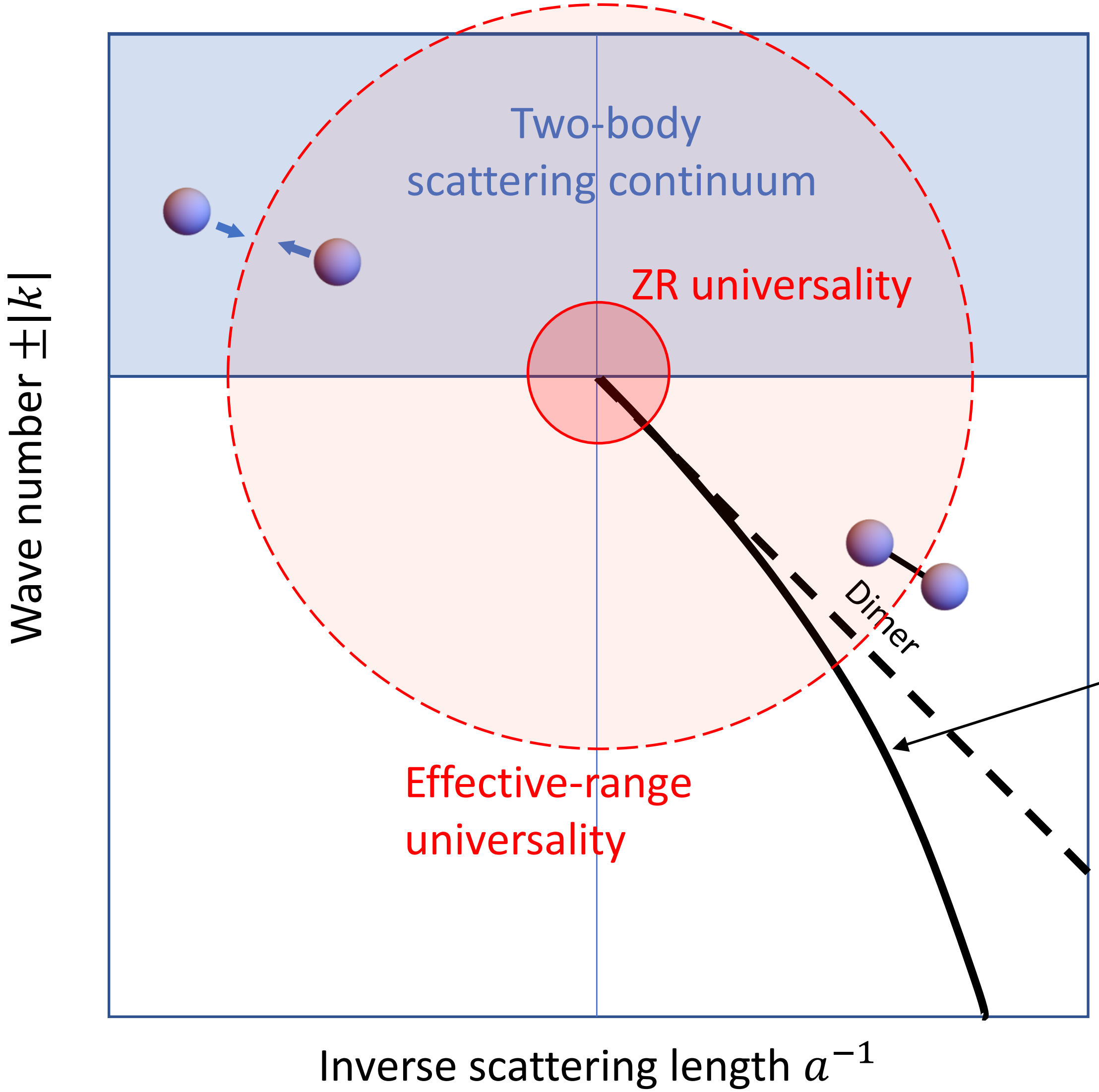
There is no zero-range limit nor universality for repulsive forces.

Other dimensions

	For $b \ll |\vec{r}| \ll k^{-1}$	Zero-range theory	Universal bound state
***D* = 3**	$\psi(r) \propto \frac{1}{|\vec{r}|} - \frac{1}{a_{3D}}$	For **attractive** forces with large scattering length $|a_{3D}| \gg b$ There is a bound state only for $a_{3D} > 0$.	$E = -\frac{\hbar^2}{2\mu a_{3D}^2} \quad (a_{3D} > 0)$
***D* = 2**	$\psi(r) \propto \ln \frac{|\vec{r}|}{a_{2D}}$	For **attractive** forces with large scattering length $a_{2D} \gg b$ The scattering length a_{2D} is always positive. There is always a bound state.	$E = -4e^{-2\gamma} \frac{\hbar^2}{2\mu a_{2D}^2}$
***D* = 1**	$\psi(r) \propto |r| - a_{1D}$	For **attractive** $(a_{1D} > 0)$ and **repulsive** $(a_{1D} < 0)$ with large $|a_{1D}| \gg b$ No regularisation: $v(r) \to g\delta(r)$ with $g = -\frac{2}{a_{1D}}$	$E = -\frac{\hbar^2}{2\mu a_{1D}^2}$

➡ Prof Doerte Blume's lecture.

Beyond zero-range theory: the effective-range theory



For $k \ll b^{-1}$, $\boxed{\dfrac{k}{\tan\delta} \approx -\dfrac{1}{a} + \dfrac{1}{2} r_e k^2 + o(k^2)}$

Scattering length

Effective range

Universal description of the low-energy two-body physics in a wider range, in terms of only **two parameters**.

Bound state:

$$\tan\delta = \frac{1}{i} \quad \Rightarrow \quad E = -\frac{\hbar^2}{m r_e^2}\left(1 - \sqrt{1 - 2r_e/a}\right)^2$$

Very useful to describe two-body systems, but less relevant to systems with more than 2 particles.

Separable theory

Zero-range potentials (contact interactions) easier to solve than finite-range potentials.
This is due to the **separability** of these potentials.

➡ *One can consider separable potentials without taking the zero-range limit!*

Local potential : $V(r)\psi(\vec{r})$ $\qquad\qquad$ $\int \frac{d^3q}{(2\pi)^3}\tilde{V}(\vec{p}-\vec{q})\tilde{\psi}(\vec{q})$

Non-local potential : $\int d^3r'\, V(\vec{r},\vec{r}\,')\psi(\vec{r}\,')$ $\qquad\qquad$ $\int \frac{d^3q}{(2\pi)^3}\tilde{V}(\vec{p},\vec{q})\tilde{\psi}(\vec{q})$

Separable potential: $\underbrace{\int d^3r'\, g}_{\text{Coupling constant}}\underbrace{\phi(\vec{r})\phi^*(\vec{r}\,')}_{\text{Form factor}}\psi(\vec{r}\,')$ $\qquad\qquad$ $\int \frac{d^3q}{(2\pi)^3} g\tilde{\phi}(\vec{p})\tilde{\phi}^*(\vec{q})\tilde{\psi}(\vec{q})$

In bra-ket notation: $\left(\hat{T} - \frac{\hbar^2}{2\mu}k^2 + g|\phi\rangle\langle\phi|\right)|\psi\rangle = 0$ with $\phi(\vec{r}) = \langle\vec{r}|\phi\rangle$ and $\tilde{\phi}(\vec{p}) = \langle\vec{p}|\phi\rangle$

- Special case: Zero-range limit

$$\tilde{\phi}(\vec{p}) = \begin{cases} 1 & \text{for } p \leq \Lambda \\ 0 & \text{for } p > \Lambda \end{cases} \quad \text{with } \Lambda \to \infty$$

Normalisation: $\int \phi(r)d^3r = \tilde{\phi}(0) \equiv 1$
So that the potential is described at low energy by the coupling constant g

Separable theory

Solution at zero energy for an attractive separable potential: $-g|\phi\rangle\langle\phi|$

$$\frac{\hbar^2 p^2}{2\mu}\tilde\psi(\vec p) - \int \frac{d^3q}{(2\pi)^3} g\tilde\phi(\vec p)\tilde\phi^*(\vec q)\tilde\psi(\vec q) = E\tilde\psi(\vec p)$$

At zero energy $E = 0$:

$$p^2\tilde\psi(\vec p) = \tilde\phi(\vec p)\underbrace{\frac{2\mu}{\hbar^2} g \int \frac{d^3q}{(2\pi)^3}\tilde\phi^*(\vec q)\tilde\psi(\vec q)}_{f}$$

$$\tilde\psi(\vec p) = (2\pi)^3\delta^3(\vec p) + \tilde\phi(\vec p)\frac{f}{p^2}$$

F.T.

$$\tilde\psi(\vec r) = 1 + f\int \frac{d^3p}{(2\pi)^3}\frac{\tilde\phi(\vec p)}{p^2}e^{i\vec p\cdot\vec r}$$

$$= 1 + \frac{f}{4\pi r} + \underbrace{f\int \frac{d^3p}{(2\pi)^3}\frac{\tilde\phi(\vec p)-1}{p^2}e^{i\vec p\cdot\vec r}}_{\xrightarrow[r\to\infty]{} 0}$$

$$f = -4\pi a$$

$$f = \frac{2\mu}{\hbar^2} g\left(1 + f\int \frac{d^3q}{(2\pi)^3}\frac{|\phi(\vec q)|^2}{q^2}\right)$$

$$\frac{\hbar^2}{2\mu}\frac{1}{g} = \frac{1}{f} + \int \frac{d^3q}{(2\pi)^3}\frac{|\phi(\vec q)|^2}{q^2}$$

$$\boxed{\frac{4\pi\hbar^2}{2\mu}\frac{1}{g} = -\frac{1}{a} + \frac{2}{\pi}\int_0^\infty dq\,|\phi(\vec q)|^2}$$

“Renormalisation relation”

Three-body physics

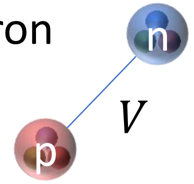
The Thomas Collapse (1935)
The Skorniakov Ter-Martirosian Equations (1955)
The Efimov breakthrough (1970)
Efimov states

The Thomas collapse (1935)

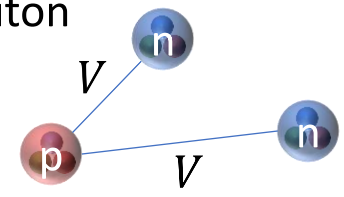
Llewellyn Thomas in 1926

"The Interaction Between a Neutron and a Proton and the Structure of H3.", Phys. Rev. 47, 903, 1935

deuteron



triton



$$H_{2B} = T_1 + T_2 + V(r_{12})$$

Ground energy: E_{2B}

$$H_{3B} = T_1 + T_2 + T_3 + V(r_{12}) + V(r_{13})$$

Ground energy: $E_{3B} \leq \frac{\langle\psi|H_{3B}|\psi\rangle}{\langle\psi|\psi\rangle}$

$$\frac{\langle\psi|H_{3B}|\psi\rangle}{\langle\psi|\psi\rangle} = -\frac{constant}{b^2}|E_{2B}|$$

For a particular ansatz ψ

Collapse:

$E_{3B} \to -\infty$ when $b \to 0$

Why? *This was a mystery*.

Estimate of *b*:

From the known ratio $\frac{E_{3B}}{E_{2B}} = 4$,

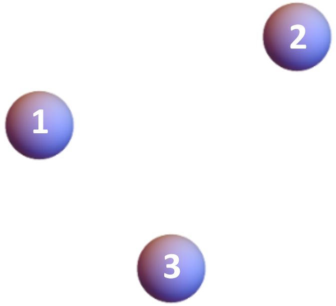
Thomas found $b \sim 6 \cdot 10^{-15}$m

The Skorniakov – Ter-Martirosian equation (1955)

Karen Avetikovich Ter-Martirosian
(undated)

G. Skorniakov and K. Ter-Martirosian, "Three Body Problem for Short Range Forces. I. Scattering of Low Energy Neutrons by Deuterons," **Sov. Phys. JETP, 4, 648, 1957**.

General three-body equation

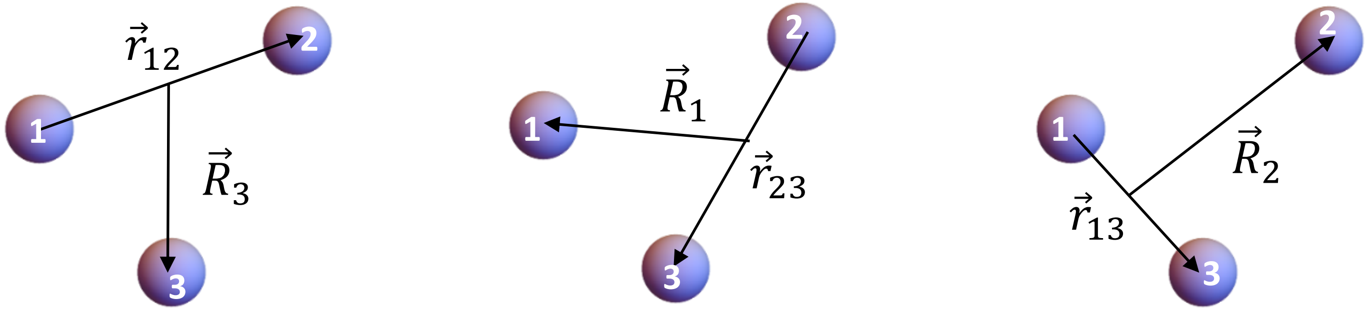


Three-body wave function: $\Psi(\vec{x}_1, \vec{x}_2, \vec{x}_3)$

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 - \frac{\hbar^2}{2m}\nabla_3^2 + V(r_{12}) + V(r_{13}) + V(r_{23})$$

1. Eliminate the centre of mass $\vec{R} = (\vec{x}_1 + \vec{x}_2 + \vec{x}_3)/3$

2. Express the remaining coordinates in terms of Jacobi coordinates: $\Psi(\vec{R}, \vec{r})$

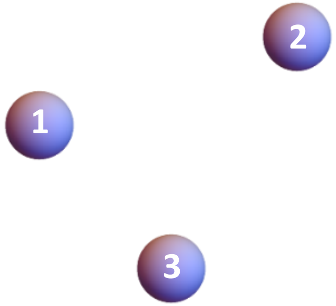


$$\vec{r}_{ij} = \vec{x}_j - \vec{x}_i$$
$$\vec{R}_k = \vec{x}_k - \frac{\vec{x}_i + \vec{x}_j}{2}$$

$$\hat{H} = -\frac{3\hbar^2}{4m}\nabla_R^2 - \frac{\hbar^2}{m}\nabla_r^2 + V(r_{12}) + V(r_{13}) + V(r_{23})$$

General three-body equation

$$\hat{H} = -\frac{3\hbar^2}{4m}\nabla_R^2 - \frac{\hbar^2}{m}\nabla_r^2 + V(r_{12}) + V(r_{13}) + V(r_{23})$$



Schrödinger equation at energy E

$$\left(-\frac{3\hbar^2}{4m}\nabla_R^2 - \frac{\hbar^2}{m}\nabla_r^2 + V(r_{12}) + V(r_{13}) + V(r_{23}) - E\right)\Psi(\vec{R},\vec{r}) = 0$$

Schrödinger equation at energy $E = -\frac{\hbar^2\kappa^2}{m} < 0$

$$\left(-\frac{3}{4}\nabla_R^2 - \nabla_r^2 + \kappa^2\right)\Psi(\vec{R},\vec{r}) \quad = -\frac{m}{\hbar^2}\big(V(r_{12}) + V(r_{13}) + V(r_{23})\big)\Psi(\vec{R},\vec{r})$$

Zero-range limit

Schrödinger equation at energy $E = -\frac{\hbar^2\kappa^2}{m} < 0$

$$\left(-\frac{3}{4}\nabla_R^2 - \nabla_r^2 + \kappa^2\right)\Psi(\vec{R},\vec{r}) \quad = -\frac{m}{\hbar^2}\left(V(r_{12}) + V(r_{13}) + V(r_{23})\right)\Psi(\vec{R},\vec{r})$$

Zero-range limit

$$\left(-\frac{3}{4}\nabla_R^2 - \nabla_r^2 + \kappa^2\right)\Psi(\vec{R},\vec{r}) = 4\pi[F_1(R_1)\delta^3(r_{23}) + F_2(R_2)\delta^3(r_{13}) + F_3(R_3)\delta^3(r_{12})]$$

with $\Psi(\vec{R}_k,\vec{r}_{ij}) \xrightarrow[r_{ij}\to 0]{} \left(\frac{1}{r_{ij}} - \frac{1}{a_{ij}}\right)F_k(\vec{R}_k)$

(Bethe-Peierls condition)

The Skorniakov & Ter-Martirosian equation

$$\left(-\frac{3}{4}\nabla_R^2 - \nabla_r^2 + \kappa^2\right)\Psi(\vec{R},\vec{r}) = 4\pi[F_1(R_1)\delta^3(r_{23}) + F_2(R_2)\delta^3(r_{13}) + F_3(R_3)\delta^3(r_{12})]$$

In momentum (Fourier) representation

with $\Psi(\vec{R}_k,\vec{r}_{ij}) \xrightarrow[r_{ij}\to 0]{} \left(\frac{1}{r_{ij}} - \frac{1}{a_{ij}}\right) F_k(\vec{R}_k)$ 2

$$\left(\frac{3}{4}P^2 + p^2 + \kappa^2\right)\tilde{\Psi}(\vec{P},\vec{p}) = 4\pi \sum_{i=1,2,3} \tilde{F}_i(\vec{P}_i)$$

$$\tilde{\Psi}(\vec{P},\vec{p}) = \frac{4\pi \sum_{i=1,2,3} \tilde{F}_i(\vec{P}_i)}{\frac{3}{4}P^2 + p^2 + \kappa^2}$$ 1

Skorniakov – Ter-Martirosian integral equations:

$$\boxed{\left(\frac{1}{a_{ij}} - \sqrt{\frac{3}{4}P^2 + \kappa^2}\right)\tilde{F}_k(\vec{P}) + 4\pi \int \frac{d^3\vec{Q}}{(2\pi)^3} \frac{\tilde{F}_i(\vec{Q}) + \tilde{F}_j(\vec{Q})}{P^2 + Q^2 + \vec{Q}\cdot\vec{P} + \kappa^2} = 0}$$

with $\{i,j,k\} = \{1,2,3\}$

Benefit of the zero-range theory:
Now, the unknown function has only 1 argument!

$\tilde{\Psi}(\vec{P},\vec{p}) \longrightarrow \tilde{F}(\vec{P})$

The Skorniakov & Ter-Martirosian equation

Detailed derivation (1/3):

$$\tilde{\Psi}(\vec{P},\vec{p}) = \frac{4\pi \sum_{i=1,2,3} \tilde{F}_i(\vec{P}_i)}{\frac{3}{4}P^2 + p^2 + \kappa^2} \quad (1)$$

with $$\Psi(\vec{R}_k,\vec{r}_{ij}) \xrightarrow[r_{ij}\to 0]{} \left(\frac{1}{r_{ij}} - \frac{1}{a_{ij}}\right) F_k(\vec{R}_k) \quad (2)$$

Consider the function $\Omega(\vec{R}_k,\vec{r}_{ij}) = \Psi(\vec{R}_k,\vec{r}_{ij}) - \frac{1}{r_{ij}} F_k(\vec{R}_k)$, which removes the $1/r_{ij}$ divergence from Ψ.

According to (2), this function goes to the finite value $-1/a_{ij}F_k(\vec{R}_k)$ when $r_{ij} \to 0$.
Let us consider its Fourier transform: $\tilde{\Omega}(\vec{P}_k,\vec{p}_{ij}) = \tilde{\Psi}(\vec{P}_k,\vec{p}_{ij}) - \frac{4\pi}{p_{ij}^2}\tilde{F}_k(\vec{P}_k)$.

We have $\Omega(\vec{R}_k,\vec{r}_{ij}) = \int \frac{d^3\vec{P}}{(2\pi)^3}\int \frac{d^3\vec{p}}{(2\pi)^3}\Omega(\vec{P},\vec{p})e^{i(\vec{P}\cdot\vec{R}_k+\vec{p}\cdot\vec{r}_{ij})}$, so $\Omega\left(\vec{R}_k,\vec{0}\right) = \int \frac{d^3\vec{P}}{(2\pi)^3}\int \frac{d^3\vec{p}}{(2\pi)^3}\Omega(\vec{P},\vec{p})e^{i\vec{P}\cdot\vec{R}_k}$, which is equal to $-1/a_{ij}F_k(\vec{R}_k)$. Taking the Fourier transform again, we arrive at $\int \frac{d^3\vec{p}}{(2\pi)^3}\Omega(\vec{P},\vec{p}) = -1/a_{ij}\tilde{F}_k(\vec{P})$
i.e.

$$\int \frac{d^3\vec{p}_{ij}}{(2\pi)^3}\left[\tilde{\Psi}(\vec{P}_k,\vec{p}_{ij}) - \frac{4\pi}{p_{ij}^2}\tilde{F}_k(\vec{P}_k)\right] = -\frac{1}{a_{ij}}\tilde{F}_k(\vec{P}_k) \quad (3)$$

The Skorniakov & Ter-Martirosian equation

Detailed derivation (2/3):

$$\tilde{\Psi}(\vec{P},\vec{p}) = \frac{4\pi \sum_{i=1,2,3} \tilde{F}_i(\vec{P}_i)}{\frac{3}{4}P^2 + p^2 + \kappa^2} \quad 1$$

with $\Psi(\vec{R}_k,\vec{r}_{ij}) \xrightarrow[r_{ij}\to 0]{} \left(\frac{1}{r_{ij}} - \frac{1}{a_{ij}}\right) F_k(\vec{R}_k)$ 2

$$\int \frac{d^3\vec{p}_{ij}}{(2\pi)^3}\left[\tilde{\Psi}(\vec{P}_k,\vec{p}_{ij}) - \frac{4\pi}{p_{ij}^2}\tilde{F}_k(\vec{P}_k)\right] = -\frac{1}{a_{ij}}\tilde{F}_k(\vec{P}_k) \quad 3$$

$$4\pi \int \frac{d^3\vec{p}_{ij}}{(2\pi)^3}\left[\frac{\tilde{F}_i(\vec{P}_i) + \tilde{F}_j(\vec{P}_j) + \tilde{F}_k(\vec{P}_k)}{\frac{3}{4}P_k^2 + p_{ij}^2 + \kappa^2} - \frac{1}{p_{ij}^2}\tilde{F}_k(\vec{P}_k)\right] = -\frac{1}{a_{ij}}\tilde{F}_k(\vec{P}_k)$$

$$\underbrace{4\pi \int \frac{d^3\vec{p}_{ij}}{(2\pi)^3}\left[\frac{1}{p_{ij}^2 + \left(\frac{3}{4}P_k^2 + \kappa^2\right)} - \frac{1}{p_{ij}^2}\right]}_{-\sqrt{\frac{3}{4}P_k^2 + \kappa^2}}\tilde{F}_k(\vec{P}_k) + 4\pi \int \frac{d^3\vec{p}_{ij}}{(2\pi)^3}\frac{\tilde{F}_i(\vec{P}_i) + \tilde{F}_j(\vec{P}_j)}{\frac{3}{4}P_k^2 + p_{ij}^2 + \kappa^2} = -\frac{1}{a_{ij}}\tilde{F}_k(\vec{P}_k)$$

The Skorniakov & Ter-Martirosian equation

Detailed derivation (3/3):

$$\left(\frac{1}{a_{ij}} - \sqrt{\frac{3}{4}P_k^2 + \kappa^2}\right)\tilde{F}_k(\vec{P}_k) + 4\pi \int \frac{d^3\vec{p}_{ij}}{(2\pi)^3} \frac{\tilde{F}_i(\vec{P}_i)}{\frac{3}{4}P_k^2 + p_{ij}^2 + \kappa^2} + 4\pi \int \frac{d^3\vec{p}_{ij}}{(2\pi)^3} \frac{\tilde{F}_j(\vec{P}_j)}{\frac{3}{4}P_k^2 + p_{ij}^2 + \kappa^2} = 0$$

Using $\vec{p}_{ij} = -\vec{P}_i - \frac{1}{2}\vec{P}_k$ and $\vec{p}_{ij} = \vec{P}_j + \frac{1}{2}\vec{P}_k$ to make a change of integration variable $\vec{p}_{ij} \to \vec{P}_i$ and $\vec{p}_{ij} \to \vec{P}_j$ in the two integrals, one obtains:

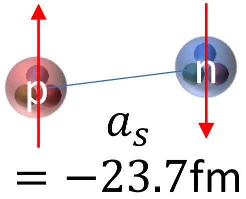
$$\left(\frac{1}{a_{ij}} - \sqrt{\frac{3}{4}P_k^2 + \kappa^2}\right)\tilde{F}_k(\vec{P}_k) + 4\pi \int \frac{d^3\vec{P}_i}{(2\pi)^3} \frac{\tilde{F}_i(\vec{P}_i)}{\frac{3}{4}P_k^2 + \left(\vec{P}_i + \frac{1}{2}\vec{P}_k\right)^2 + \kappa^2} + 4\pi \int \frac{d^3\vec{P}_j}{(2\pi)^3} \frac{\tilde{F}_j(\vec{P}_j)}{\frac{3}{4}P_k^2 + \left(\vec{P}_j + \frac{1}{2}\vec{P}_k\right)^2 + \kappa^2} = 0$$

Finally, relabelling the integration variables $\vec{P}_i$ and $\vec{P}_j$ as $\vec{Q}$, one arrives at the Skorniakov – Ter-Martirosian equations:

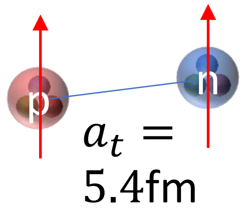
$$\boxed{\left(\frac{1}{a_{ij}} - \sqrt{\frac{3}{4}P^2 + \kappa^2}\right)\tilde{F}_k(\vec{P}) + 4\pi \int \frac{d^3\vec{Q}}{(2\pi)^3} \frac{\tilde{F}_i\left(\vec{Q}\right) + \tilde{F}_j\left(\vec{Q}\right)}{P^2 + Q^2 + \vec{Q}\cdot\vec{P} + \kappa^2} = 0}$$

Application to nucleons:

Nucleon interaction:



Singlet $S = 0$
$|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$

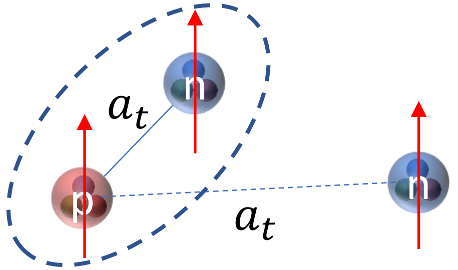


Triplet $S = 1$
$|\uparrow\rangle|\uparrow\rangle$
$|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle$
$|\downarrow\rangle|\downarrow\rangle$

Deuteron-neutron scattering:

deuteron ($s' = 1$) neutron (s$= 1/2$)

$S = 3/2$



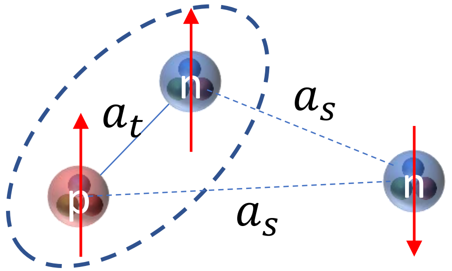
$F_3 = -F_2 \equiv F$ $F_1 = 0$

$$\boxed{\left(\frac{1}{a_t} - \sqrt{\frac{3}{4}P^2 + \kappa^2}\right)\tilde{F}(\vec{P}) - 4\pi\int\frac{d^3\vec{Q}}{(2\pi)^3}\frac{\tilde{F}(\vec{Q})}{P^2 + Q^2 + \vec{Q}\cdot\vec{P} + \kappa^2} = 0}$$

Neutron-deuteron scatt. Length: $\boxed{a_{nd} = 1.18a_t}$

Universal result

$S = 1/2$



Simplified version: $a_t = a_s$ $F_1 = F_2 = F_3 \equiv F$

(equivalent to the problem of 3 identical bosons)

$$\boxed{\left(\frac{1}{a_t} - \sqrt{\frac{3}{4}P^2 + \kappa^2}\right)\tilde{F}(\vec{P}) + 8\pi\int\frac{d^3\vec{Q}}{(2\pi)^3}\frac{\tilde{F}(\vec{Q})}{P^2 + Q^2 + \vec{Q}\cdot\vec{P} + \kappa^2} = 0}$$

Problem: energy is not bound from below!

The analytical solution of Minlos-Faddeev (1961)

Skorniakov – Ter-Martirosian integral equation for three bosons at unitarity ($a \to \infty$):

$$\left(\cancel{\frac{1}{a}} - \sqrt{\frac{3}{4}P^2 + \kappa^2}\right)\tilde{F}(P) + \frac{2}{\pi}\int_0^\infty \frac{Q}{P}dQ \ln\frac{Q^2 + P^2 + PQ + \kappa^2}{Q^2 + P^2 - PQ + \kappa^2}\tilde{F}(Q) = 0$$

1. Extension of integration to $[-\infty, \infty]$

$$\left(0 - \sqrt{\frac{3}{4}P^2 + \kappa^2}\right)\tilde{F}(P) + \frac{1}{\pi}\int_{-\infty}^\infty \frac{Q}{P}dQ \ln\frac{Q^2 + P^2 + PQ + \kappa^2}{Q^2 + P^2 - PQ + \kappa^2}\tilde{F}(Q) = 0$$

2. Change of variables: $P = \frac{1}{\sqrt{3}}\kappa\left(z - \frac{1}{z}\right)$ $\quad Q = \frac{1}{\sqrt{3}}\kappa(z' - 1/z')$

$$-g(z) + \frac{4}{\pi\sqrt{3}}\int_0^\infty \frac{dz'}{z'}\ln\left(\frac{z'^2 + z^2 + zz'}{z'^2 + z^2 - zz'}\right)g(z') = 0$$

where $g(z) \equiv \left(z^2 - \frac{1}{z^2}\right)\tilde{F}(P)$

The analytical solution of Minlos-Faddeev (1961)

$$-g(z) + \frac{4}{\pi\sqrt{3}} \int_0^\infty \frac{dz'}{z'} \ln\left(\frac{z'^2 + z^2 + zz'}{z'^2 + z^2 - zz'}\right) g(z') = 0$$

where $g(z) \equiv \left(z^2 - \frac{1}{z^2}\right) F(P)$

Scale invariance: $z \to \lambda z$ (if $g(z)$ is a solution, then $g(\lambda z)$ is also a solution)

3. Solution of the form: $g(z) = z^s$

$$-z^s + \frac{4}{\pi\sqrt{3}} \int_0^\infty dz' \ln\left(\frac{z'^2 + z^2 + zz'}{z'^2 + z^2 - zz'}\right) z'^{s-1} = 0$$

$$-1 + \frac{4}{\pi\sqrt{3}} \underbrace{\int_0^\infty dx' \ln\left(\frac{1 + x^2 + x}{1 + x^2 - x}\right) x^{s-1}}_{\frac{2\pi}{s}\frac{\sin\left(\frac{\pi}{6}s\right)}{\cos\left(\frac{\pi}{2}s\right)}} = 0$$

for $-1 < \mathrm{Re}(s) < 1$

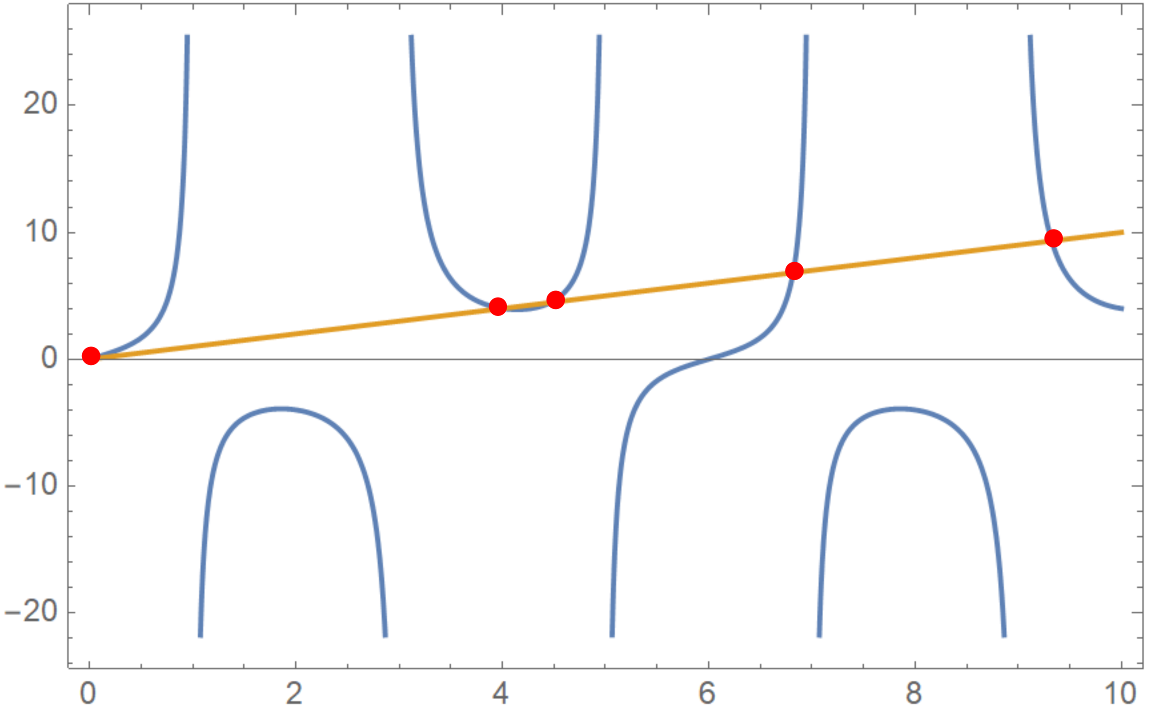
Transcendental equation:

$$\boxed{s = \frac{8\pi \sin\left(\frac{\pi}{6}s\right)}{\sqrt{3}\cos\left(\frac{\pi}{2}s\right)}}$$

3. Solution of the form: $g(z) = z^s$

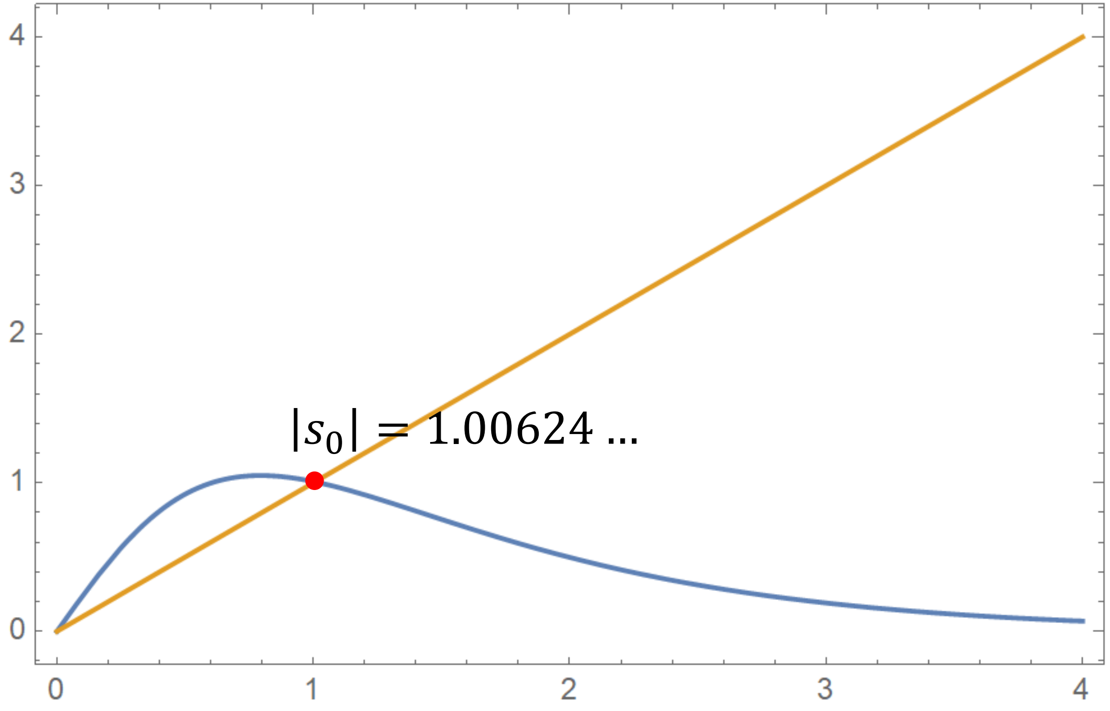
$$\boxed{s = \frac{8\pi \sin\left(\frac{\pi}{6} s\right)}{\sqrt{3}\cos\left(\frac{\pi}{2} s\right)}}$$

Real solutions $s = \pm|s|$



$|s|$

Imaginary solutions $s = \pm i|s|$



$|s|$

3. Solution of the form: $g(z) = z^s$

$$\boxed{s = \frac{8\pi \sin\left(\frac{\pi}{6}s\right)}{\sqrt{3}\cos\left(\frac{\pi}{2}s\right)}}$$

$$g(z) = C_+ z^{i|s_0|} + C_- z^{-i|s_0|}$$

Since $g(z) = (z^2 - z^{-2})\tilde{F}(P)$, we have $g(1) = 0$, therefore $C_+ = -C_- \equiv C$.

$$g(z) = C\left(z^{i|s_0|} - z^{-i|s_0|}\right)$$

4. Going back to the original variables, we obtain the solution:

$$\boxed{\tilde{F}(P) \propto \frac{1}{P\sqrt{1+\frac{3P^2}{4\kappa^2}}} \sin\left(|s_0| \operatorname{arcsinh}\frac{\sqrt{3}P}{2\kappa}\right)}$$

Solution valid for any κ, i.e. any negative energy!
(consistent with Thomas collapse)

Obviously something is wrong

Vitaly Efimov in 1977

Efimov's breakthrough (1970)

V. Efimov, "Weakly-bound states of three resonantly-interacting particles," **Yad. Fiz., 12, 1080–1091**, November 1970, [**Sov. J. Nucl. Phys. 12, 589-595 (1971)**].

V. Efimov, "Energy levels arising from resonant two-body forces in a three-body system." **Physics Letters B, 33, 563 – 564, 1970**.

Derivation for three identical bosons

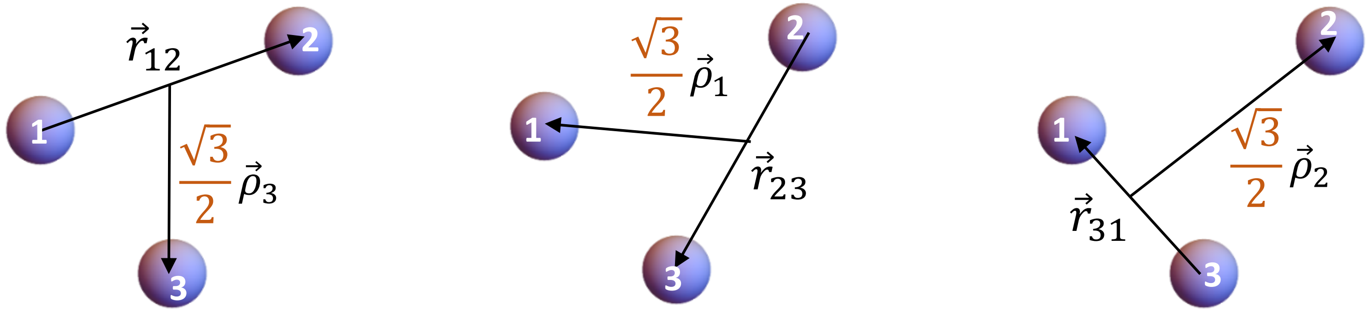
Hamiltonian in coordinate representation:

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 - \frac{\hbar^2}{2m}\nabla_3^2$$

with the two-body condition $\Psi \xrightarrow[r_{ij}\to 0]{} \propto \frac{1}{r_{ij}} - \frac{1}{a}$

1. Eliminate the centre of mass $\vec{R} = \vec{x}_1 + \vec{x}_2 + \vec{x}_3$

2. Express the remaining coordinates in terms of Jacobi coordinates:



$$\vec{r}_{ij} = \vec{x}_j - \vec{x}_i$$

$$\frac{\sqrt{3}}{2}\vec{\rho}_k = \vec{x}_k - \frac{\vec{x}_i + \vec{x}_j}{2}$$

Schrödinger equation:

$$\left(-\nabla_{r_{12}}^2 - \nabla_{\rho_3}^2 - k^2\right)\Psi = 0$$

For a total energy $E = \hbar^2 k^2/m$

Derivation for three identical bosons

3. Make the Faddeev decomposition:

$$\Psi = \chi(\vec{r}_{12}, \vec{\rho}_3) + \chi(\vec{r}_{23}, \vec{\rho}_1) + \chi(\vec{r}_{31}, \vec{\rho}_2)$$
$$= \chi(\vec{r}, \vec{\rho}) + \chi\left(-\tfrac{1}{2}\vec{r} + \tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{\sqrt{3}}{2}\vec{r} - \tfrac{1}{2}\vec{\rho}\right) + \chi\left(-\tfrac{1}{2}\vec{r} - \tfrac{\sqrt{3}}{2}\vec{\rho}, \tfrac{\sqrt{3}}{2}\vec{r} - \tfrac{1}{2}\vec{\rho}\right)$$

Where χ satisfies

$$\left(-\nabla_r^2 - \nabla_\rho^2 - k^2\right)\chi(\vec{r}, \vec{\rho}) = 0$$

4. Apply the two-body condition $\Psi \xrightarrow[r_{ij} \to 0]{} \propto \frac{1}{r_{ij}} - \frac{1}{a}$ ⟺ $\frac{\partial}{\partial r}(r\Psi) = -\frac{1}{a}(r\Psi)$ for $r \to 0$

Derivation for three identical bosons

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$$\left[\frac{\partial}{\partial r}\left(r\chi(\vec{r},\vec{\rho})\right)\right]_{r\to 0} + \left[\frac{\partial}{\partial r}\left(r\chi\left(\cancel{-\tfrac{1}{2}\vec{r}} + \tfrac{\sqrt{3}}{2}\vec{\rho}, \cancel{-\tfrac{\sqrt{3}}{2}\vec{r}} - \tfrac{1}{2}\vec{\rho}\right)\right)\right]_{r\to 0} + \left[\frac{\partial}{\partial r}\left(r\chi\left(\cancel{-\tfrac{1}{2}\vec{r}} - \tfrac{\sqrt{3}}{2}\vec{\rho}, \cancel{\tfrac{\sqrt{3}}{2}\vec{r}} - \tfrac{1}{2}\vec{\rho}\right)\right)\right]_{r\to 0}$$

$$= -\frac{1}{a}\left[r\left(\chi(\vec{r},\vec{\rho}) + \chi\left(\cancel{-\tfrac{1}{2}\vec{r}} + \tfrac{\sqrt{3}}{2}\vec{\rho}, \cancel{-\tfrac{\sqrt{3}}{2}\vec{r}} - \tfrac{1}{2}\vec{\rho}\right) + \chi\left(\cancel{-\tfrac{1}{2}\vec{r}} - \tfrac{\sqrt{3}}{2}\vec{\rho}, \cancel{\tfrac{\sqrt{3}}{2}\vec{r}} - \tfrac{1}{2}\vec{\rho}\right)\right)\right]_{r\to 0}$$

$$\left[\frac{\partial}{\partial r}\left(r\chi(\vec{r},\vec{\rho})\right)\right]_{r\to 0} + \chi\left(\tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{1}{2}\vec{\rho}\right) + \chi\left(-\tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{1}{2}\vec{\rho}\right)$$

$$= -\frac{1}{a}\left[r\left(\chi(\vec{r},\vec{\rho}) + \cancel{\chi\left(\tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{1}{2}\vec{\rho}\right)} + \cancel{\chi\left(-\tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{1}{2}\vec{\rho}\right)}\right)\right]_{r\to 0}$$

Derivation for three identical bosons

3. Make the Faddeev decomposition:

$$\Psi = \chi(\vec{r}_{12},\vec{\rho}_3) + \chi(\vec{r}_{23},\vec{\rho}_1) + \chi(\vec{r}_{31},\vec{\rho}_2)$$
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Where χ satisfies

$$\left(-\nabla_r^2 - \nabla_\rho^2 - k^2\right)\chi(\vec{r},\vec{\rho}) = 0$$

4. Apply the two-body condition $\Psi \xrightarrow[r_{ij}\to 0]{} \propto \frac{1}{r_{ij}} - \frac{1}{a}$ ⟺ $\frac{\partial}{\partial r}(r\Psi) = -\frac{1}{a}(r\Psi)$ for $r \to 0$

$$\left[\frac{\partial}{\partial r}\left(r\chi(\vec{r},\vec{\rho})\right)\right]_{r\to 0} + \left[\frac{\partial}{\partial r}\left(r\chi\left(-\cancel{\tfrac{1}{2}\vec{r}} + \tfrac{\sqrt{3}}{2}\vec{\rho}, -\cancel{\tfrac{\sqrt{3}}{2}\vec{r}} - \tfrac{1}{2}\vec{\rho}\right)\right)\right]_{r\to 0} + \left[\frac{\partial}{\partial r}\left(r\chi\left(-\cancel{\tfrac{1}{2}\vec{r}} - \tfrac{\sqrt{3}}{2}\vec{\rho}, \cancel{\tfrac{\sqrt{3}}{2}\vec{r}} - \tfrac{1}{2}\vec{\rho}\right)\right)\right]_{r\to 0}$$

$$= -\frac{1}{a}\left[r\left(\chi(\vec{r},\vec{\rho}) + \chi\left(-\cancel{\tfrac{1}{2}\vec{r}} + \tfrac{\sqrt{3}}{2}\vec{\rho}, -\cancel{\tfrac{\sqrt{3}}{2}\vec{r}} - \tfrac{1}{2}\vec{\rho}\right) + \chi\left(-\cancel{\tfrac{1}{2}\vec{r}} - \tfrac{\sqrt{3}}{2}\vec{\rho}, \cancel{\tfrac{\sqrt{3}}{2}\vec{r}} - \tfrac{1}{2}\vec{\rho}\right)\right)\right]_{r\to 0}$$

$$\left[\frac{\partial}{\partial r}\left(r\chi(\vec{r},\vec{\rho})\right)\right]_{r\to 0} + \chi\left(\tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{1}{2}\vec{\rho}\right) + \chi\left(-\tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{1}{2}\vec{\rho}\right)$$

$$= -\frac{1}{a}\left[r\chi(\vec{r},\vec{\rho})\right]_{r\to 0}$$

Derivation for three identical bosons

Equation:

$$\left(-\nabla_r^2 - \nabla_\rho^2 - k^2\right)\chi(\vec{r},\vec{\rho}) = 0$$

Boundary condition $r \to 0$:

$$\left[\frac{\partial}{\partial r}\left(r\chi(\vec{r},\vec{\rho})\right)\right]_{r\to 0} + \chi\left(\tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{1}{2}\vec{\rho}\right) + \chi\left(-\tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{1}{2}\vec{\rho}\right) = -\frac{1}{a}\left[r\chi(\vec{r},\vec{\rho})\right]_{r\to 0}$$

5. Expand χ in partial waves. For a total angular momentum $L = 0$,

$$\chi(\vec{r},\vec{\rho}) = \frac{\chi_0(r,\rho)}{r\rho}$$

$$\Rightarrow \left(-\frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{1}{\rho}\frac{\partial^2}{\partial \rho^2}\rho - k^2\right)\frac{\chi_0(r,\rho)}{r\rho} = 0 \quad \text{with} \quad \left[\frac{\partial}{\partial r}\frac{\chi_0(r,\rho)}{\rho}\right]_{r\to 0} + 2\times\frac{\chi_0\left(\frac{\sqrt{3}}{2}\rho, \frac{1}{2}\rho\right)}{\frac{\sqrt{3}}{2}\rho\cdot\frac{1}{2}\rho} = -\frac{1}{a}\frac{\chi_0(0,\rho)}{\rho}$$

$$\Rightarrow \left(-\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial \rho^2} - k^2\right)\chi_0(r,\rho) = 0 \quad \text{with} \quad \left[\frac{\partial}{\partial r}\chi_0(r,\rho)\right]_{r\to 0} + \frac{8}{\sqrt{3}\rho}\chi_0\left(\tfrac{\sqrt{3}}{2}\rho, \tfrac{1}{2}\rho\right) = -\frac{1}{a}\chi_0(0,\rho)$$

Derivation for three identical bosons

Equation:

$$\left(-\frac{\partial^2}{\partial r^2}-\frac{\partial^2}{\partial \rho^2}-k^2\right)\chi_0(r,\rho)=0$$

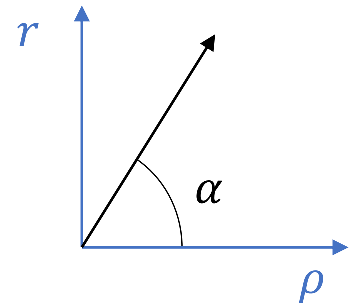
Boundary condition $r \to 0$:

$$\left[\frac{\partial}{\partial r}\chi_0(r,\rho)\right]_{r\to 0}+\frac{8}{\sqrt{3}\rho}\chi_0\left(\tfrac{\sqrt{3}}{2}\rho,\tfrac{1}{2}\rho\right)=-\frac{1}{a}\chi_0(0,\rho)$$

6. Change the coordinates (r,ρ) to polar coordinates (R,α)

$r = R\sin\alpha$ $R=\sqrt{r^2+\rho^2}$ (hyper-radius)

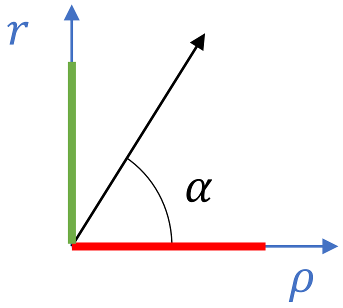
$\rho = R\cos\alpha$ $\alpha = \arctan r/\rho$ (hyper-angle)



$$\left(-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\frac{\partial^2}{\partial \alpha^2}-k^2\right)\chi_0(R,\alpha)=0$$

with $\left[\frac{\partial}{\partial \alpha}\chi_0(R,\alpha)\right]_{\alpha\to 0}+\frac{8}{\sqrt{3}}\chi_0\left(R,\frac{\pi}{3}\right)=-\frac{R}{a}\chi_0(R,0)$

Derivation for three identical bosons



Equation:

$$\left(-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\frac{\partial^2}{\partial \alpha^2}-k^2\right)\chi_0(R,\alpha)=0$$

Boundary condition $\alpha \to 0$:

$$\left[\frac{\partial}{\partial\alpha}\chi_0(R,\alpha)\right]_{\alpha\to 0}+\frac{8}{\sqrt{3}}\chi_0\left(R,\frac{\pi}{3}\right)=-\frac{R}{a}\chi_0(R,0)$$

Boundary condition $\alpha \to \frac{\pi}{2}$: $\quad \chi_0\left(R,\frac{\pi}{2}\right)=0$

$$\chi(\vec{r},\vec{\rho})=\frac{\chi_0(r,\rho)}{r\rho} \implies [\chi_0(r,\rho)]_{\rho\to 0}=0$$

7. **At unitarity** $a\to\infty$, the two boundary conditions are independent of R.
Therefore, the problem becomes separable in R and α

Solutions of the form: $\quad \chi_0(R,\alpha)=F_n(R)\phi_n(\alpha)$

Boundary condition $\alpha \to \frac{\pi}{2}$: OK

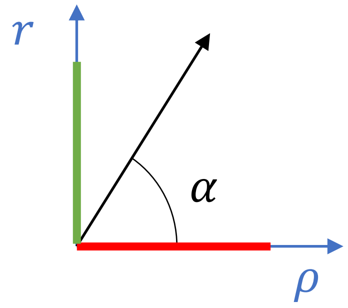
Eigenfunctions of $-\partial^2/\partial\alpha^2$:

$$-\frac{\partial^2}{\partial\alpha^2}\phi_n(\alpha)=s_n^2\phi_n(\alpha) \quad \Rightarrow \quad \phi_n(\alpha)=\sin\left(s_n\left(\frac{\pi}{2}-\alpha\right)\right)$$

Boundary condition $\alpha \to 0$:

$$s_n\cos\left(\frac{s_n\pi}{2}\right)+\frac{8}{\sqrt{3}}\sin\left(\frac{s_n\pi}{6}\right)=0$$

Derivation for three identical bosons



Equation:

$$\left(-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\frac{\partial^2}{\partial \alpha^2}-k^2\right)\chi_0(R,\alpha)=0$$

Solutions of the form: $\chi_0(R,\alpha)=F_n(R)\phi_n(\alpha)$

Eigenfunctions of $-\partial^2/\partial\alpha^2$:

$$-\frac{\partial^2}{\partial\alpha^2}\phi_n(\alpha)=s_n^2\phi_n(\alpha) \quad \Rightarrow \quad \phi_n(\alpha)=\sin\left(s_n\left(\frac{\pi}{2}-\alpha\right)\right) \qquad s_n\cos\left(\frac{s_n\pi}{2}\right)+\frac{8}{\sqrt{3}}\sin\left(\frac{s_n\pi}{6}\right)=0$$

$$\left(-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}+\frac{s_n^2}{R^2}-k^2\right)F_n(R)=0$$

$$\left(-\frac{\partial^2}{\partial R^2}+\underbrace{\boxed{\frac{s_n^2-\frac{1}{4}}{R^2}}}_{V_n(R)}-k^2\right)\sqrt{R}F_n(R)=0$$

Derivation for three identical bosons

Effective Schrödinger equation for the hyper-radius R

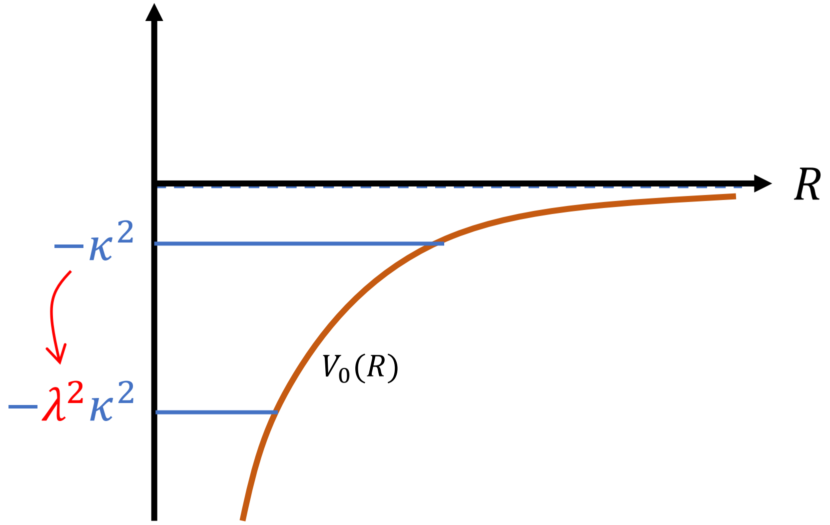
$$\left(-\frac{\partial^2}{\partial R^2}+\underbrace{\boxed{\frac{s_n^2-\frac{1}{4}}{R^2}}}_{V_n(R)}-k^2\right)\sqrt{R}F_n(R)=0$$

$$s_n\cos\left(\frac{s_n\pi}{2}\right)+\frac{8}{\sqrt{3}}\sin\left(\frac{s_n\pi}{6}\right)=0$$

All s_n are real, except one: $s_0=\pm i1.00624$

For $n=0$, one gets the Efimov attractive potential $\boxed{V_0(R)=-\frac{|s_0|^2+\frac{1}{4}}{R^2}}$

Scale invariance
$R\to\lambda R$



Solution F at energy $-\kappa^2$

$$\left(-\frac{\partial^2}{\partial R^2}+\frac{s_n^2-\frac{1}{4}}{R^2}+\kappa^2\right)\sqrt{R}F_n(R)=0$$

Solution F at energy $-\lambda^2\kappa^2$

$$\left(-\frac{\partial^2}{\partial R^2}+\frac{s_n^2-\frac{1}{4}}{R^2}+\lambda^2\kappa^2\right)\sqrt{\lambda R}F_n(\lambda R)=0$$

Without boundary condition at short hyper-radius, the Efimov attraction allows any negative energy : **this illustrates the Thomas collapse!**

Derivation for three identical bosons

Effective Schrödinger equation for the hyper-radius R

$$\left(-\frac{\partial^2}{\partial R^2}+\underbrace{\boxed{\frac{s_n^2-\frac{1}{4}}{R^2}}}_{V_n(R)}-k^2\right)\sqrt{R}F_n(R)=0$$

$$s_n\cos\left(\frac{s_n\pi}{2}\right)+\frac{8}{\sqrt{3}}\sin\left(\frac{s_n\pi}{6}\right)=0$$

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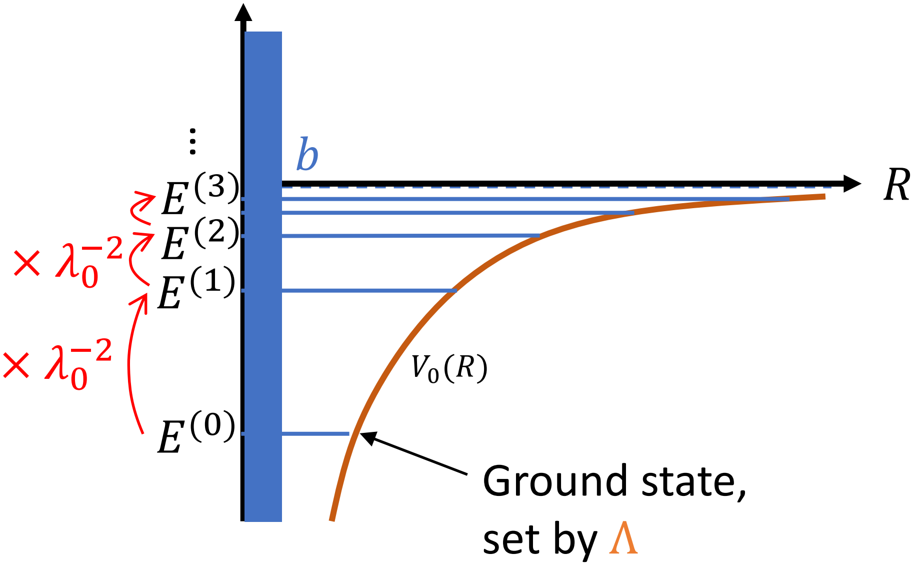
Discrete scale invariance
$R\to\lambda_0 R$



The problem is due to the zero-range approximation. In reality, when the three particles come within distances of the order of b, the interaction potential sets a boundary condition for R.

For small R, $F_0(R)=\alpha R^{i|s_0|}+\beta R^{-i|s_0|}\propto\cos(|s_0|\ln\Lambda R)$

Three-body parameter

$$F_0(\lambda R)\propto\cos(|s_0|\ln\Lambda\lambda R)=\cos(|s_0|\ln\Lambda R+\underbrace{|s_0|\ln\lambda}_{\pi})\quad\propto F_0(R)$$

$$\boxed{\lambda_0=e^{\pi/|s_0|}\approx 22.7}\Rightarrow\boxed{E^{(n)}=E^{(0)}\lambda_0^{-2n}}$$

Generalised discrete scaling away from unitarity

Suppose we have a solution χ_0 of:

Equation:

$$\left(-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\frac{\partial^2}{\partial \alpha^2}+\kappa^2\right)\chi_0(R,\alpha)=0$$

Boundary condition $\alpha \to 0$:

$$\left[\frac{\partial}{\partial \alpha}\chi_0(R,\alpha)\right]_{\alpha\to 0}+\frac{8}{\sqrt{3}}\chi_0\left(R,\frac{\pi}{3}\right)=-\frac{R}{a}\chi_0(R,0)$$

Discrete scaling: $R \to R/\lambda_0$

$$\left(-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\frac{\partial^2}{\partial \alpha^2}+\lambda_0^{-2}\kappa^2\right)\chi_0(\lambda_0 R,\alpha)=0$$

$$\left[\frac{\partial}{\partial \alpha}\chi_0(\lambda_0 R,\alpha)\right]_{\alpha\to 0}+\frac{8}{\sqrt{3}}\chi_0\left(\lambda_0 R,\frac{\pi}{3}\right)=-\frac{R}{\lambda_0 a}\chi_0(\lambda_0 R,0)$$

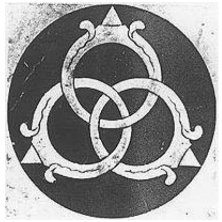
i.e. we have a new solution for: $a \to \lambda_0 a$, $\kappa \to \kappa/\lambda_0$

$$(a^{-1},\kappa)\to(a^{-1},\kappa)/\lambda_0$$

"Efimov spectrum"

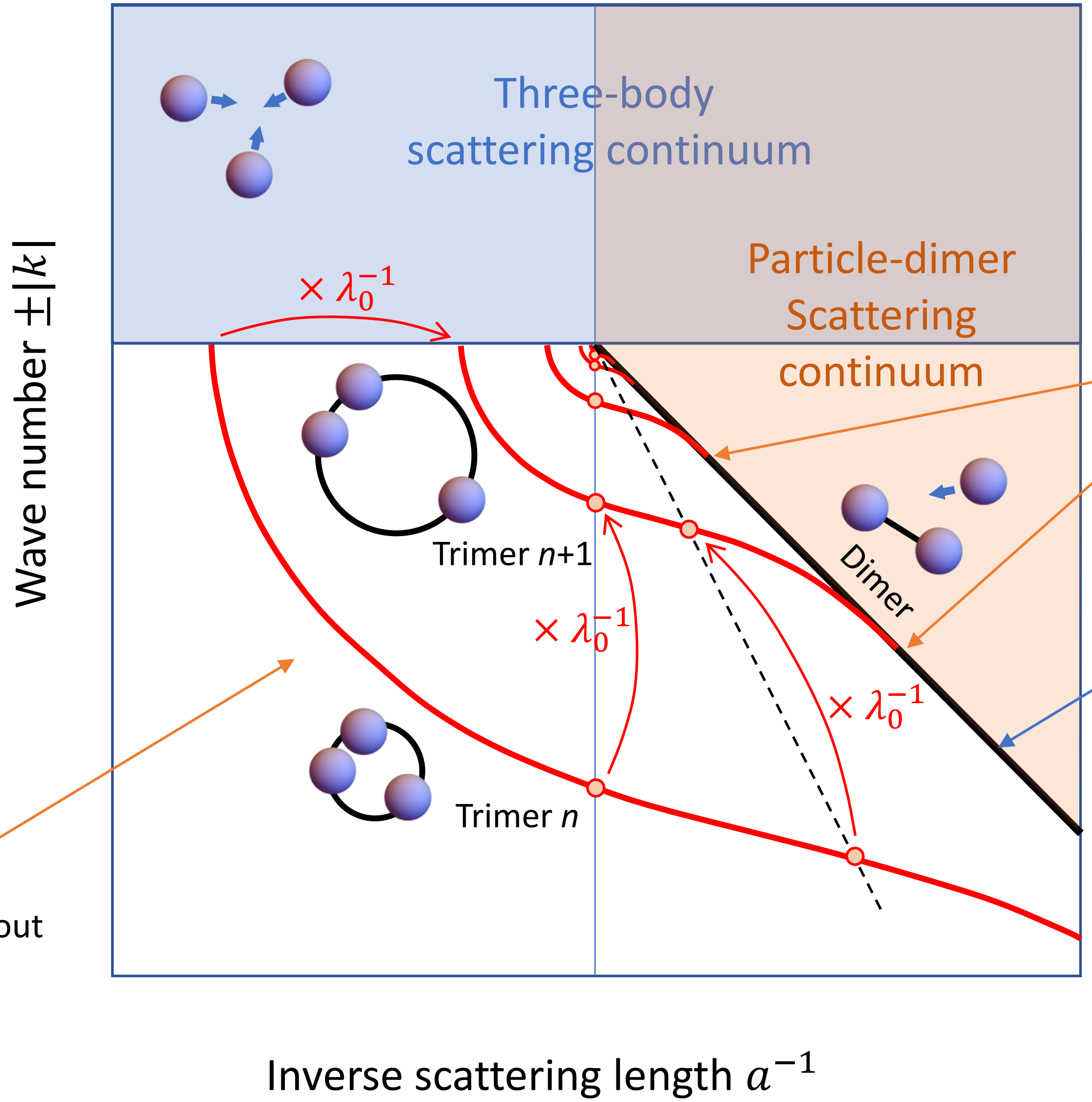
1. Discrete scale invariance

Infinite number of three-body bound states.



2. Borromean states

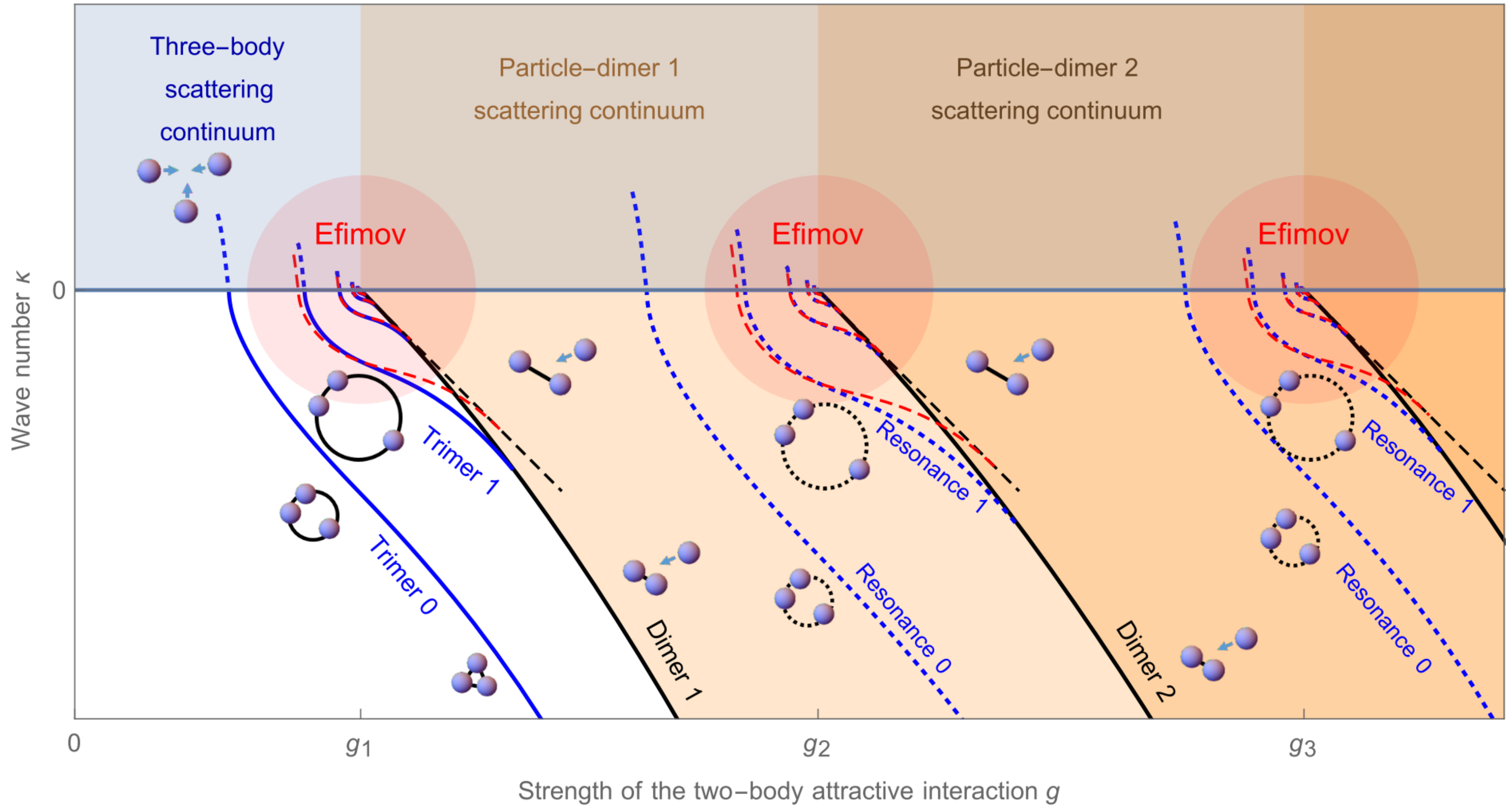
Three-body bound states without two-body bound states.

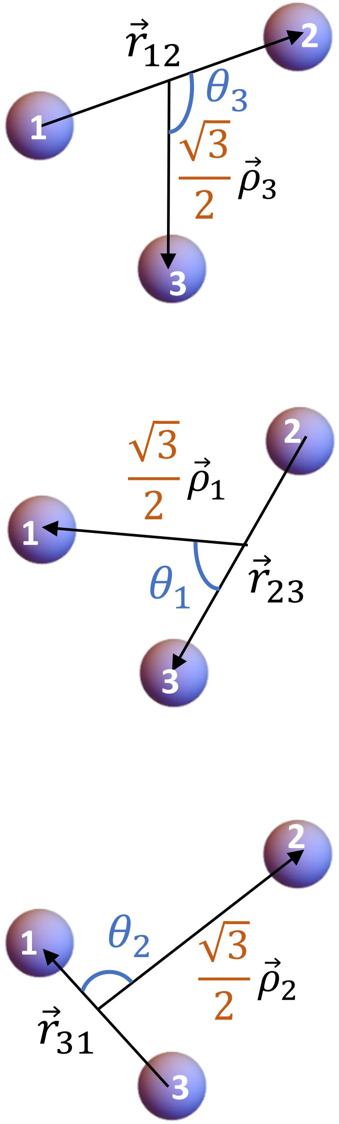


3. Trimer dissociation with increasing interaction

$$E_2 = -\frac{\hbar^2\kappa^2}{m}$$
with $\kappa = a^{-1}$

Three bosons





What is the shape of an Efimov state?

No definite shape (fluctuating) but a tendency to form elongated triangles

$$\Psi = \chi(\vec{r}_{12},\vec{\rho}_3) + \chi(\vec{r}_{23},\vec{\rho}_1) + \chi(\vec{r}_{31},\vec{\rho}_2)$$

$$= \frac{\chi_0(r_{12},\rho_3)}{r_{12}\,\rho_3} + \frac{\chi_0(r_{23},\rho_1)}{r_{23}\,\rho_1} + \frac{\chi_0(r_{31},\rho_2)}{r_{31}\,\rho_2}$$

Partial wave $L = 0$

$$= \frac{2}{R^2}\left(\frac{\chi_0(R,\alpha_3)}{\sin 2\alpha_3} + \frac{\chi_0(R,\alpha_1)}{\sin 2\alpha_1} + \frac{\chi_0(R,\alpha_2)}{\sin 2\alpha_2}\right)$$

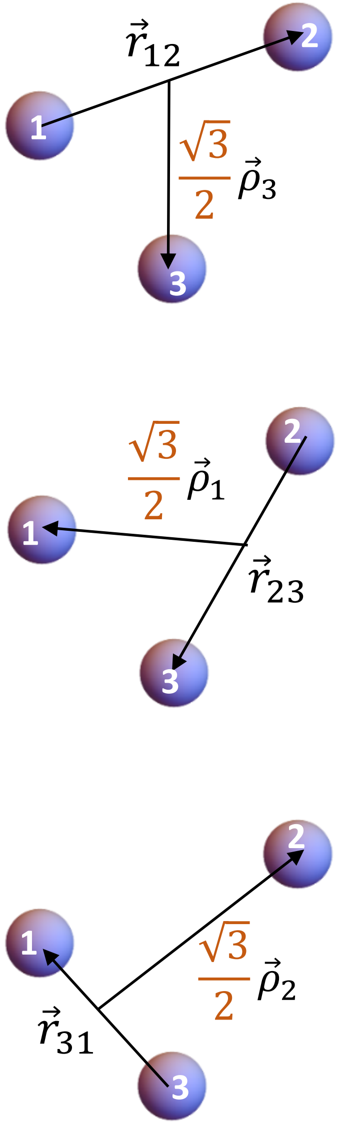
Hyperspherical coordinates:

$$r = R\sin\alpha$$
$$\rho = R\cos\alpha$$

$$= \underbrace{\frac{2F(R)}{R^2}}_{\substack{\text{Hyper-radial} \\ \text{(size)}}}\underbrace{\left(\frac{\phi_0(\alpha_3)}{\sin 2\alpha_3} + \frac{\phi_0(\alpha_1)}{\sin 2\alpha_1} + \frac{\phi_0(\alpha_2)}{\sin 2\alpha_2}\right)}_{\substack{\text{Hyper-angular} \\ \Phi_0(\alpha_3,\theta_3) \\ \text{(shape)}}}$$

$\chi_0(R,\alpha) = F(R)\phi_0(\alpha)$ at unitarity

with $\phi_0(\alpha) = \sinh\left(|s_0|\left(\frac{\pi}{2} - \alpha\right)\right)$



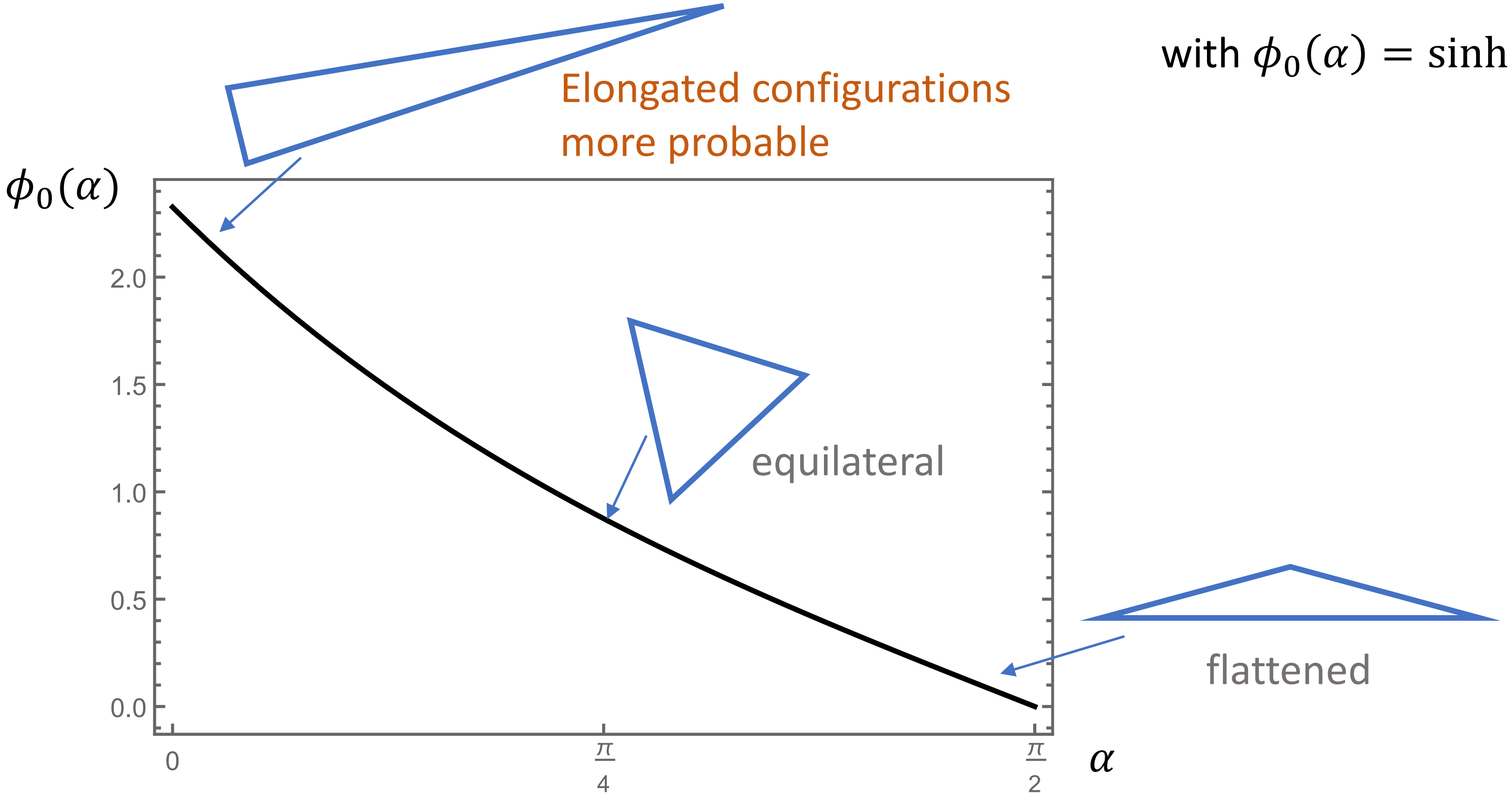
What is the shape of an Efimov state?

No definite shape (fluctuating) but a tendency to form elongated triangles

$$\Psi = \frac{2F(R)}{R^2}\left(\frac{\phi_0(\alpha_3)}{\sin 2\alpha_3} + \frac{\phi_0(\alpha_1)}{\sin 2\alpha_1} + \frac{\phi_0(\alpha_2)}{\sin 2\alpha_2}\right)$$

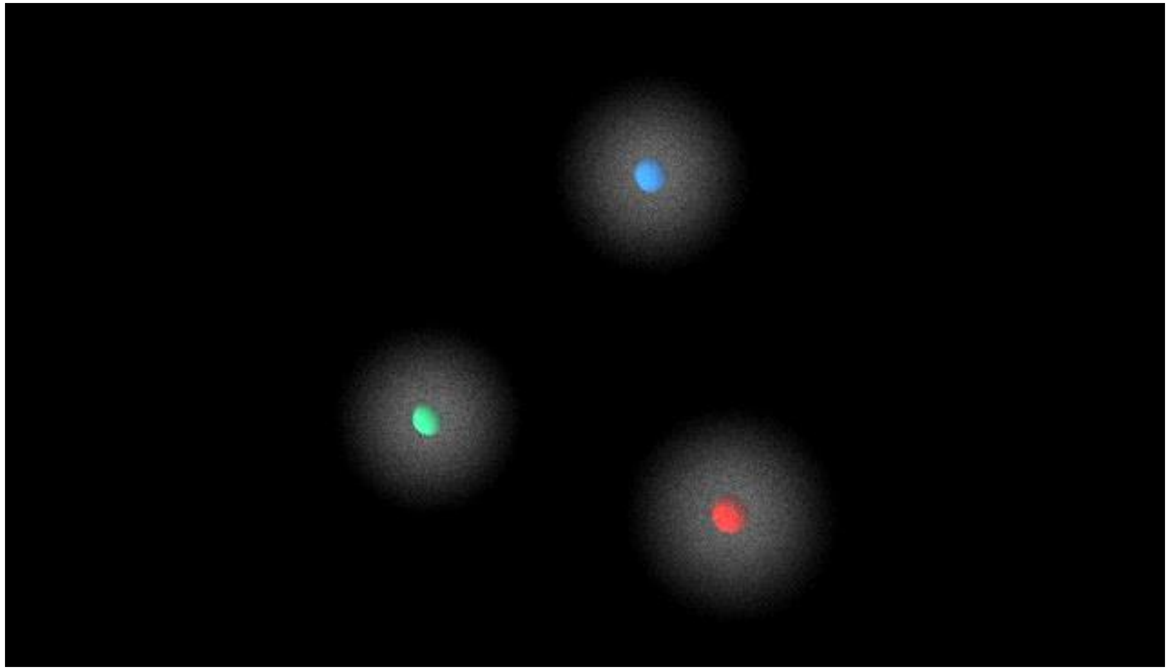
$$r = R\sin\alpha$$
$$\rho = R\cos\alpha$$

with $\phi_0(\alpha) = \sinh\left(|s_0|\left(\frac{\pi}{2} - \alpha\right)\right)$



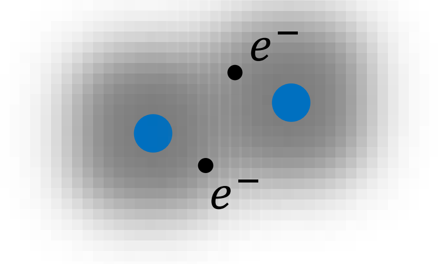
Why long range ? in spite of short-range two-body interactions?

The Efimov attraction may be viewed as an interaction between two particles mediated by a third particle

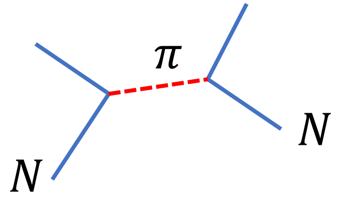


Similar to:

Chemical covalent bonding
(exchange of electron)



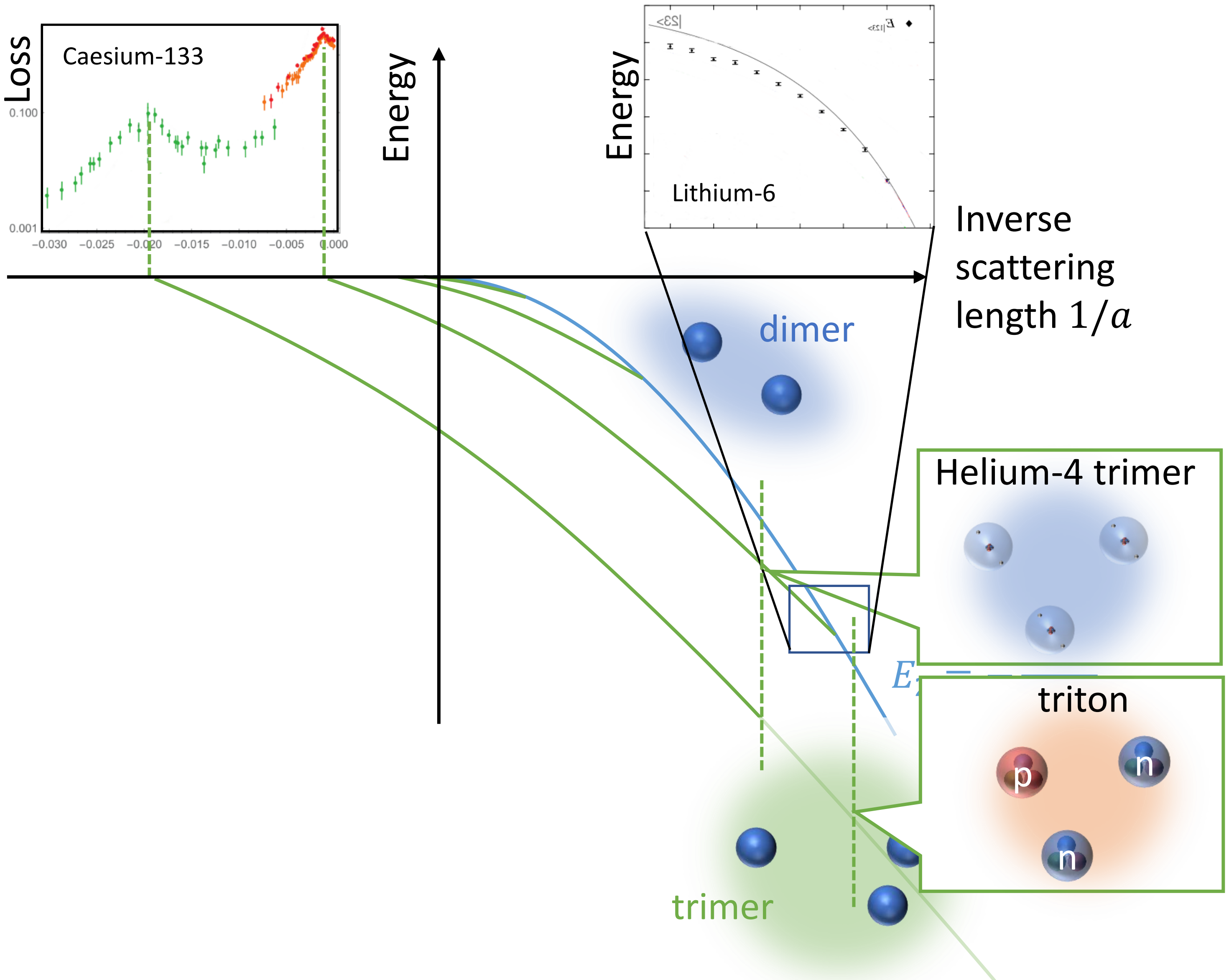
Nuclear force
(exchange of virtual meson)



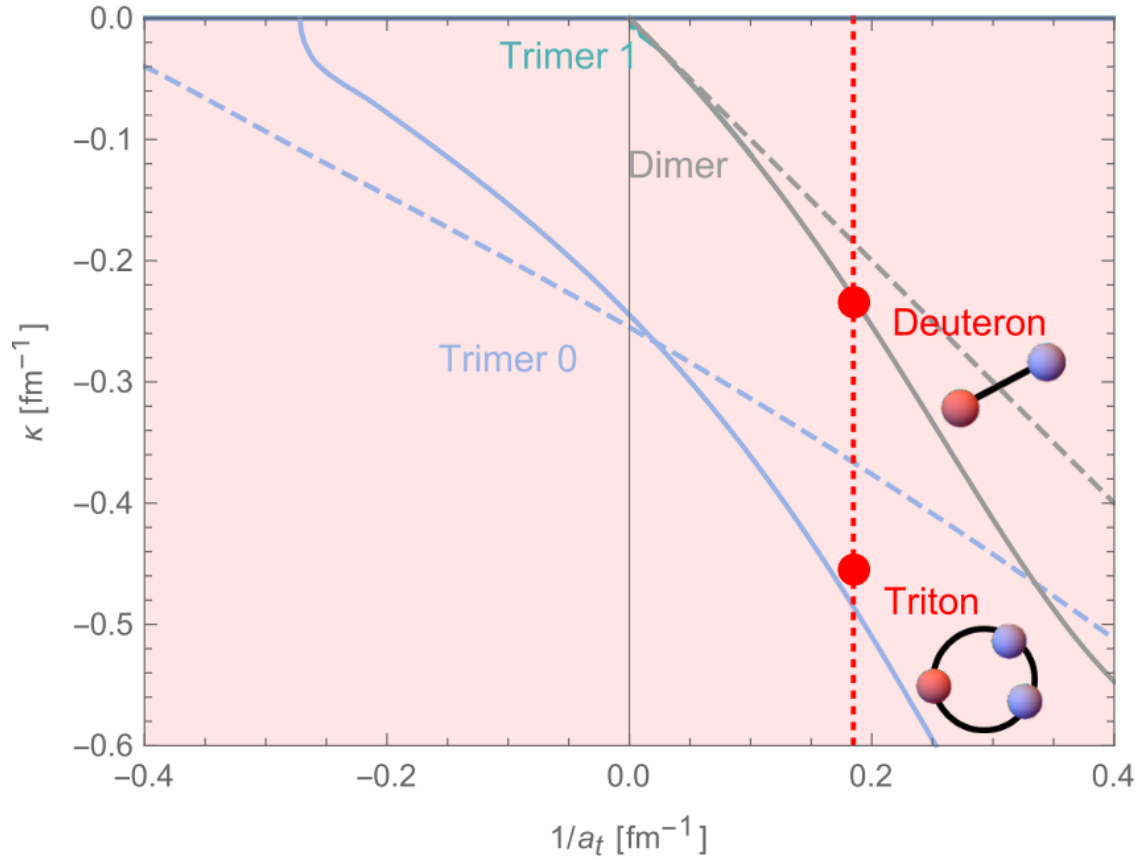
Overview of universal clusters

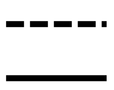
- Experimental observations
  - The triton
  - The Hoyle state
  - The helium trimers
  - Observations in cold atomic gases
- Mixtures of particles
  - 1+2 particles : the Born approximation
  - Mixtures of two two kinds of bosons
  - Mixtures of two kinds of fermions
  - Halo nuclei

Three bosons



The triton (2 neutrons + 1 proton)

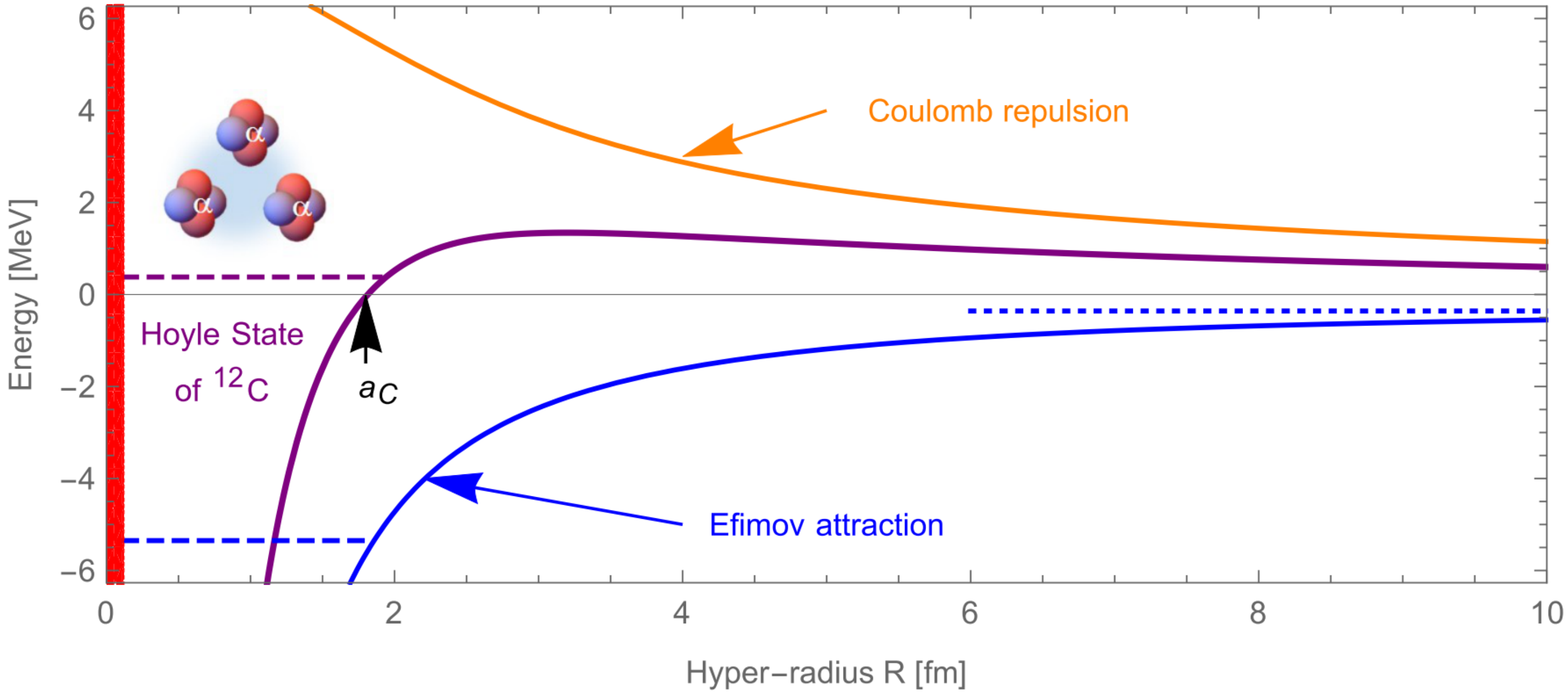




Qualitatively consistent,
but not the best example.

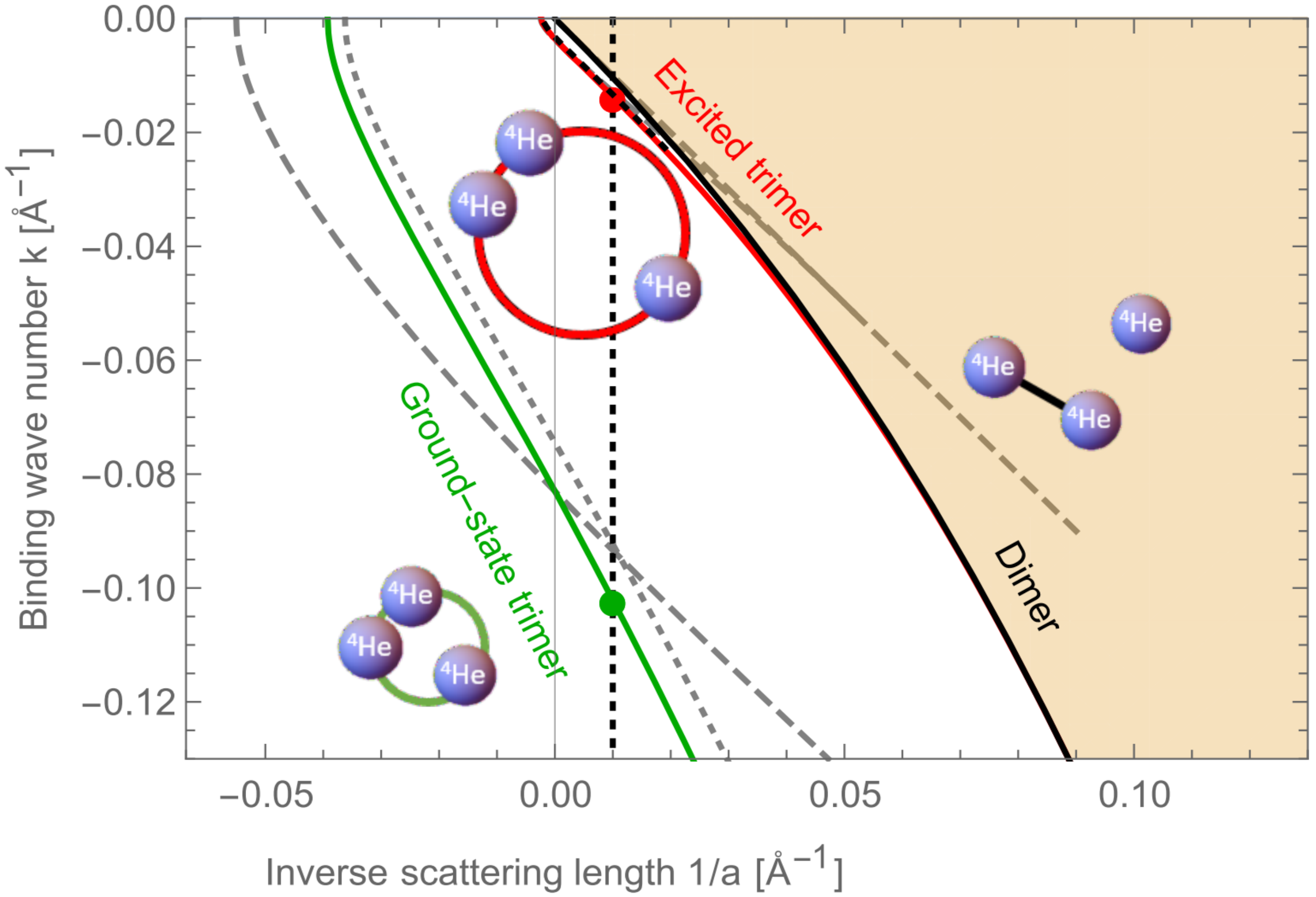
The Hoyle state of carbon-12

Originally suggested in V. Efimov's first paper

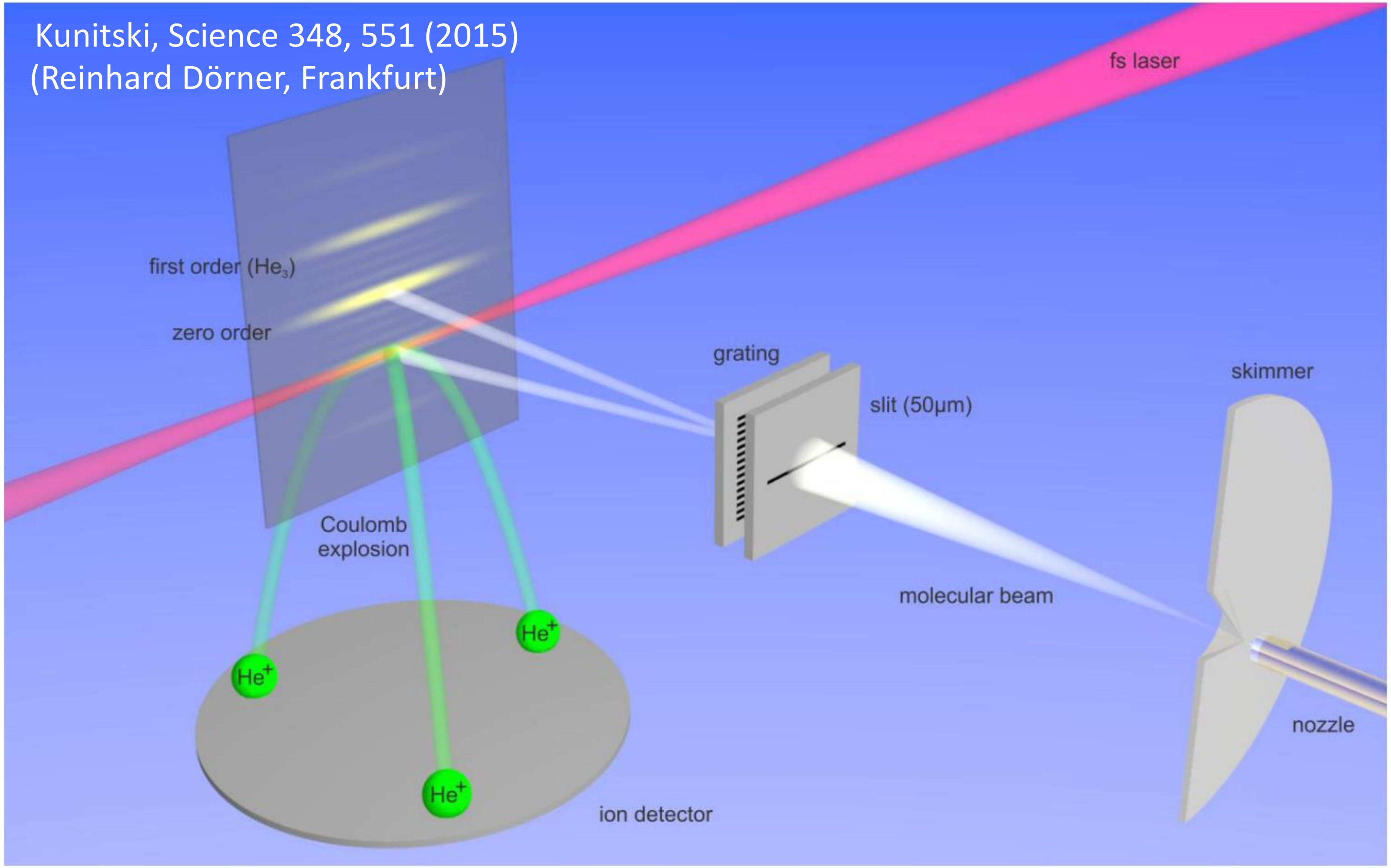


No clear evidence.

The helium triatomic molecules $^4He_3$

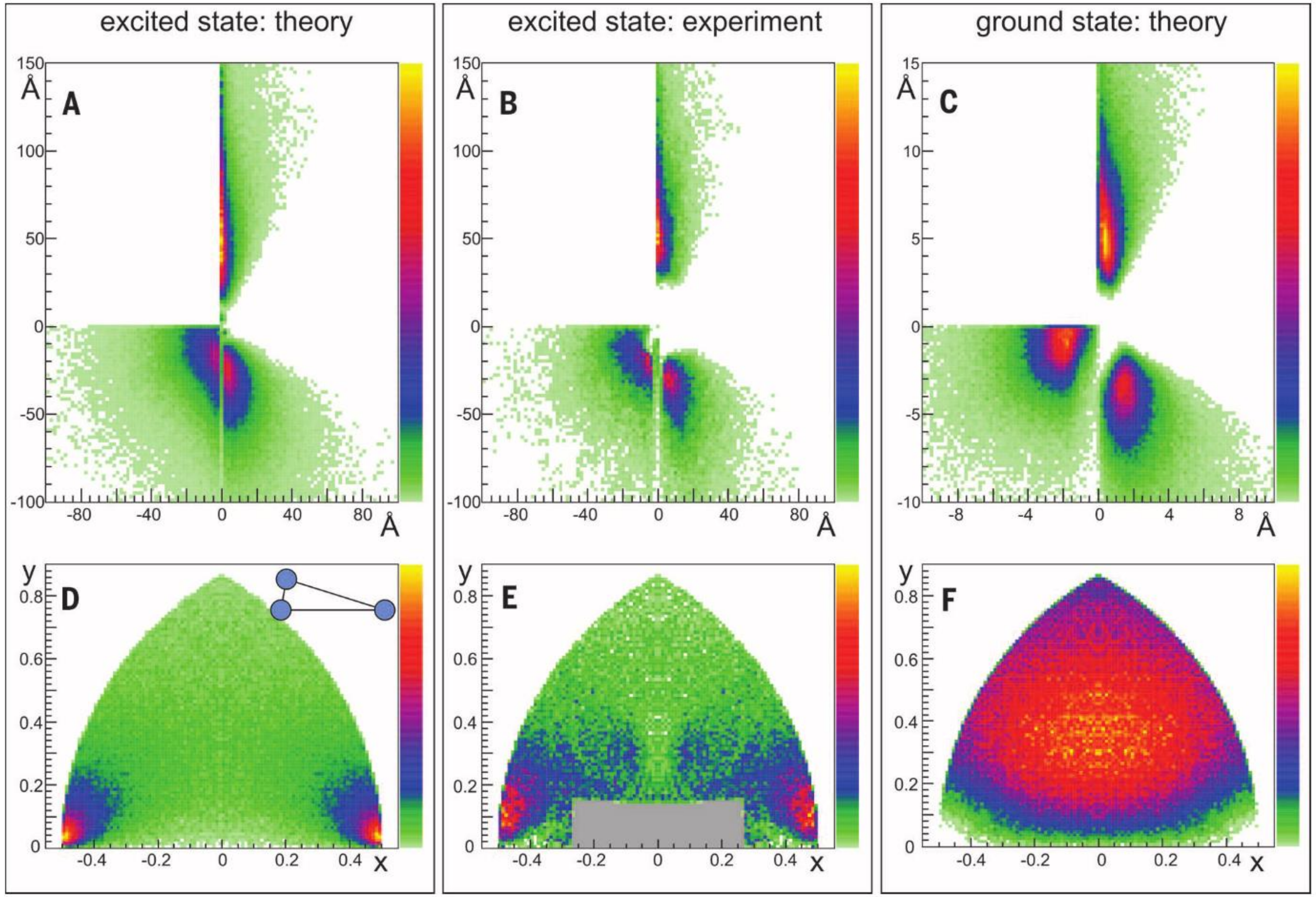


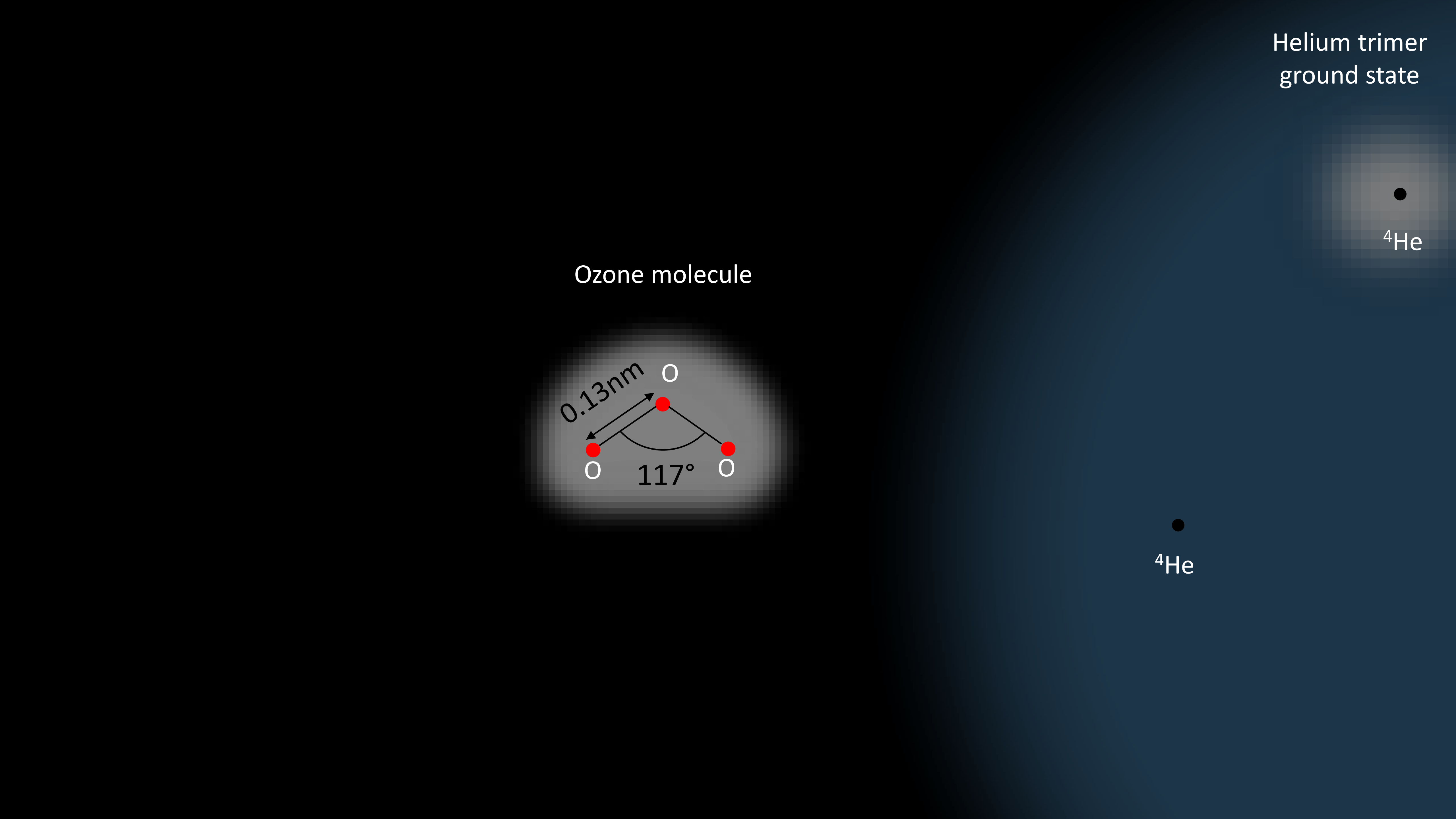
The helium triatomic molecules $^4He_3$

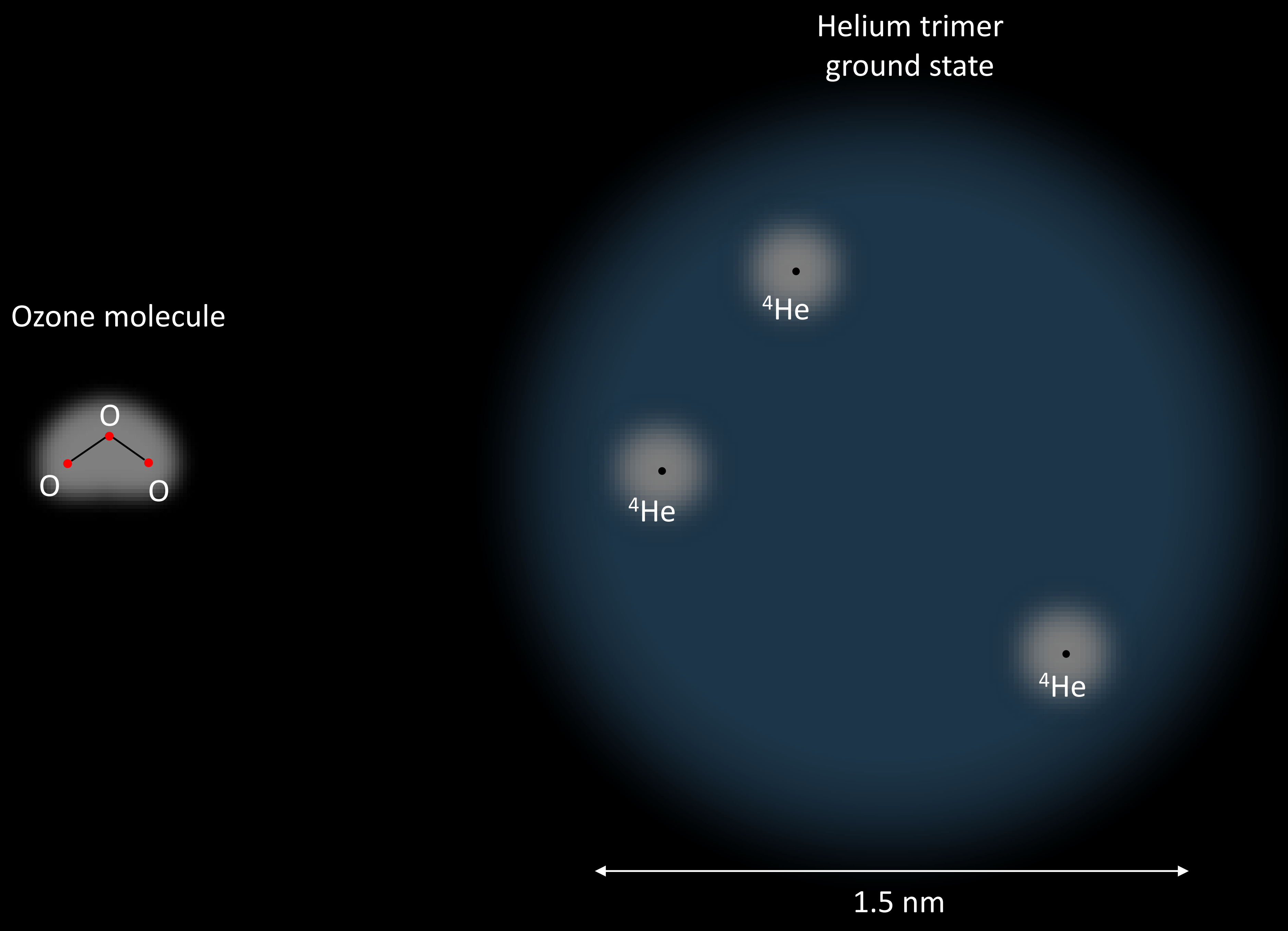


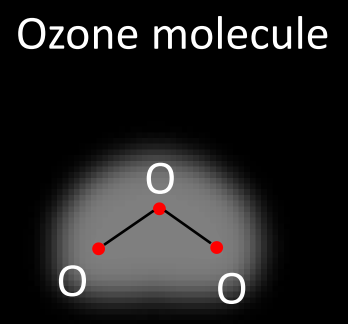
The helium triatomic molecules $^4He_3$

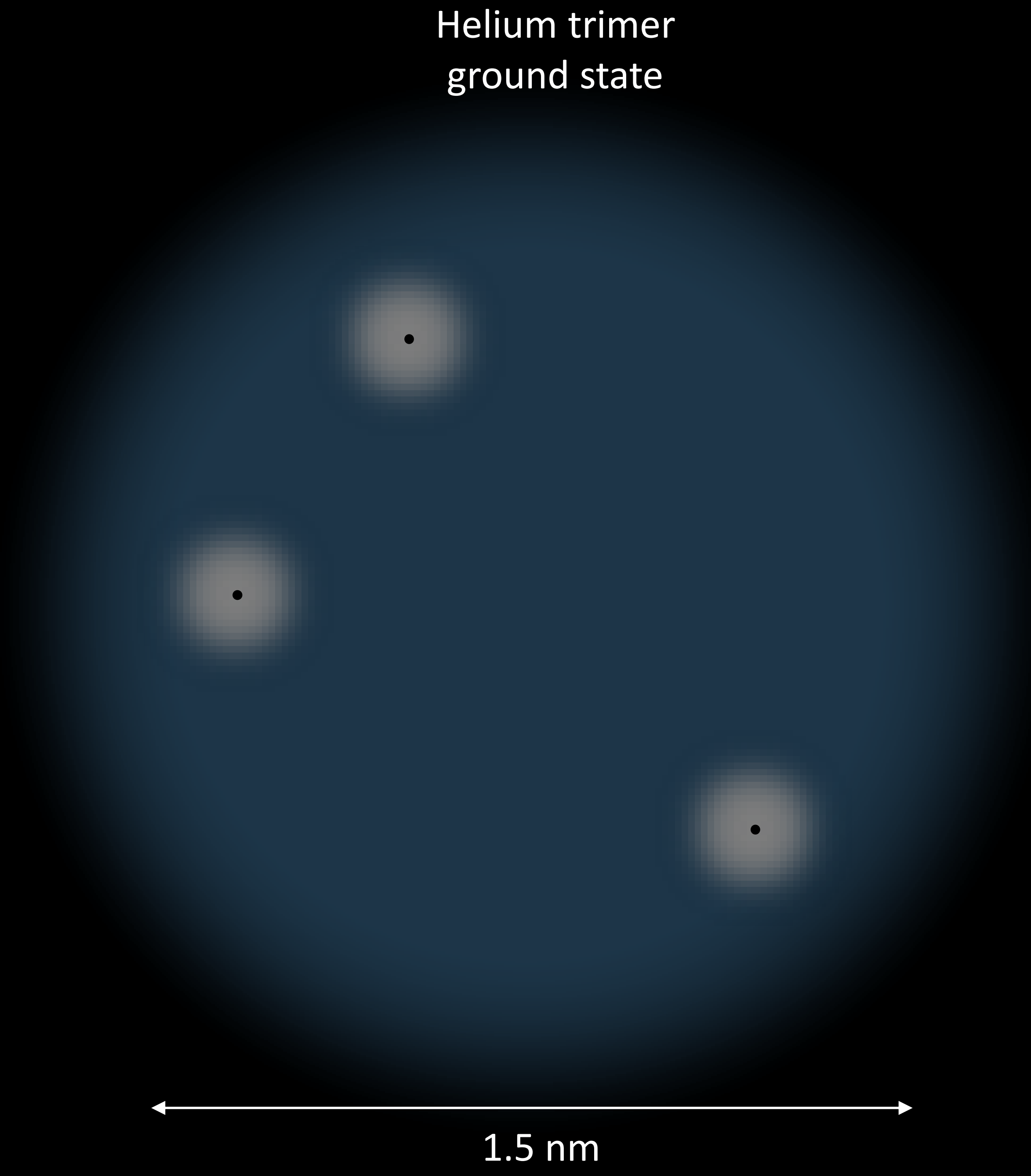
Kunitski, Science 348, 551 (2015)
(Reinhard Dörner, Frankfurt)

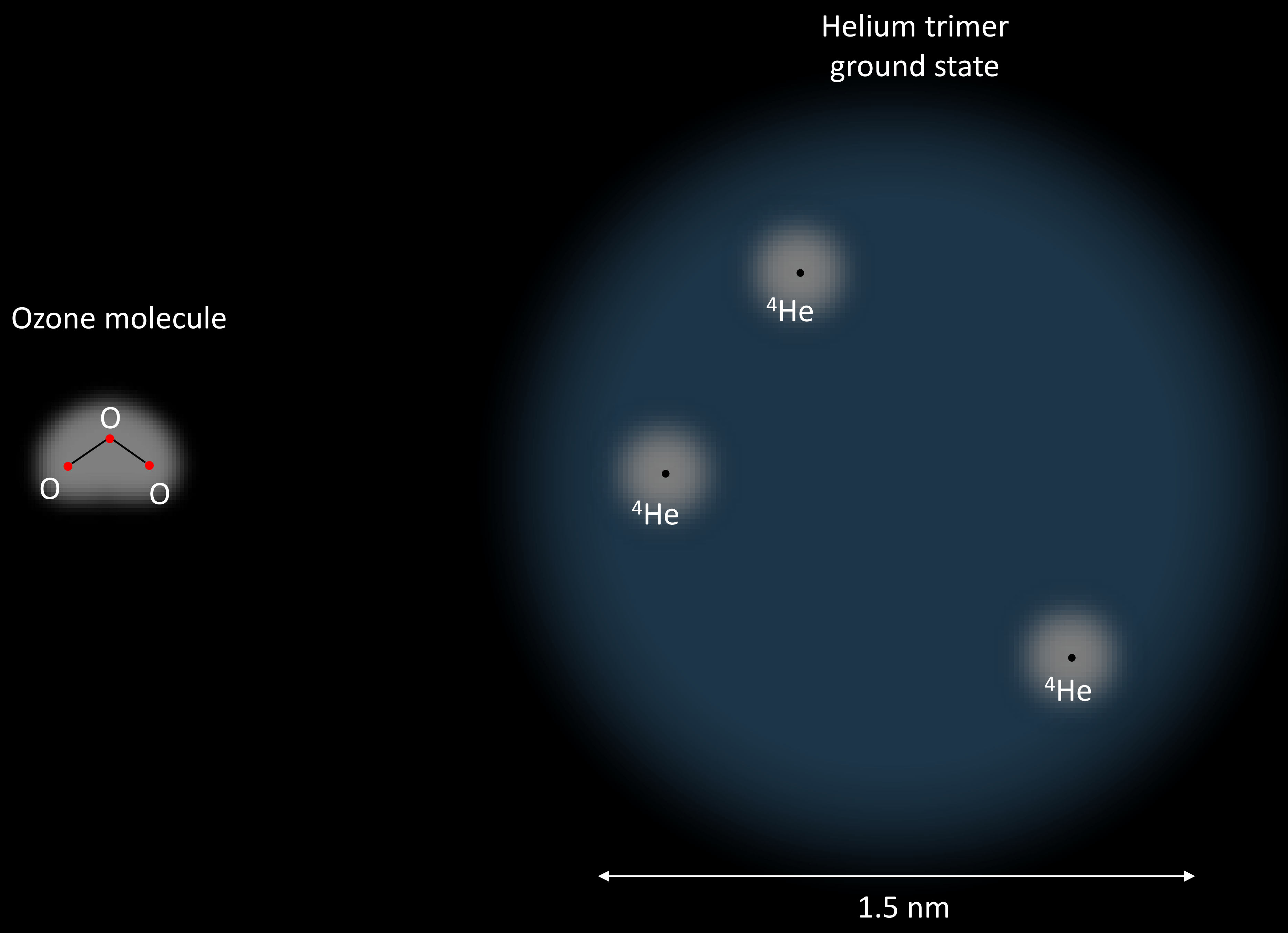



Helium trimer
ground state
^{4}He
Ozone molecule
O
0.13nm
O
117°
O
^{4}He

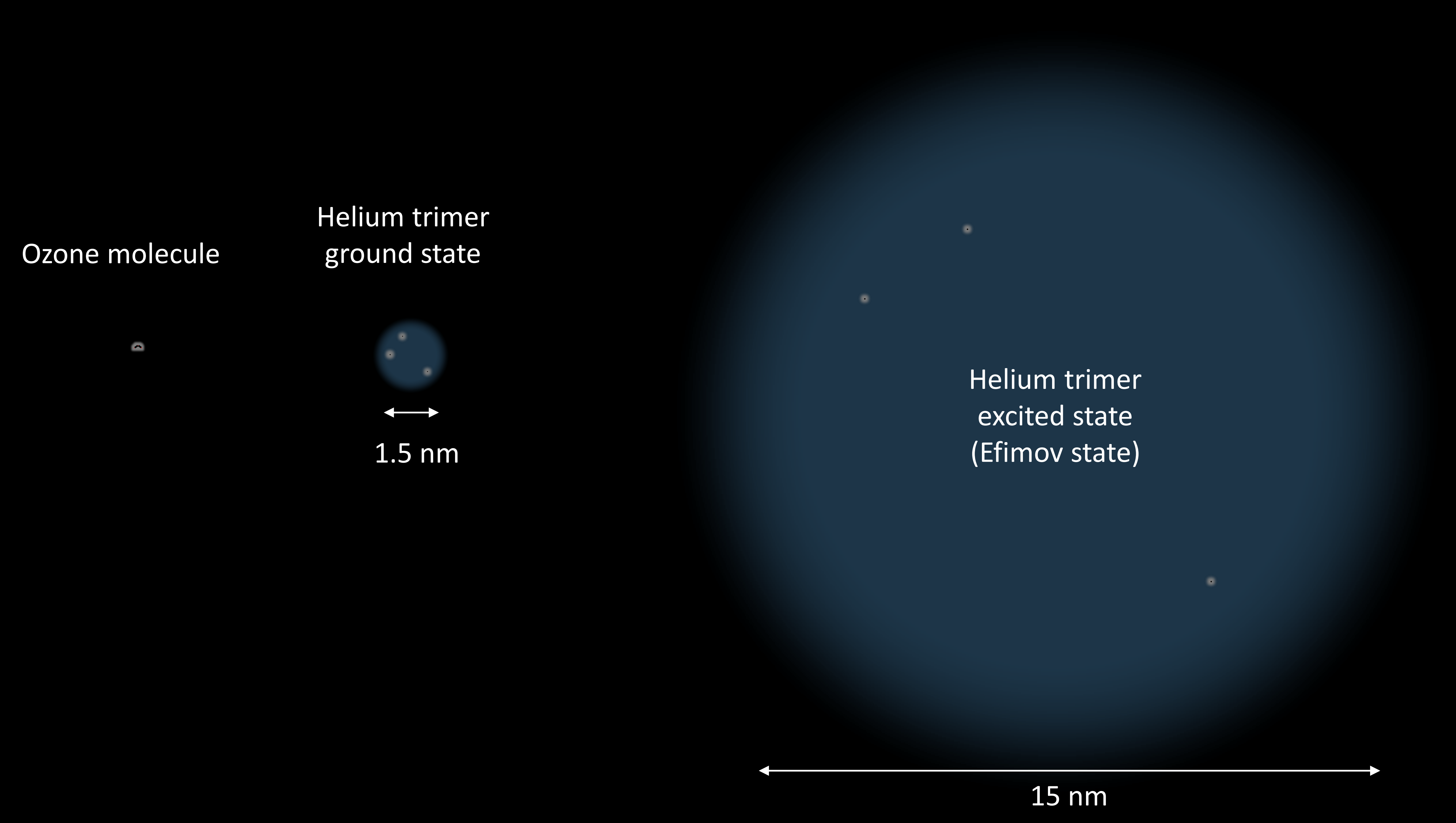

Helium trimer
ground state
Ozone molecule
O
O
O
^{4}He
^{4}He
^{4}He
1.5 nm


Ozone molecule
O
O
O

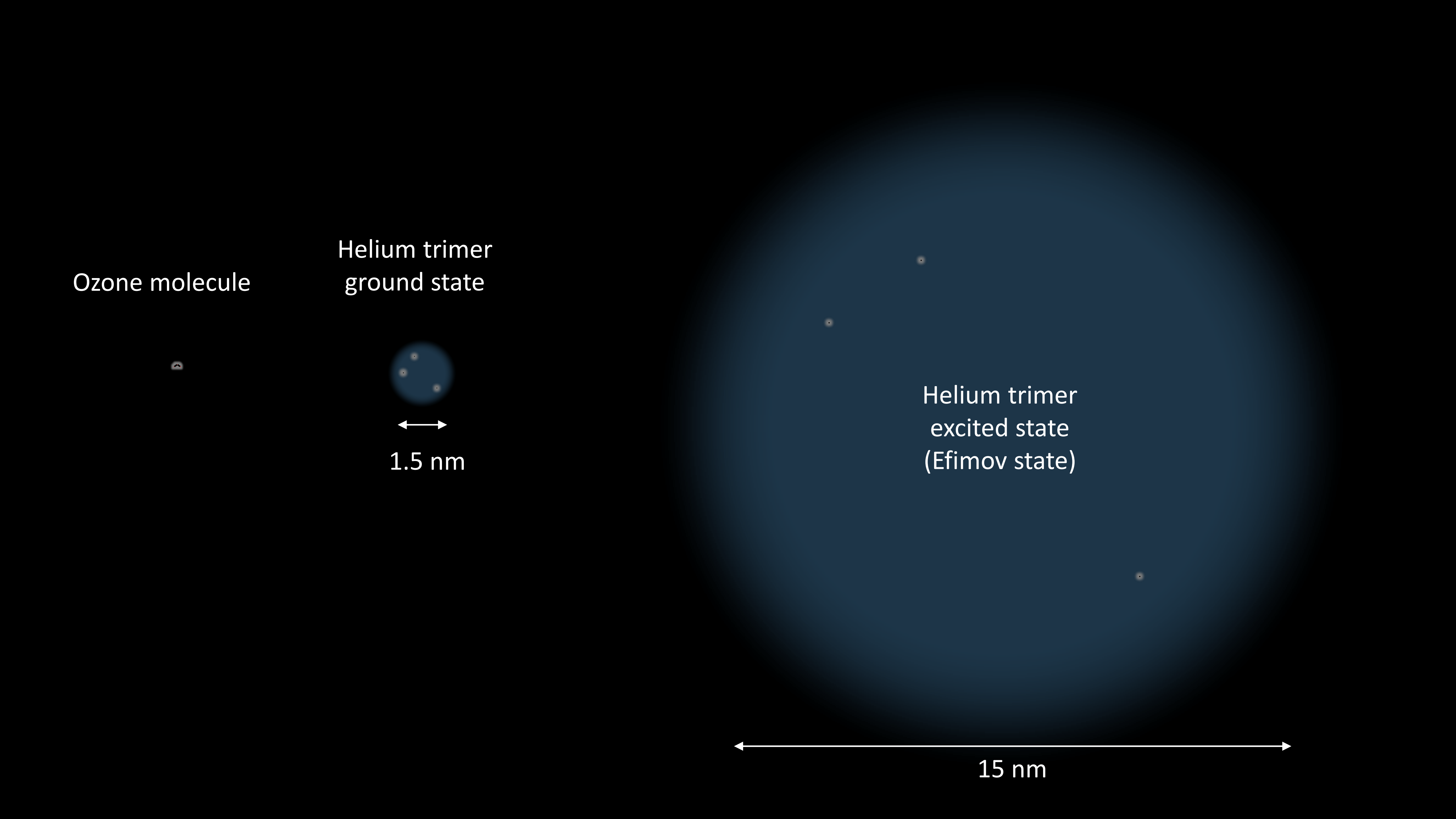

Helium trimer
ground state
1.5 nm



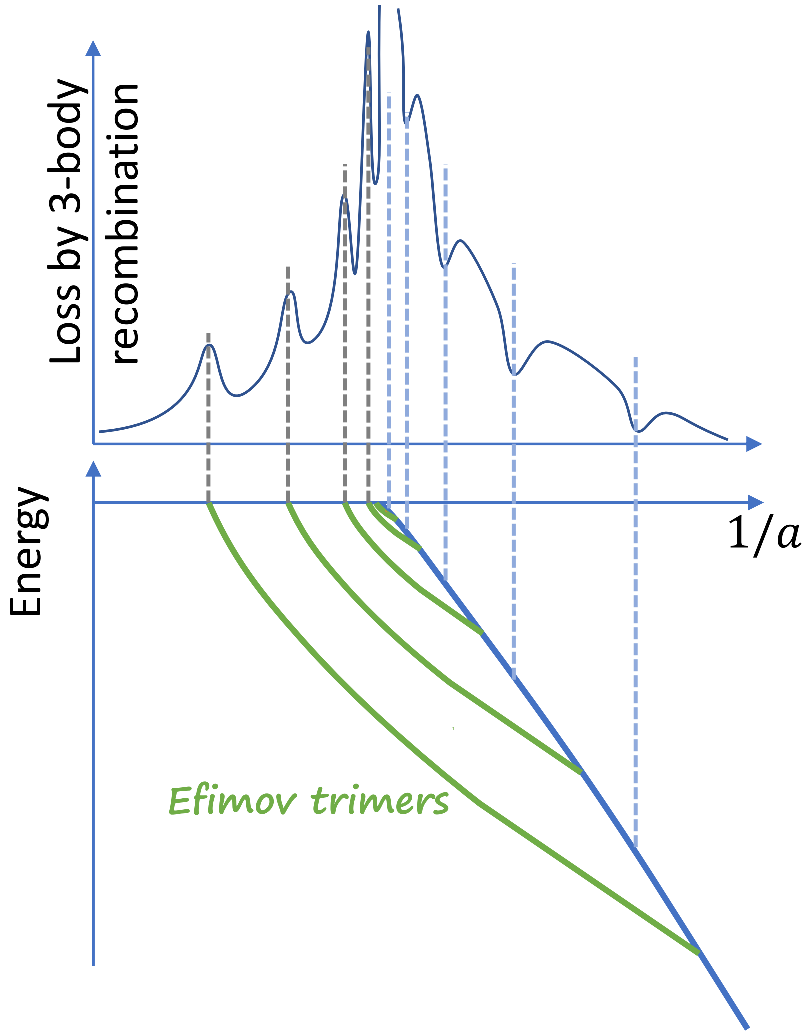
Ozone molecule
O
O
O
Helium trimer
ground state
^{4}He
^{4}He
^{4}He
1.5 nm

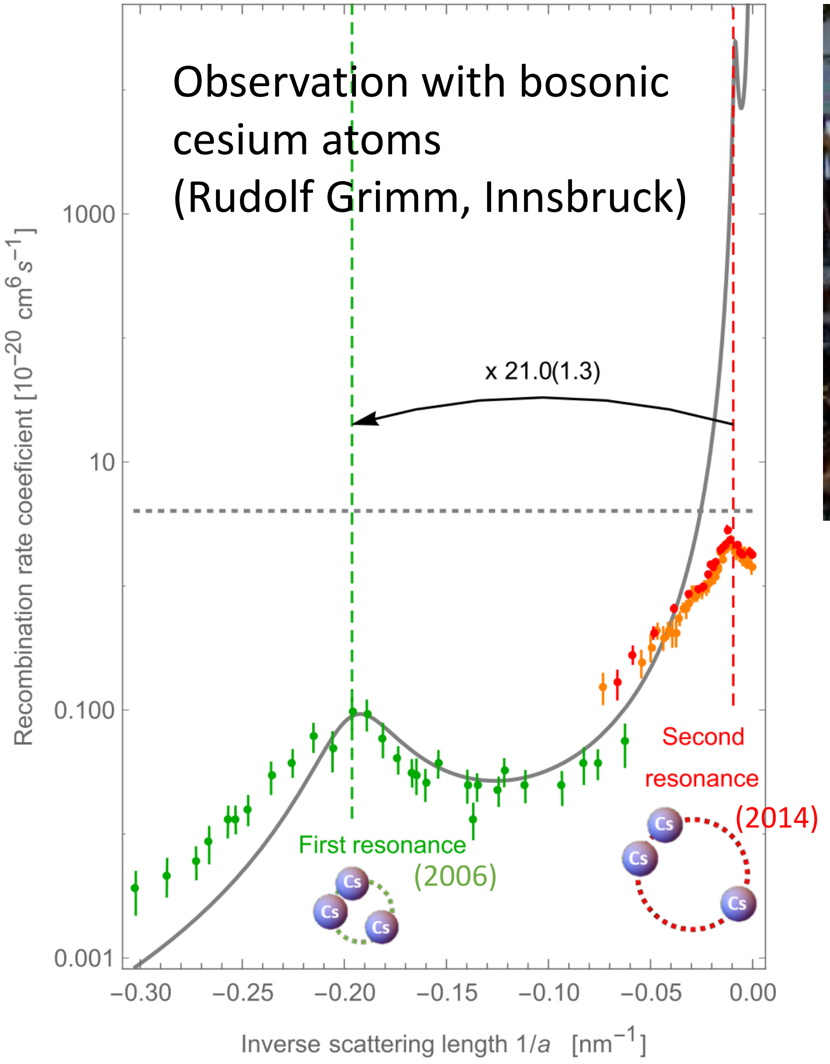


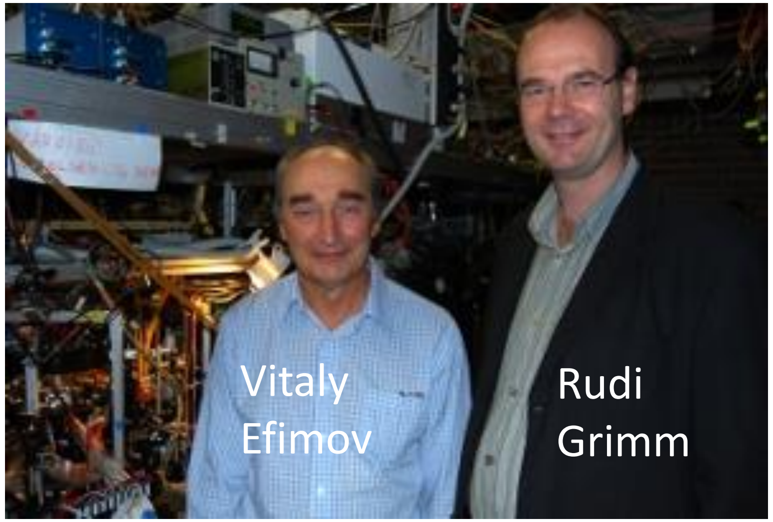
Ozone molecule
Helium trimer
ground state
1.5 nm
Helium trimer
excited state
(Efimov state)
15 nm


Ozone molecule
Helium trimer
ground state
1.5 nm
Helium trimer
excited state
(Efimov state)
15 nm

Observations in ultra-cold atomic gases





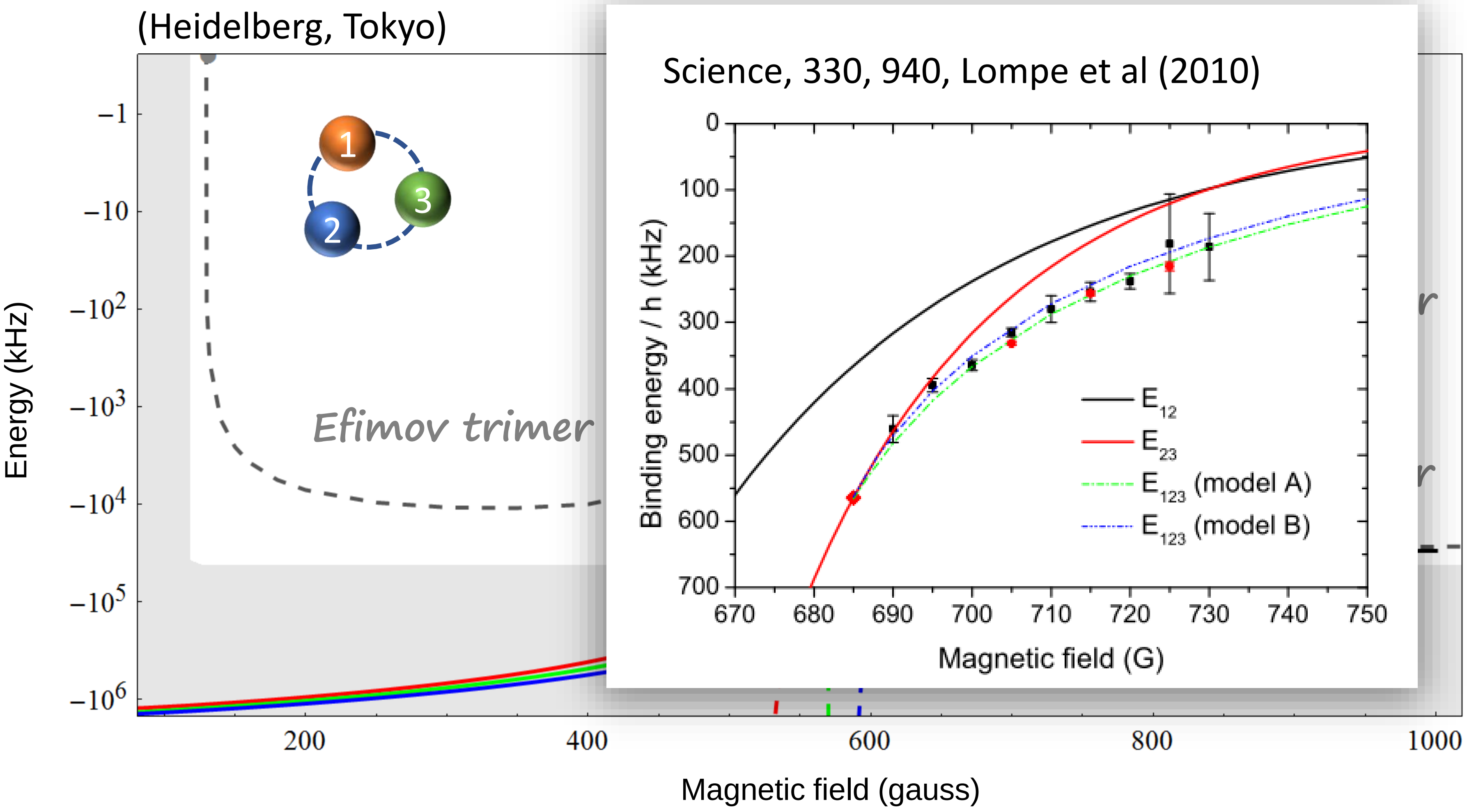


Vitaly Efimov and Rudolf Grimm receive the first Faddeev medal in Caen (July 11, 2018)

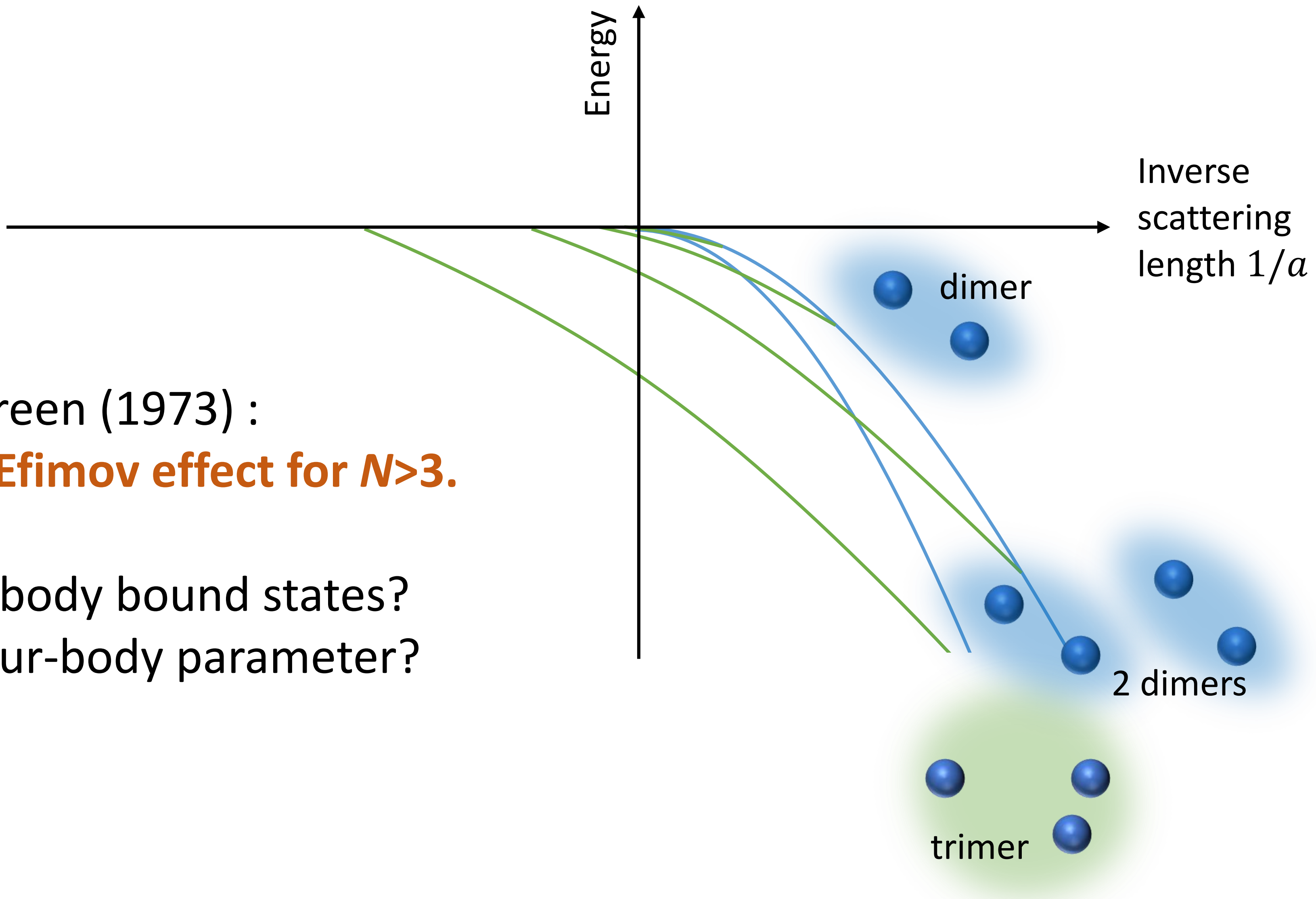
Vitaly Efimov’s speech after receiving the prize

Observations in ultra-cold atomic gases

Observation with three distinguishable states of lithium atoms
(Heidelberg, Tokyo)



Four bosons



Amado & Green (1973) :
No *N*-body Efimov effect for *N*>3.

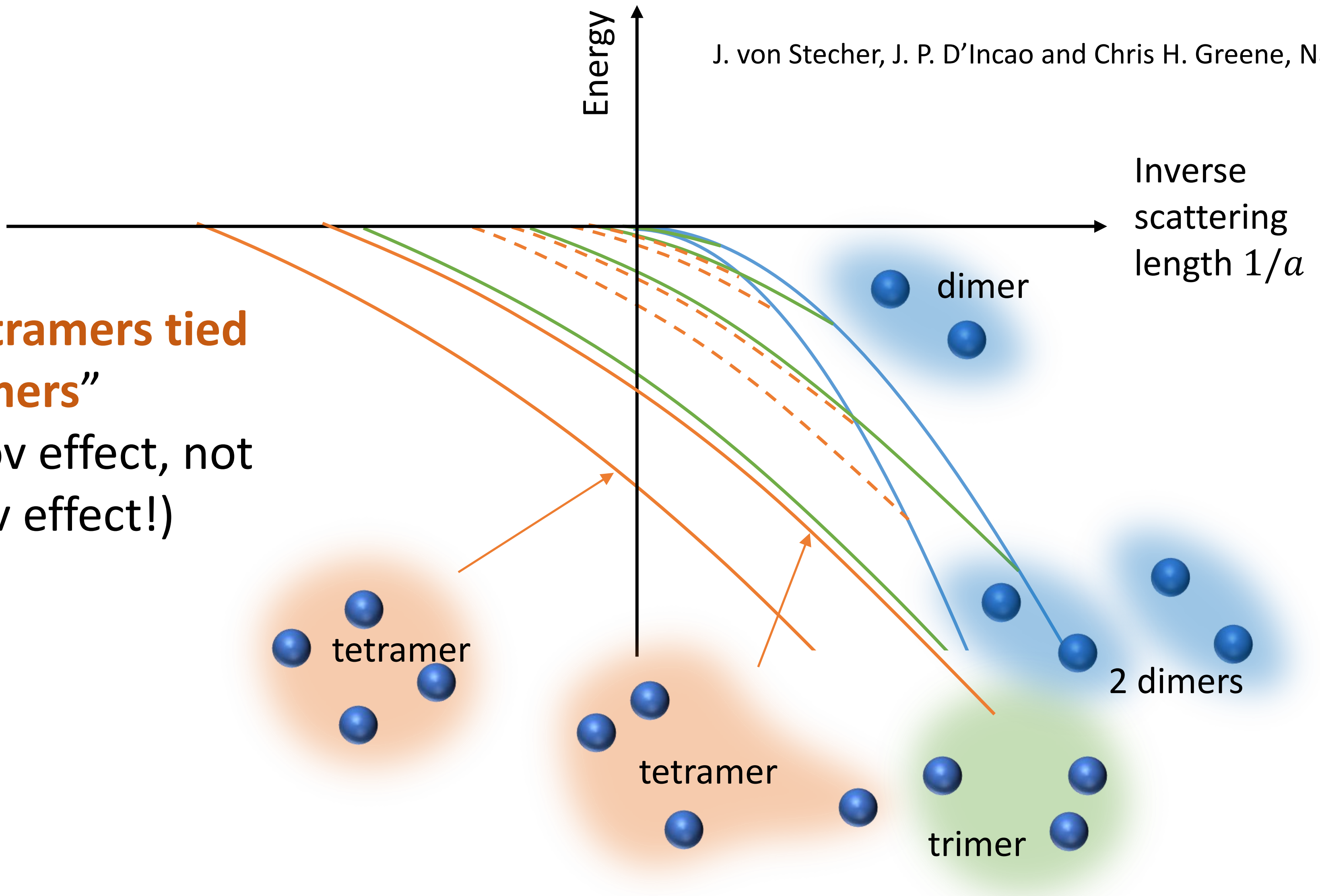
Universal 4-body bound states?
Is there a four-body parameter?

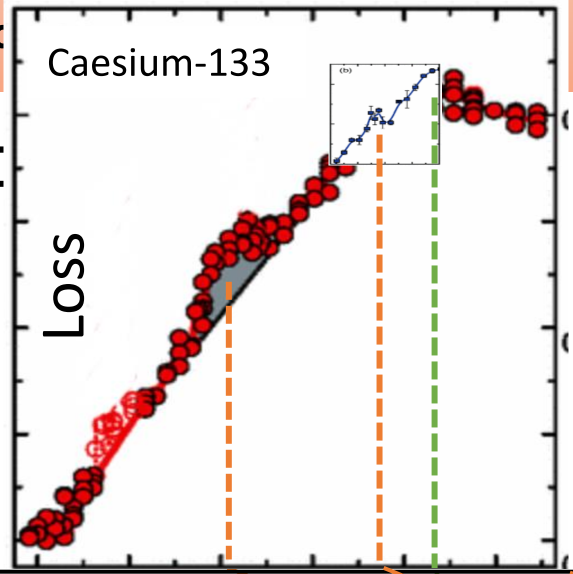
Four bosons

H-W Hammer L. Platter, Eur Phys J A 32, 113 (2007)

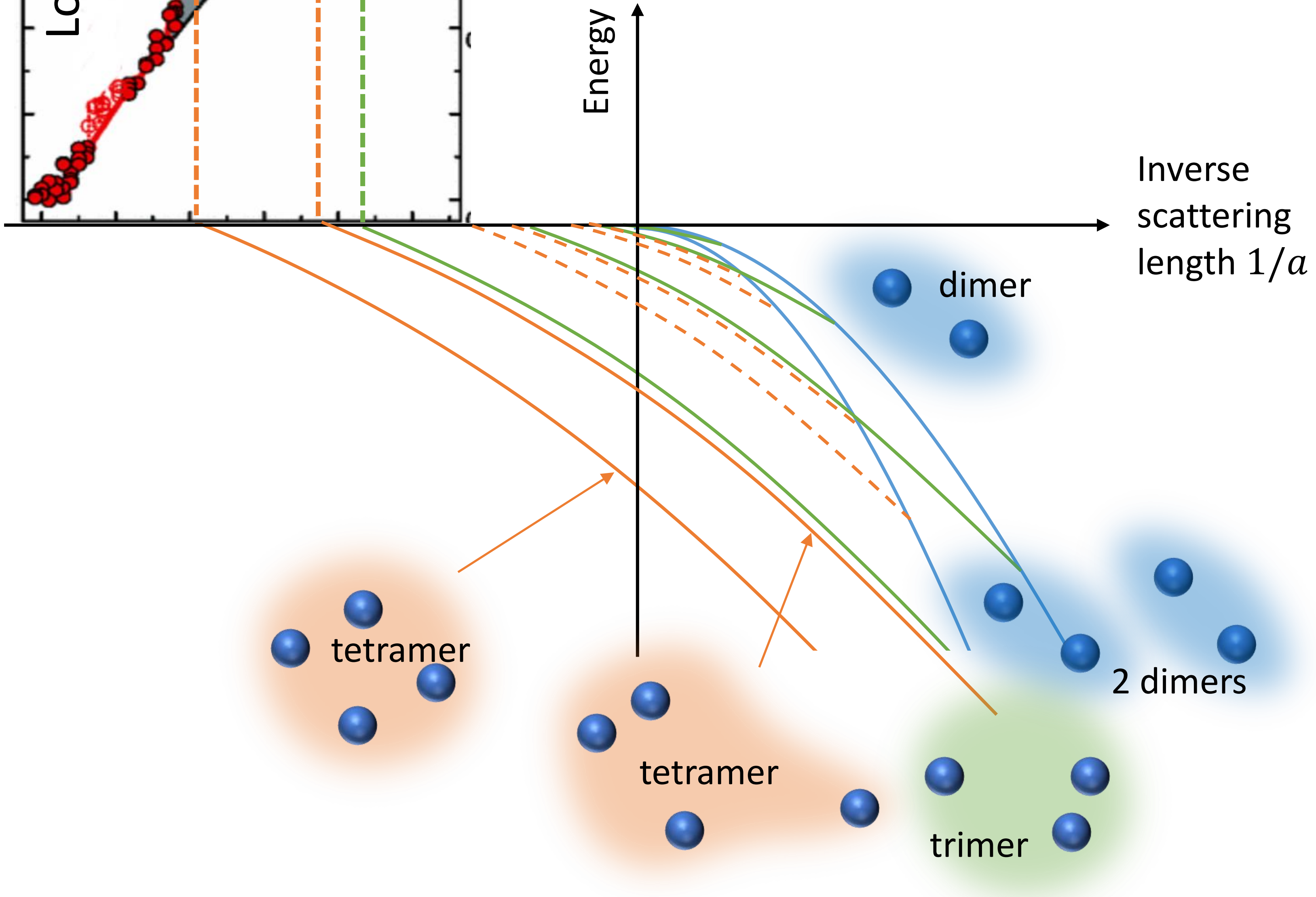
J. von Stecher, J. P. D'Incao and Chris H. Greene, Nat Phys 5, 417 (2009)

“**Universal tetramers tied to Efimov trimers**”
(3-body Efimov effect, not 4-body Efimov effect!)



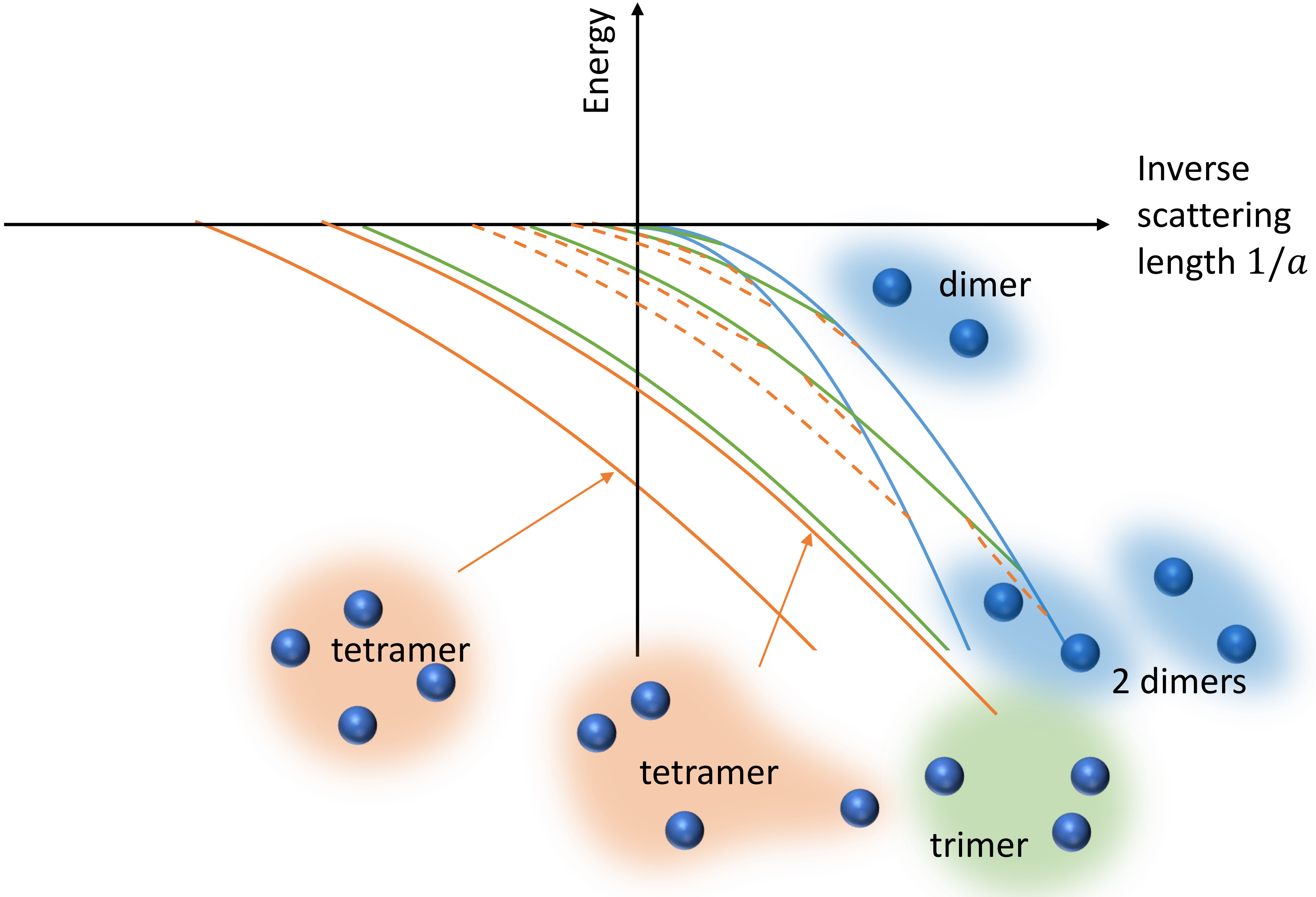


Ferlaino et al, Phys. Rev. Lett., 102, 140401 (2009)

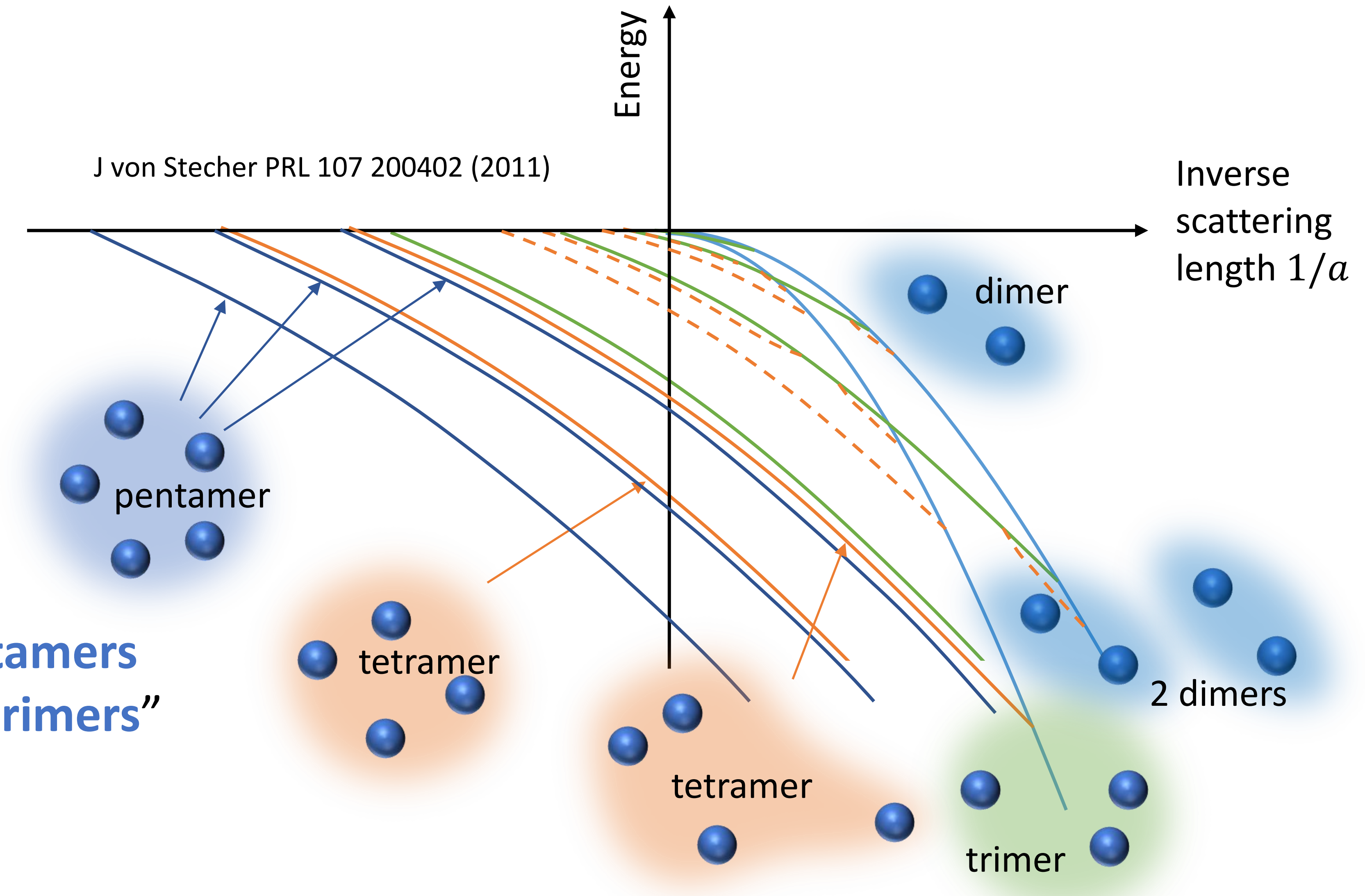


Four bosons

A. Deltuva, Eur Phys Lett 95 43002 (2011)



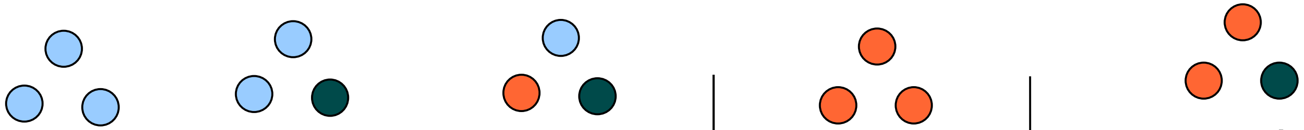
More than 4 bosons



“Universal pentamers tied to Efimov trimers”

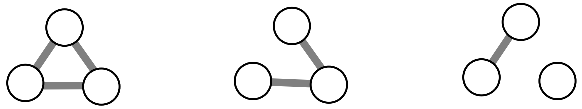
Mixtures

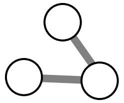
Particles of different statistics (bosons, fermions)
Particles of different masses
Particles in different spin states



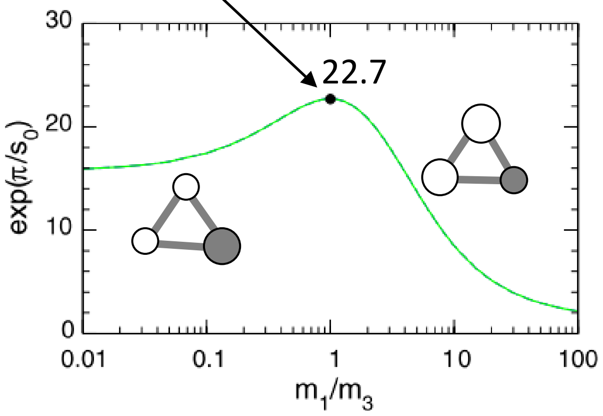
	3 Bosons	2 Bosons + 1 particle	3 Distin-guishable	3 Fermions	2 Fermions + 1 particle
Efimov effect	Yes	Yes	No	No	Yes (For mass ratio > **13.607**)
Scaling factor $e^{\pi/\lvert s_0\rvert}$ for equal masses	22.7	1986.1			

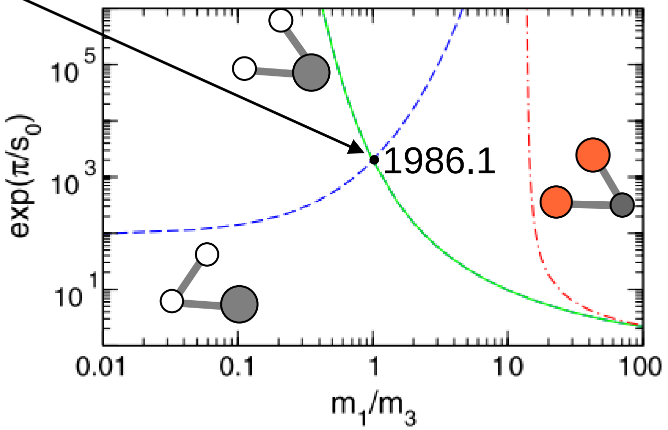
Number of resonant pairs



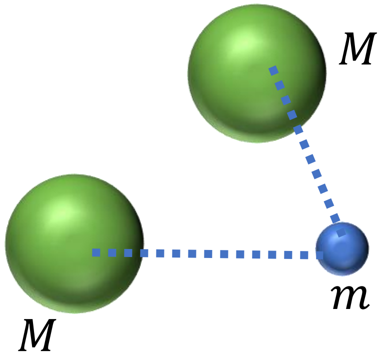


Scaling factor $e^{\pi/\lvert s_0\rvert}$ for unequal masses



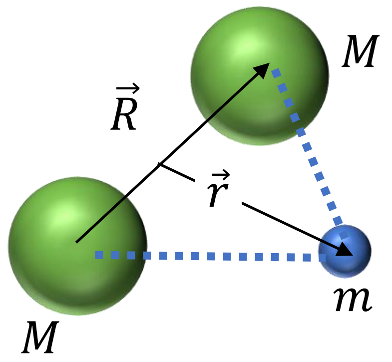


1 particle interacting with 2 identical particles



(no interaction between the two identical particles)

The Born-Oppenheimer approximation



$$\Psi(\vec{R},\vec{r}) = F(\vec{R})\phi(\vec{r};\vec{R})$$ (like an electron with two nuclei)

Solve the motion of the light particle in presence of the 2 heavy particles:

$$-\frac{\hbar^2}{2m}\nabla_r^2\phi(\vec{r};\vec{R}) = \underbrace{-\frac{\hbar^2\kappa(R)^2}{2m}}_{\text{Born-Oppenheimer potential}}\phi(\vec{r};\vec{R})$$ and the two-body conditions

Lowest energy solution:

$$\phi(\vec{r};\vec{R}) = \frac{\exp(-\kappa|\vec{r}-\vec{R}/2|)}{|\vec{r}-\frac{\vec{R}}{2}|} + \frac{\exp(-\kappa|\vec{r}+\vec{R}/2|)}{|\vec{r}+\frac{\vec{R}}{2}|}$$

$$\kappa - \frac{e^{-\kappa R}}{R} = 1/a \xrightarrow{a\to\infty} \boxed{\kappa(R) = \frac{\Omega}{R}}$$

$\Omega = W(1) \approx 0.567$

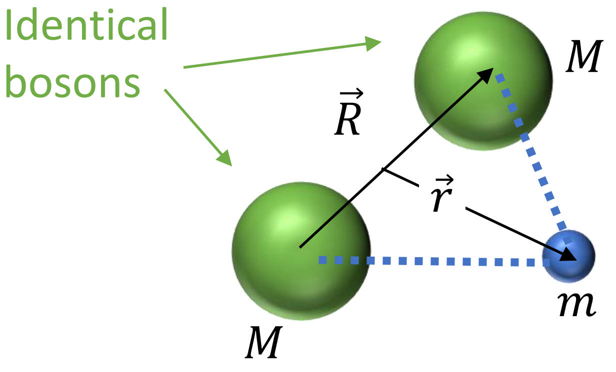
Motion of the heavy particles:

$$\left(-\frac{\hbar^2}{M}\nabla_R^2 - \frac{\hbar^2\kappa(R)^2}{2m}\right)F(\vec{R}) = EF(\vec{R})$$

For a certain angular momentum L between the two heavy particles:
$F(\vec{R}) = \frac{u_L(R)}{R}P_L(\cos\theta)$

$$\left(-\frac{d^2}{dR^2} + \frac{L(L+1)}{R^2} - \frac{M}{2m}\frac{\Omega^2}{R^2}\right)u_L(R) = \frac{ME}{\hbar^2}u_L(R)$$

The Born-Oppenheimer approximation



Competition

Centrifugal repulsion

Efimov attraction

$$\left(-\frac{d^2}{dR^2}+\underbrace{\frac{L(L+1)}{R^2}-\frac{M}{2m}\frac{\Omega^2}{R^2}}_{V_0(r)=-\frac{|s_0|^2+\frac{1}{4}}{R^2}}\right)u_L(R) \quad =\frac{ME}{\hbar^2}u_L(R)$$

$$|s_0|^2=\frac{M}{2m}\Omega^2-L(L+1)-\frac{1}{4}$$

For $L=0$

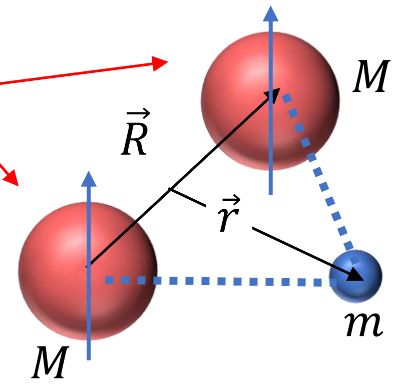
No centrifugal repulsion. Efimov attraction wins. ➡ **Efimov effect**

physics depend on a and Λ

Stronger for large mass ratio M/m

The Born-Oppenheimer approximation

Identical fermions
(in the same spin state)



Competition

Centrifugal repulsion — Efimov attraction

$$\left(-\frac{d^2}{dR^2}+\underbrace{\frac{L(L+1)}{R^2}-\frac{M}{2m}\frac{\Omega^2}{R^2}}_{V_0(r)\,=\,-\frac{|s_0|^2+\frac{1}{4}}{R^2}}\right)u_L(R) \quad = \frac{ME}{\hbar^2}u_L(R)$$

$$|s_0|^2=\frac{M}{2m}\Omega^2-L(L+1)-\frac{1}{4}$$

For $L=1$

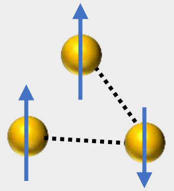
Critical mass ratio such that $|s_0|=0$: $\frac{M}{m}=13.990296\ldots$ (exact result: $13.607\ldots$)

$\frac{M}{m}>13.6$: Efimov attraction wins. ➡ **Efimov effect**
physics depend on a and Λ

$\frac{M}{m}<13.6$: Efimov attraction loses. ➡ **no Efimov effect**
physics depend only on a

Particular case

No Efimov effect for 3 fermions with spin 1/2



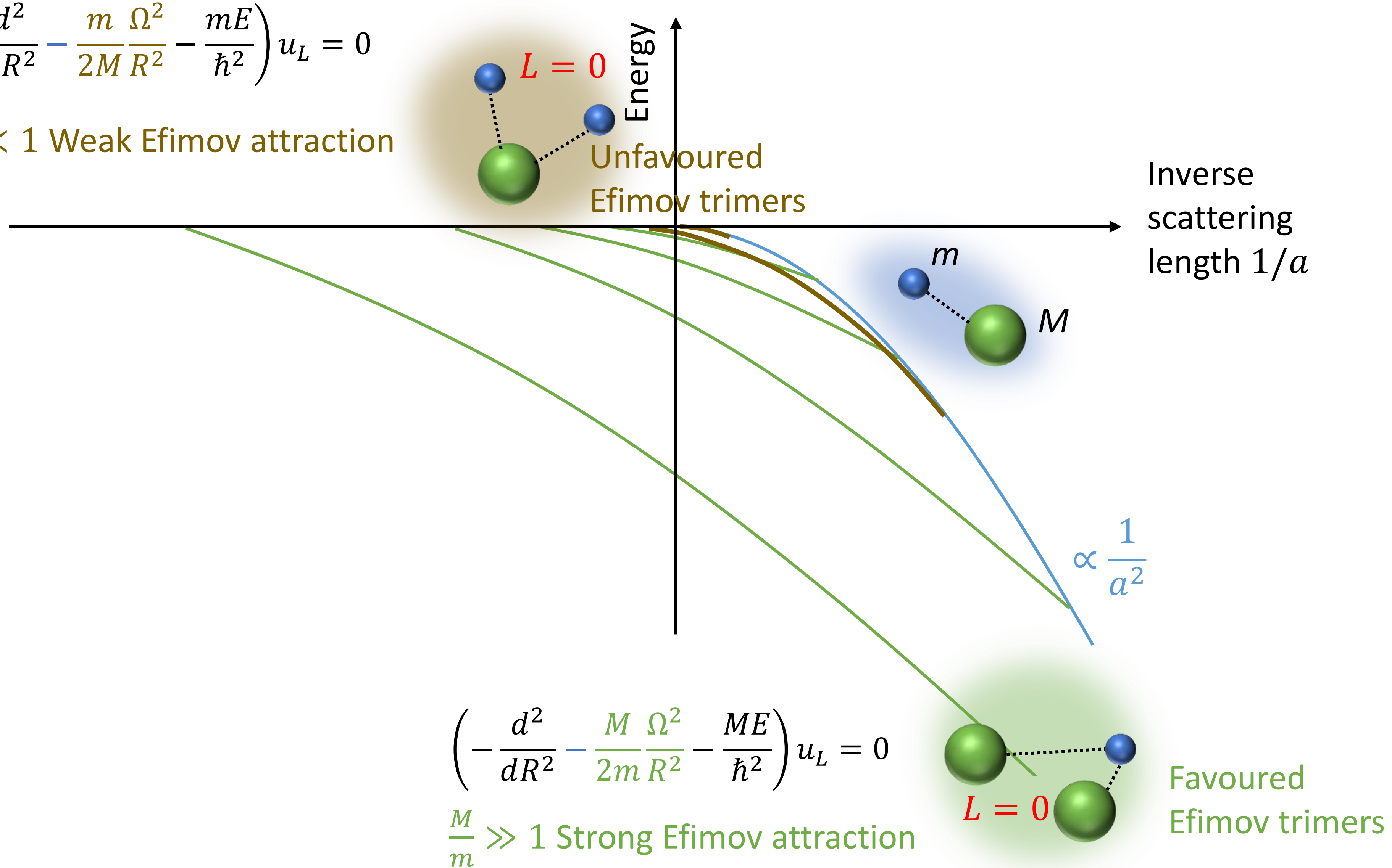
$\frac{M}{m}=1$

e.g. no trineutron

Mixtures of two kinds of bosons

$$\left(-\frac{d^2}{dR^2} - \frac{m}{2M}\frac{\Omega^2}{R^2} - \frac{mE}{\hbar^2}\right)u_L = 0$$

$\frac{m}{M} \ll 1$ Weak Efimov attraction



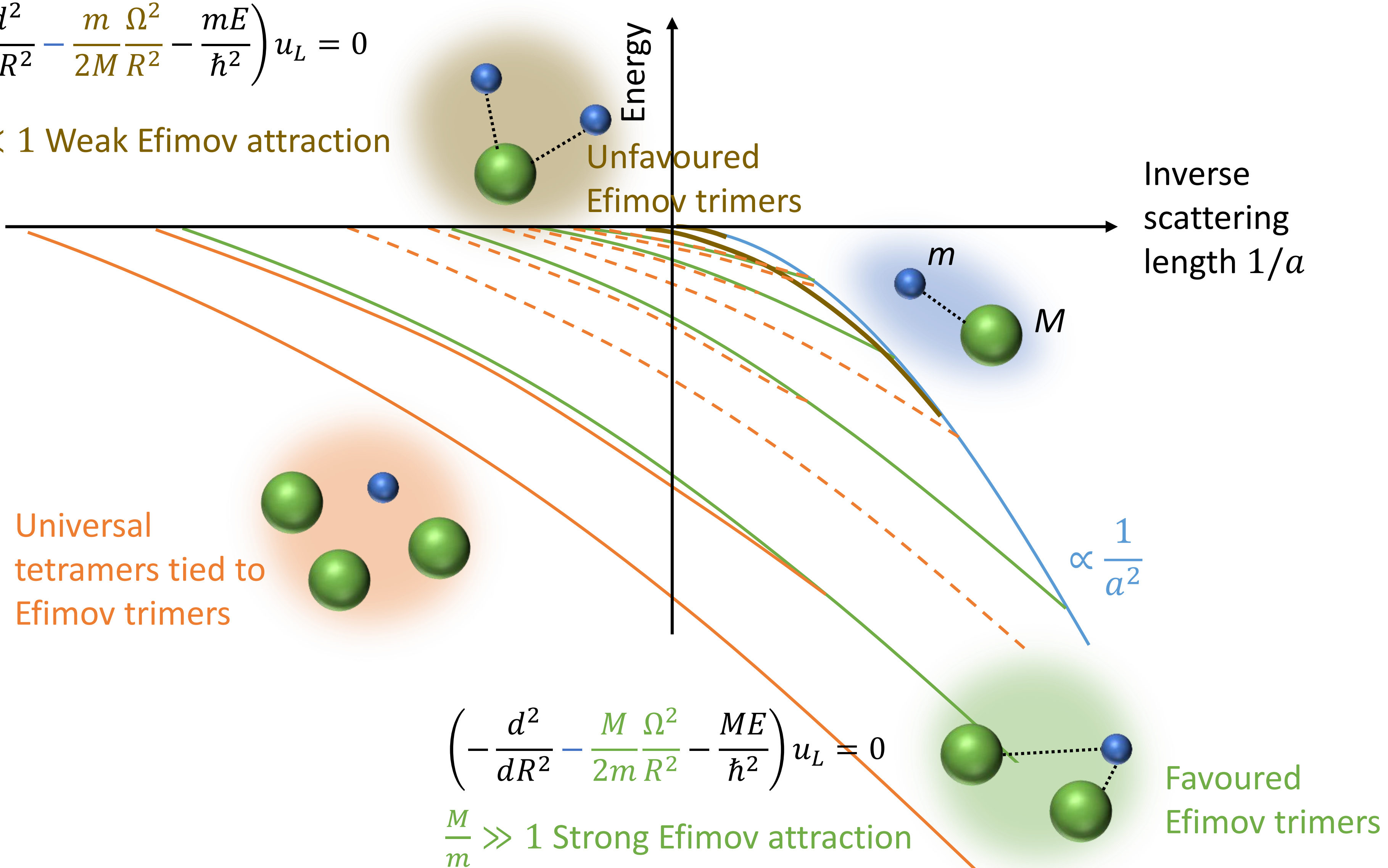
$$\left(-\frac{d^2}{dR^2} - \frac{M}{2m}\frac{\Omega^2}{R^2} - \frac{ME}{\hbar^2}\right)u_L = 0$$

$\frac{M}{m} \gg 1$ Strong Efimov attraction

Mixtures of two kinds of bosons

$$\left(-\frac{d^2}{dR^2} - \frac{m}{2M}\frac{\Omega^2}{R^2} - \frac{mE}{\hbar^2}\right)u_L = 0$$

$\frac{m}{M} \ll 1$ Weak Efimov attraction



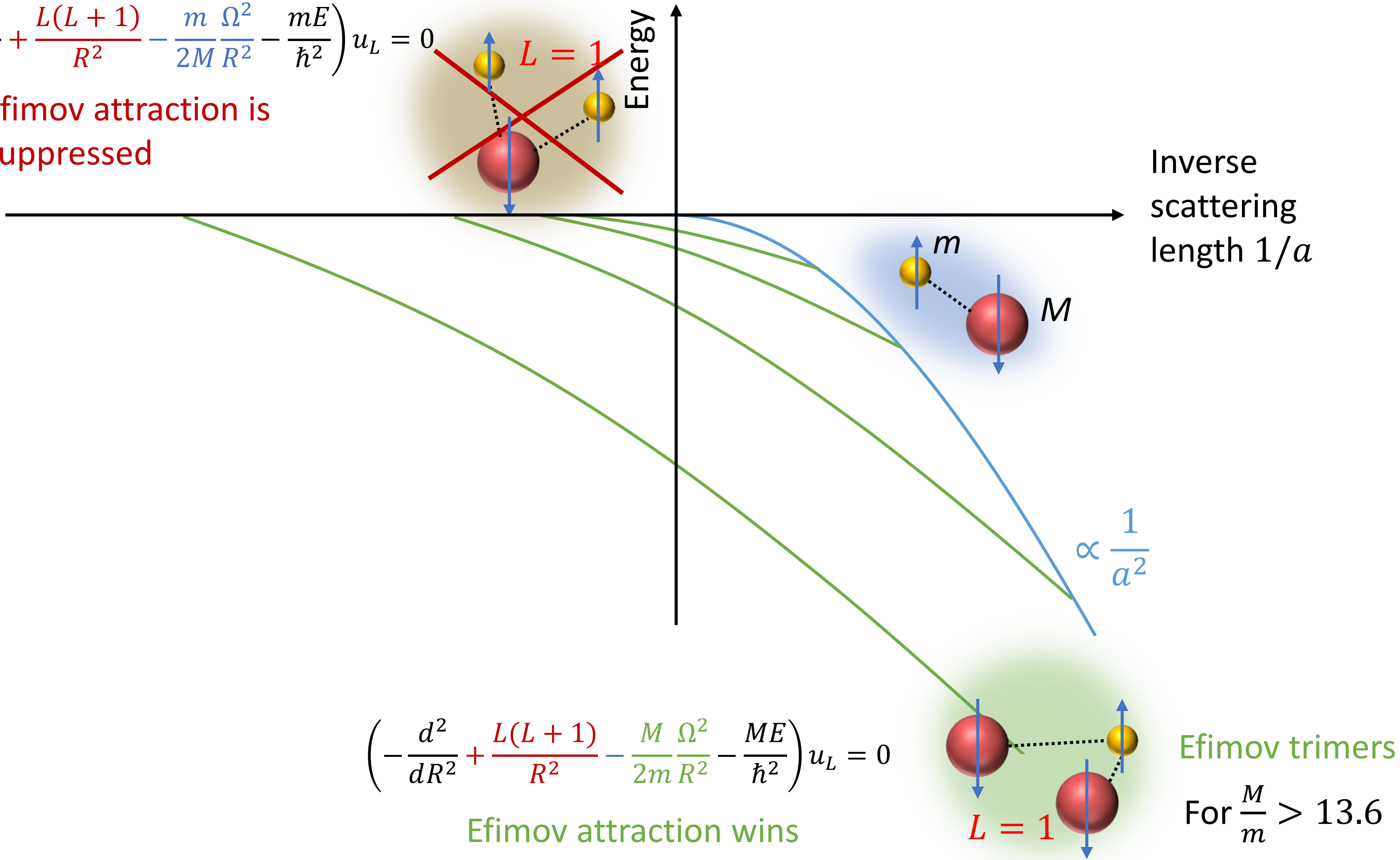
$$\left(-\frac{d^2}{dR^2} - \frac{M}{2m}\frac{\Omega^2}{R^2} - \frac{ME}{\hbar^2}\right)u_L = 0$$

$\frac{M}{m} \gg 1$ Strong Efimov attraction

Mixtures of two kinds of fermions (polarised = spinless)

$$\left(-\frac{d^2}{dR^2}+\frac{L(L+1)}{R^2}-\frac{m}{2M}\frac{\Omega^2}{R^2}-\frac{mE}{\hbar^2}\right)u_L=0$$

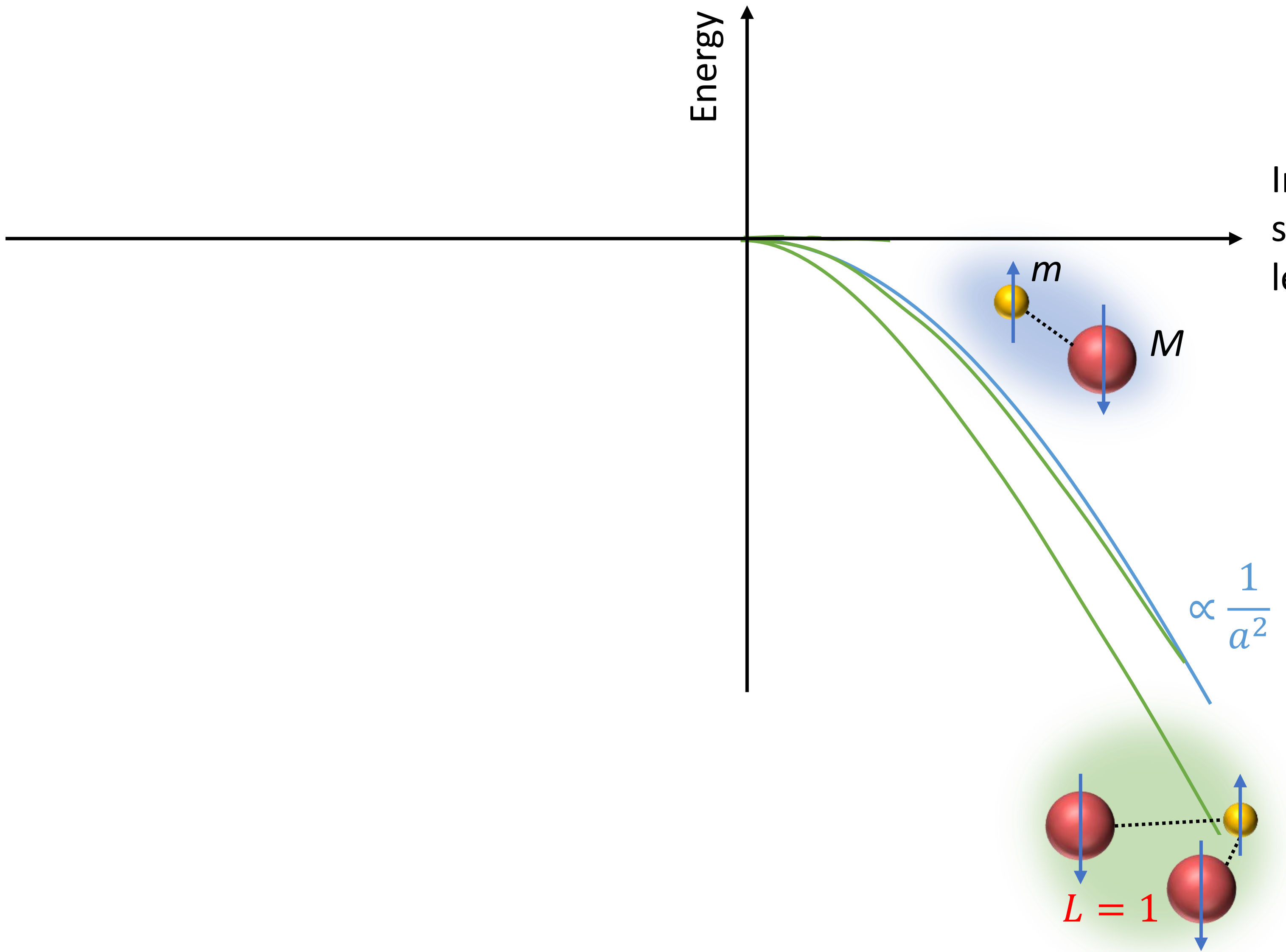
Efimov attraction is suppressed



$$\left(-\frac{d^2}{dR^2}+\frac{L(L+1)}{R^2}-\frac{M}{2m}\frac{\Omega^2}{R^2}-\frac{ME}{\hbar^2}\right)u_L=0$$

Efimov attraction wins

Mixtures of two kinds of fermions (polarised = spinless)

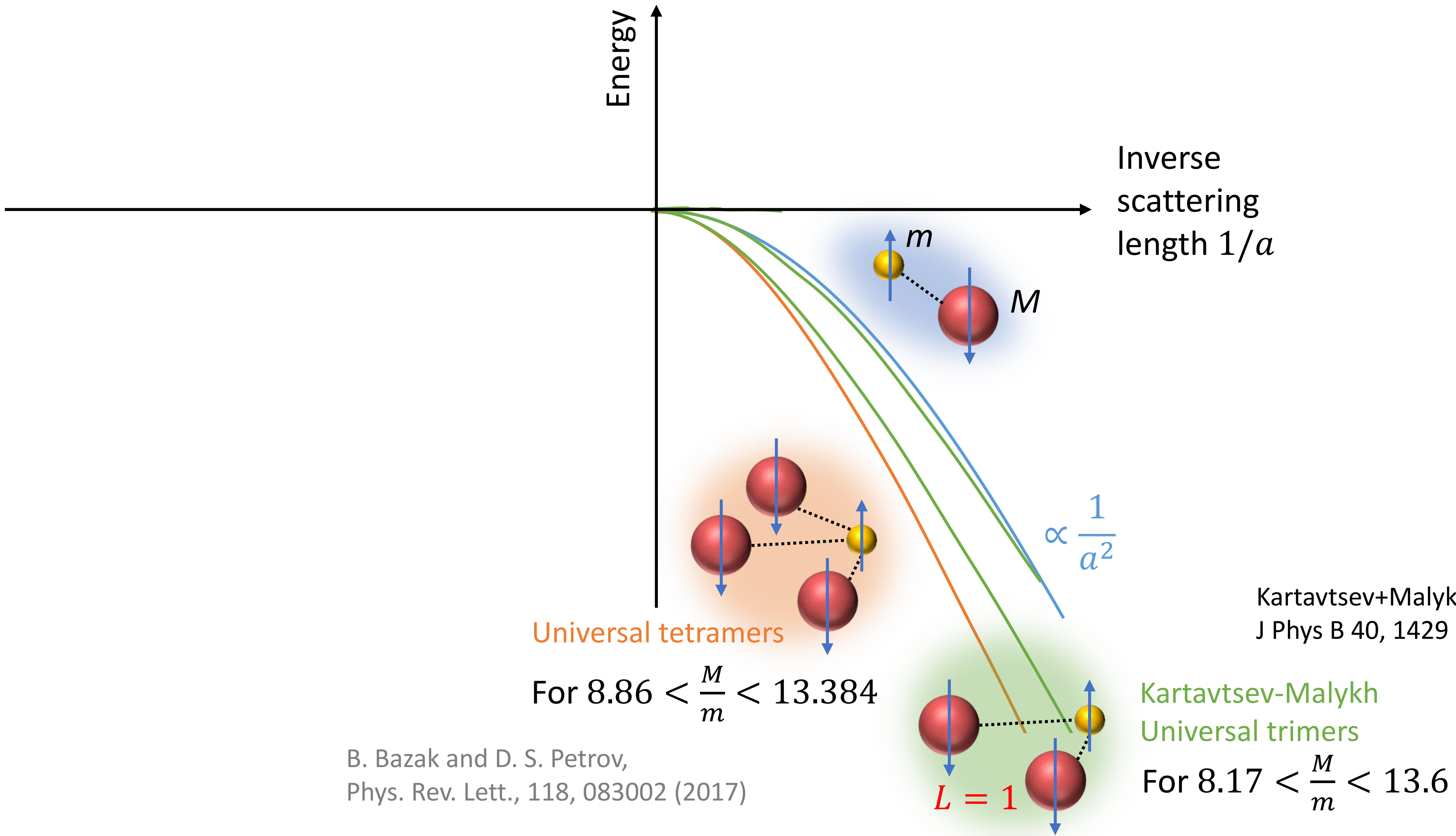


Kartavtsev+Malykh
J Phys B 40, 1429 (2007)

Kartavtsev-Malykh
Universal trimers

For $8.17 < \frac{M}{m} < 13.6$

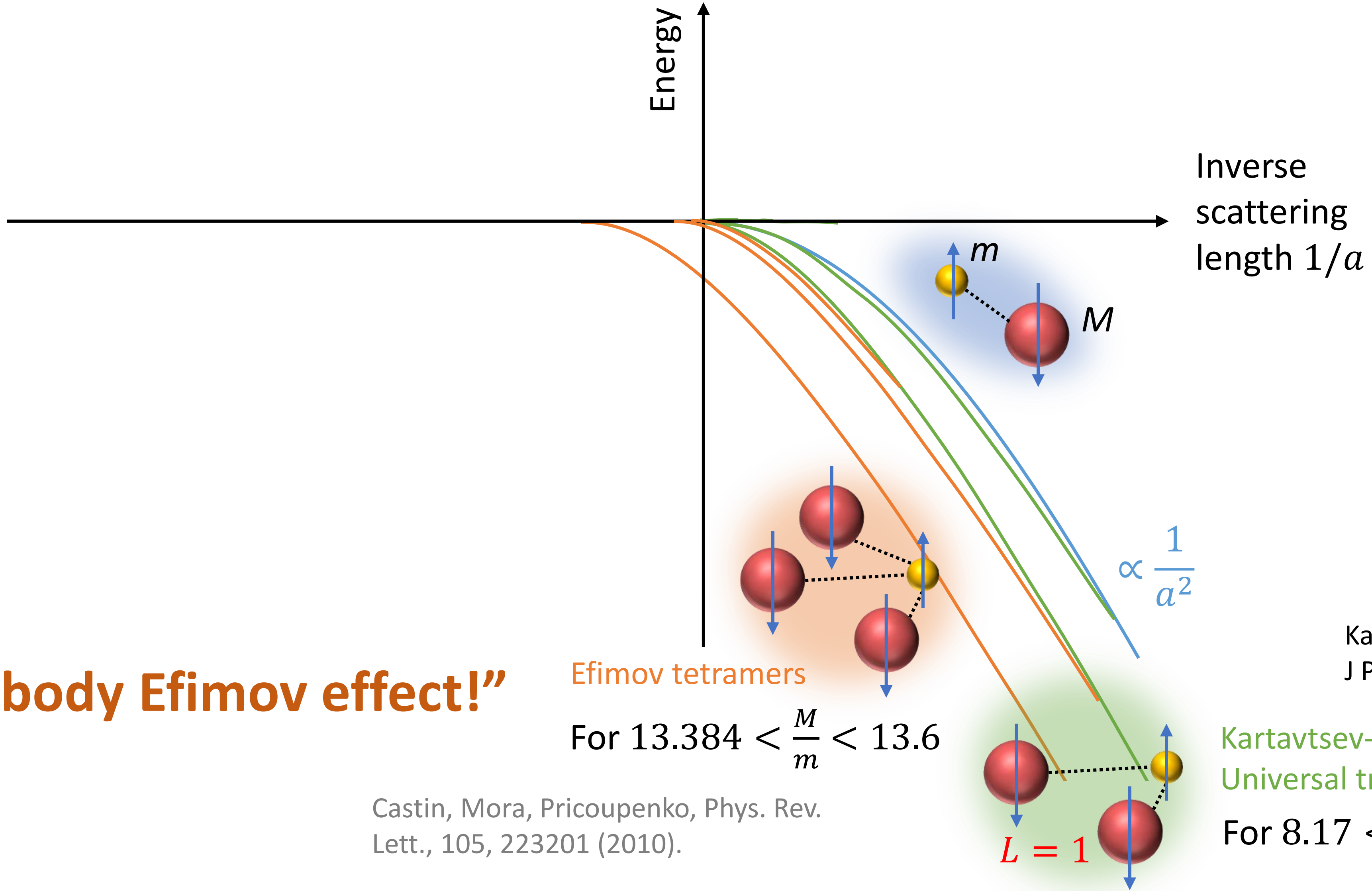
Mixtures of two kinds of fermions (polarised = spinless)



Kartavtsev+Malykh
J Phys B 40, 1429 (2007)

B. Bazak and D. S. Petrov,
Phys. Rev. Lett., 118, 083002 (2017)

Mixtures of two kinds of fermions (polarised = spinless)



"4-body Efimov effect!"

Efimov tetramers

For $13.384 < \frac{M}{m} < 13.6$

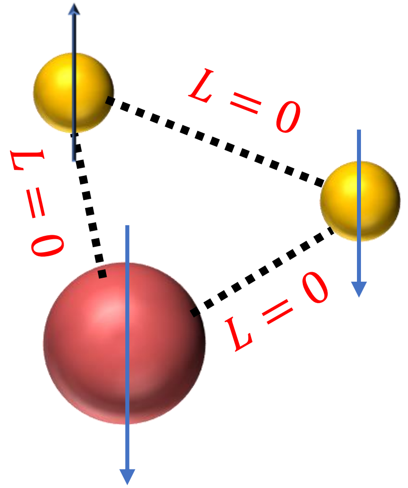
Castin, Mora, Pricoupenko, Phys. Rev. Lett., 105, 223201 (2010).

Kartavtsev+Malykh
J Phys B 40, 1429 (2007)

Kartavtsev-Malykh
Universal trimers

For $8.17 < \frac{M}{m} < 13.6$

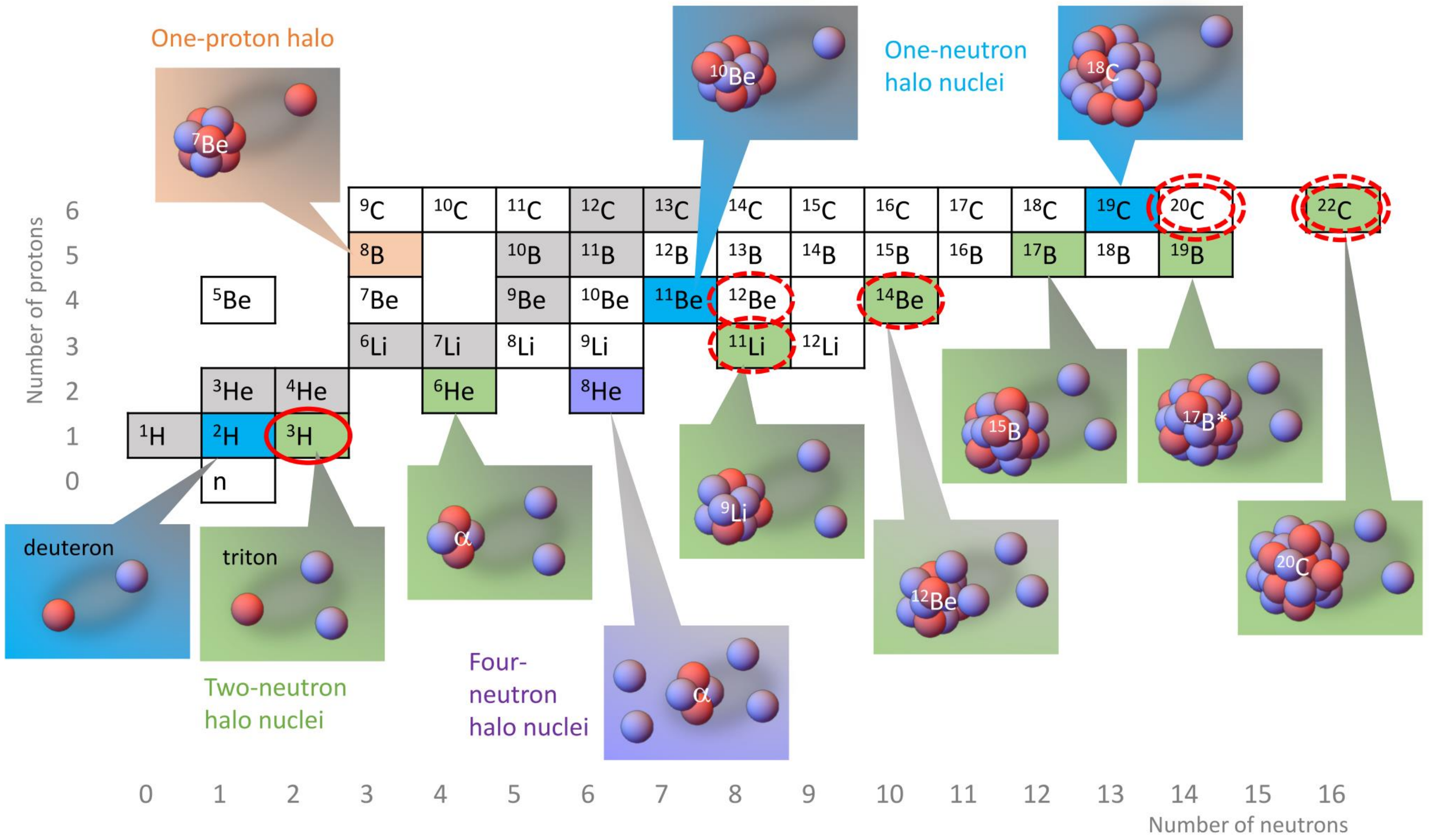
Two kinds of fermions with spin



(All pairs can interact in the s wave)

→ Two-neutron halo nuclei?

Halo nuclei



Efimov trimer candidates: Ground state

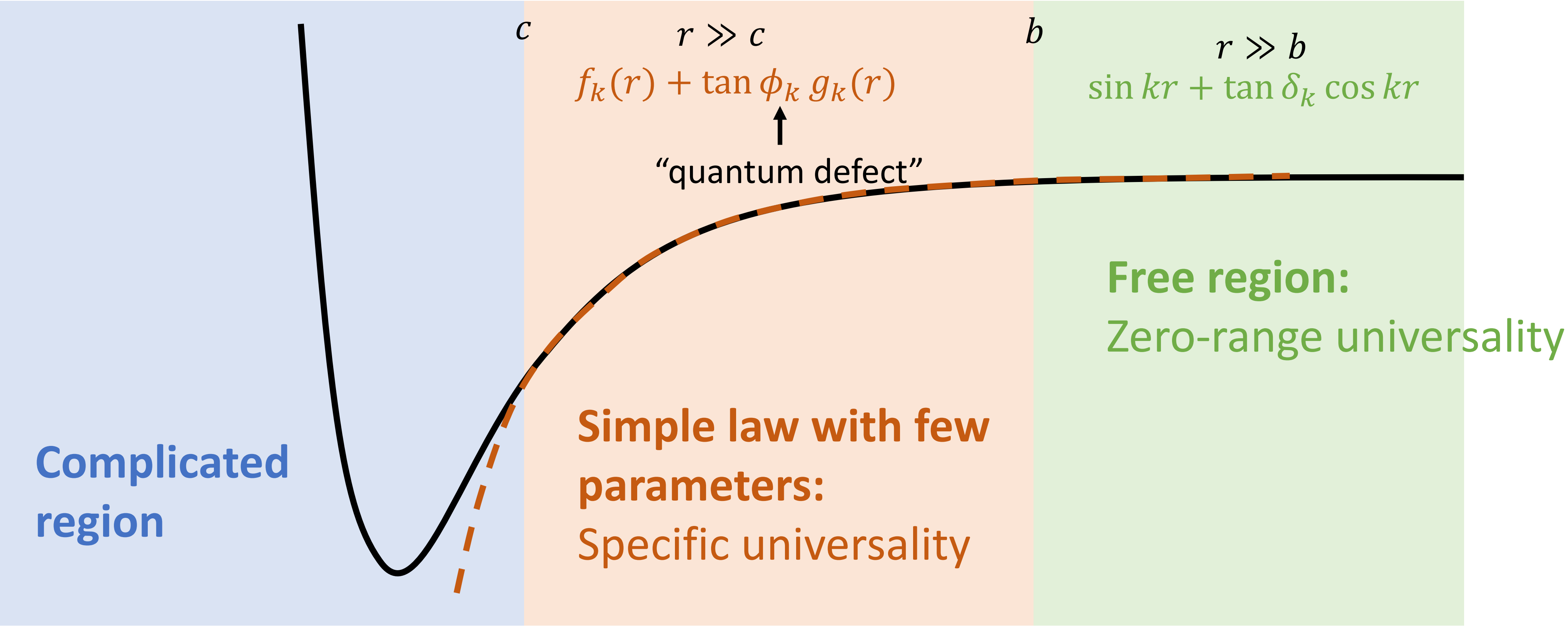
First excited state

(to be confirmed)

Van der Waals universality

Two-body van der Waals universality
The three-body parameter and its van der Waals universality

Other classes of universalities



Examples: Electron in a Rydberg atom $V(r) = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$

Two neutral atoms $V(r) = -C_6/r^6$

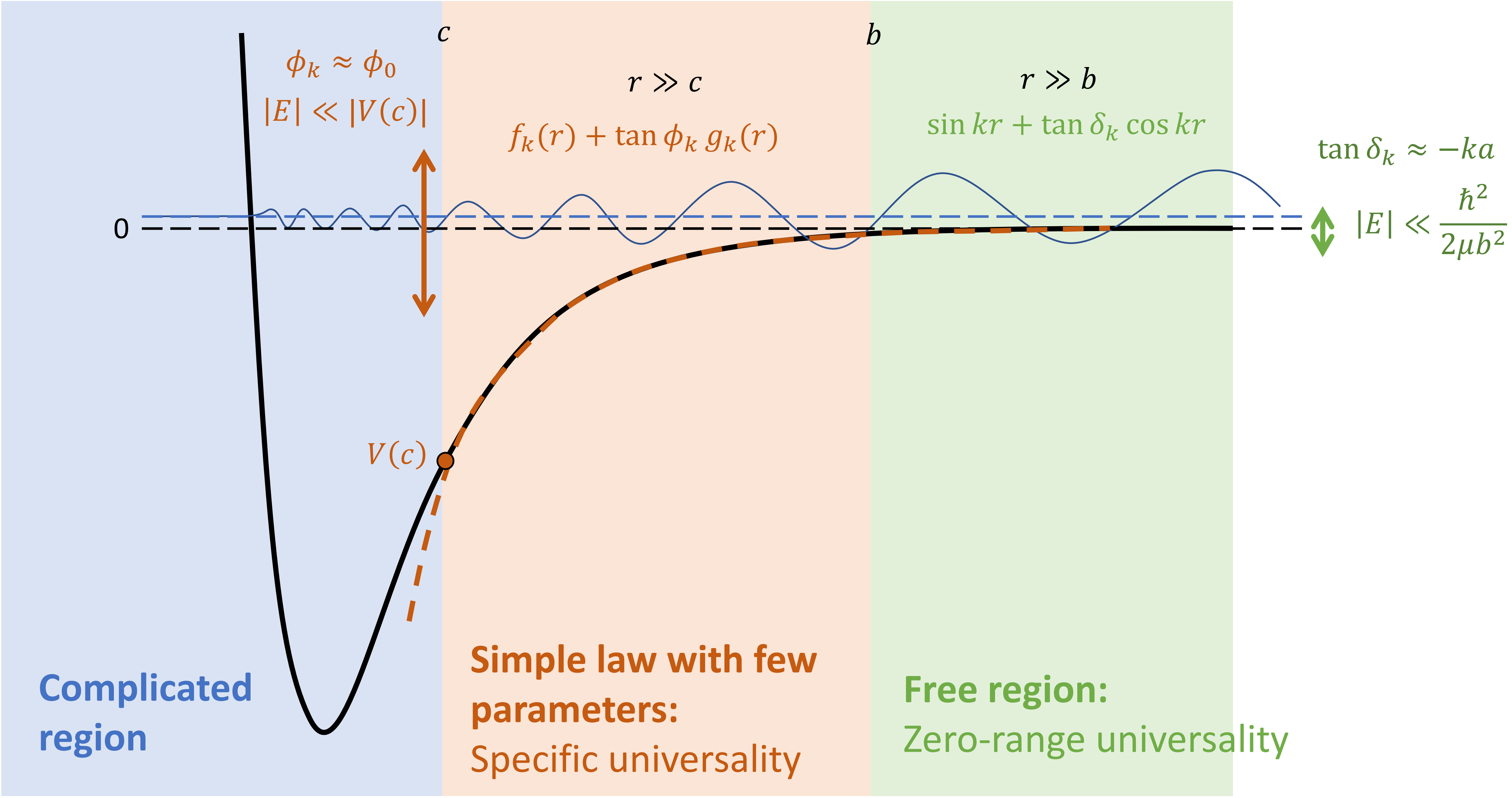
$V(r) \sim \frac{\hbar^2}{2\mu r^2}$

Interaction $\sim$ kinetic

Bohr radius: $a_0 = \frac{4\pi\epsilon_0}{e^2}\frac{\hbar^2}{2\mu}$

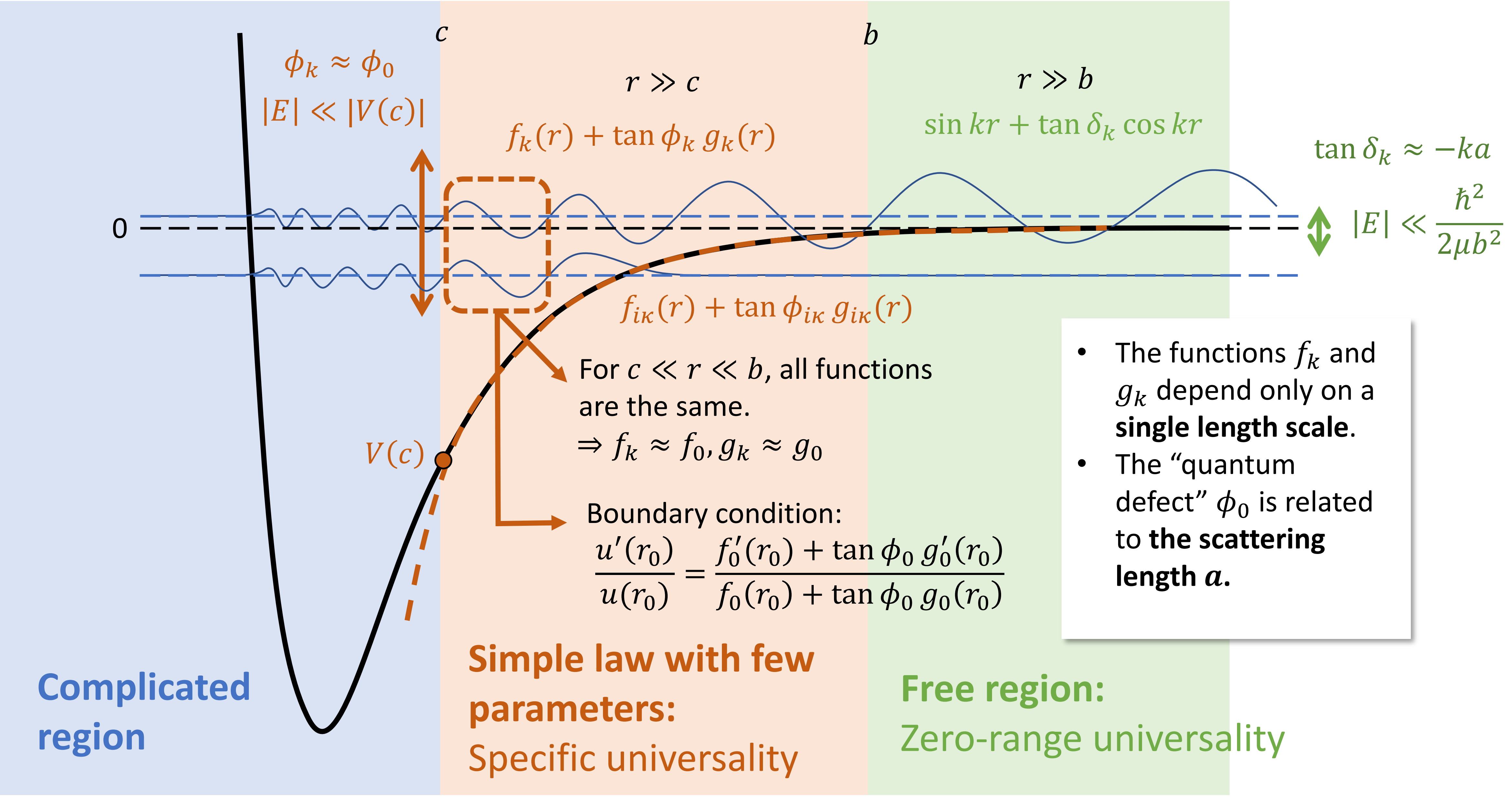
Van der Waals length: $l_{\mathrm{vdW}} = \frac{1}{2}\left(\frac{2\mu C_6}{\hbar^2}\right)^{1/4}$

(not radius)



c
b
$\phi_k \approx \phi_0$
$|E| \ll |V(c)|$
$r \gg c$
$f_k(r) + \tan\phi_k\, g_k(r)$
$r \gg b$
$\sin kr + \tan\delta_k \cos kr$
$\tan\delta_k \approx -ka$
0
$|E| \ll \frac{\hbar^2}{2\mu b^2}$
$V(c)$
Complicated region
Simple law with few parameters:
Specific universality
Free region:
Zero-range universality

5. Van der Waals universality



- The functions f_k and g_k depend only on a **single length scale**.
- The "quantum defect" ϕ_0 is related to **the scattering length $\boldsymbol{a}$.**

Two neutral atoms

Solution of the $-C_6/r^6$ potential

Radial equation in the s wave:

$$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2}-\frac{C_6}{r^6}u=Eu$$

$$-\frac{d^2u}{dr^2}-\frac{2\mu C_6}{\hbar^2 r^6}u=\frac{2\mu E}{\hbar^2}u$$

Dimensionless variable:

$x \equiv r/\ell_{\mathrm{vdW}}$

$$-\frac{1}{\ell_{\mathrm{vdW}}^2}\frac{d^2u}{dx^2}-\frac{1}{\ell_{\mathrm{vdW}}^6}\frac{2\mu C_6}{\hbar^2 x^6}u=\frac{2\mu E}{\hbar^2}u$$

$$-\frac{d^2u}{dx^2}-\frac{1}{\ell_{\mathrm{vdW}}^4}\frac{2\mu C_6}{\hbar^2 x^6}u=\frac{2\mu E\ell_{\mathrm{vdW}}^2}{\hbar^2}u$$

We set

$$\varepsilon \equiv \frac{2\mu E\ell_{\mathrm{vdW}}^2}{\hbar^2}$$

$$l_{\mathrm{vdW}} \equiv \frac{1}{2}\left(\frac{2\mu C_6}{\hbar^2}\right)^{1/4}$$

$$\boxed{-u''(x)-\frac{16}{x^6}u(x)=\varepsilon u(x)}$$

Solution of the $-C_6/r^6$ potential at zero energy

$$u''(x)+\frac{16}{x^6}u(x)=0$$

Solution of the $-C_6/r^6$ potential at zero energy

$$u''(x) + \frac{16}{x^6}u(x) = 0$$

Change of variable: $u(x) \equiv x^{1/2} v(\underbrace{2x^{-2}}_{y})$

$$u'(x) = x^{\frac{1}{2}} \times (-4x^{-3})v' + \frac{1}{2}x^{-\frac{1}{2}} \times v$$

$$u'(x) = -4x^{-5/2}v' + \frac{1}{2}x^{-\frac{1}{2}}v$$

$$u''(x) = -4x^{-5/2} \times (-4x^{-3})\, v'' + 10x^{-7/2}v' + \frac{1}{2}x^{-\frac{1}{2}}(-4x^{-3})v' - \frac{1}{4}x^{-\frac{3}{2}} \times v$$

$$u''(x) = 16x^{-\frac{11}{2}}v'' + 8x^{-\frac{7}{2}}v' - \frac{1}{4}x^{-\frac{3}{2}}v$$

$$16x^{-\frac{11}{2}}v'' + 8x^{-\frac{7}{2}}v' - \frac{1}{4}x^{-\frac{3}{2}}v + 16x^{-11/2}v = 0$$

$\times \frac{1}{4}x^{3/2}$

$$4x^{-4}v'' + 2x^{-2}v' - \frac{1}{16}v + 4x^{-4}v = 0$$

$$y^2v'' + yv' - \frac{1}{16}v + y^2v = 0$$

$$\boxed{y^2v'' + yv' + (y^2 - \alpha^2)v = 0}$$

Bessel equation with $\alpha = \pm 1/4$

The solutions are linear combinations of Bessel functions:

$$v(y) = \alpha J_{-\frac{1}{4}}(y) + \beta J_{\frac{1}{4}}(y)$$

$$\boxed{u(x) = \sqrt{x}\left(\alpha J_{-\frac{1}{4}}(2x^{-2}) + \beta J_{\frac{1}{4}}(2x^{-2})\right)}$$

Solution of the $-C_6/r^6$ potential at zero energy

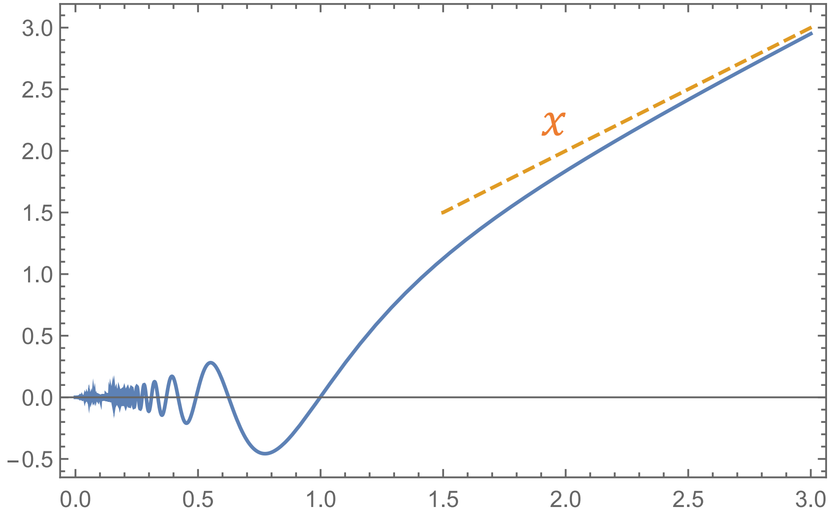
$$u(r) = f(r) - \frac{a}{l_{vdW}} g(r)$$

Van der Waals length

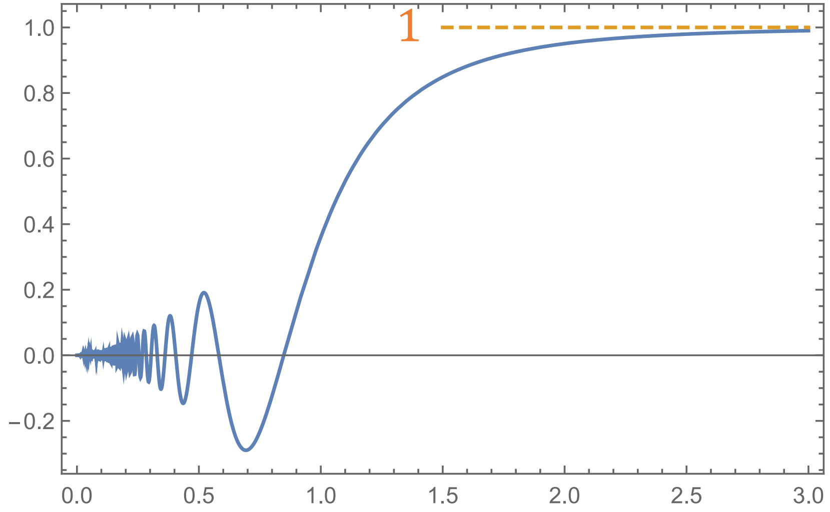
$$l_{vdW} = \frac{1}{2}\left(\frac{mC_6}{\hbar^2}\right)^{1/4}$$

$$f(r) = \Gamma(3/4)\sqrt{x}J_{-\frac{1}{4}}(2x^{-2})$$

$$g(r) = \Gamma(5/4)\sqrt{x}J_{\frac{1}{4}}(2x^{-2})$$

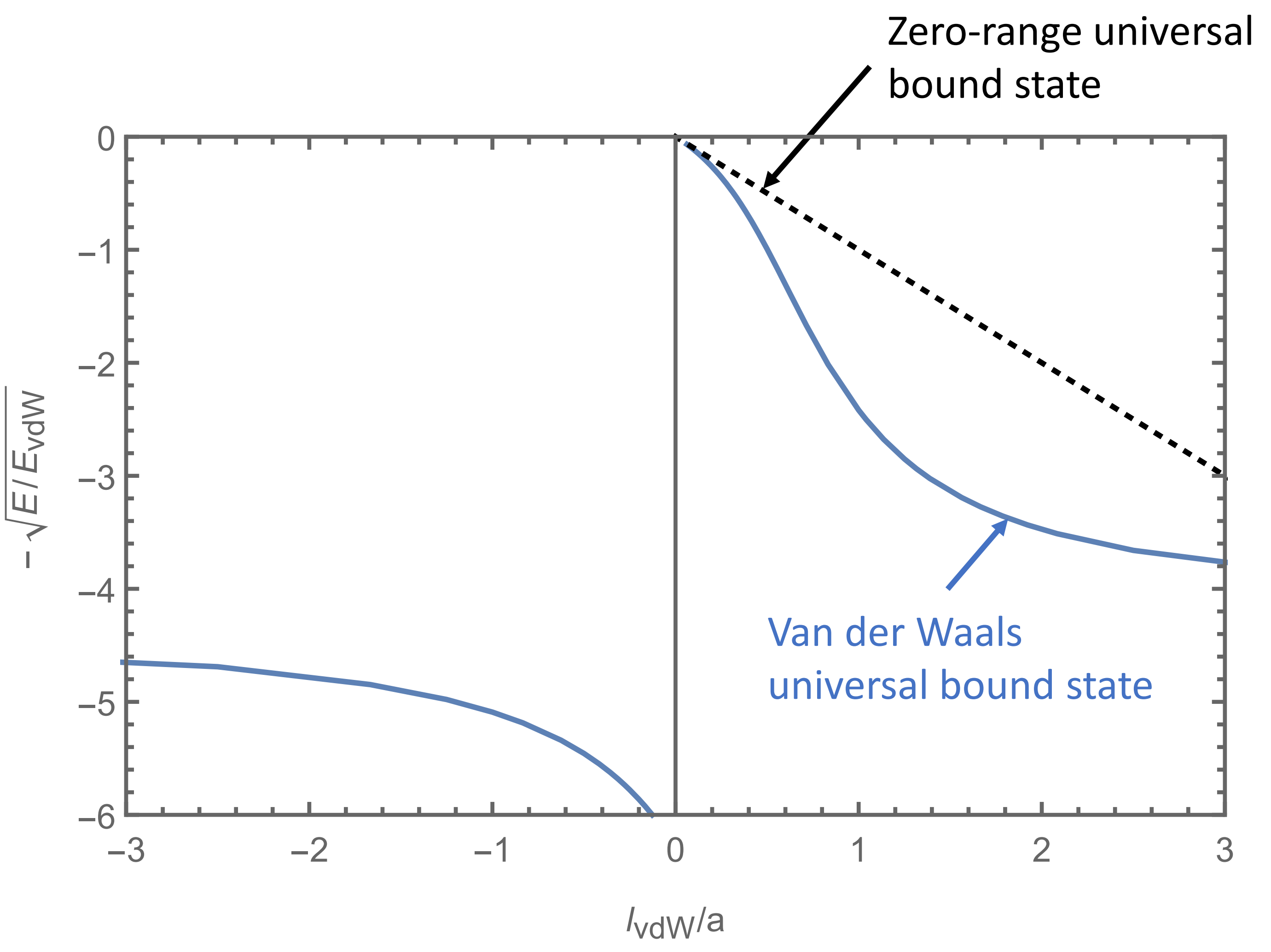


$$x = r/\ell_{vdW}$$



$$x = r/\ell_{vdW}$$

Solution of the $-C_6/r^6$ potential at negative energy



Van der Waals length

$$l_{vdW} = \frac{1}{2}\left(\frac{mC_6}{\hbar^2}\right)^{1/4}$$

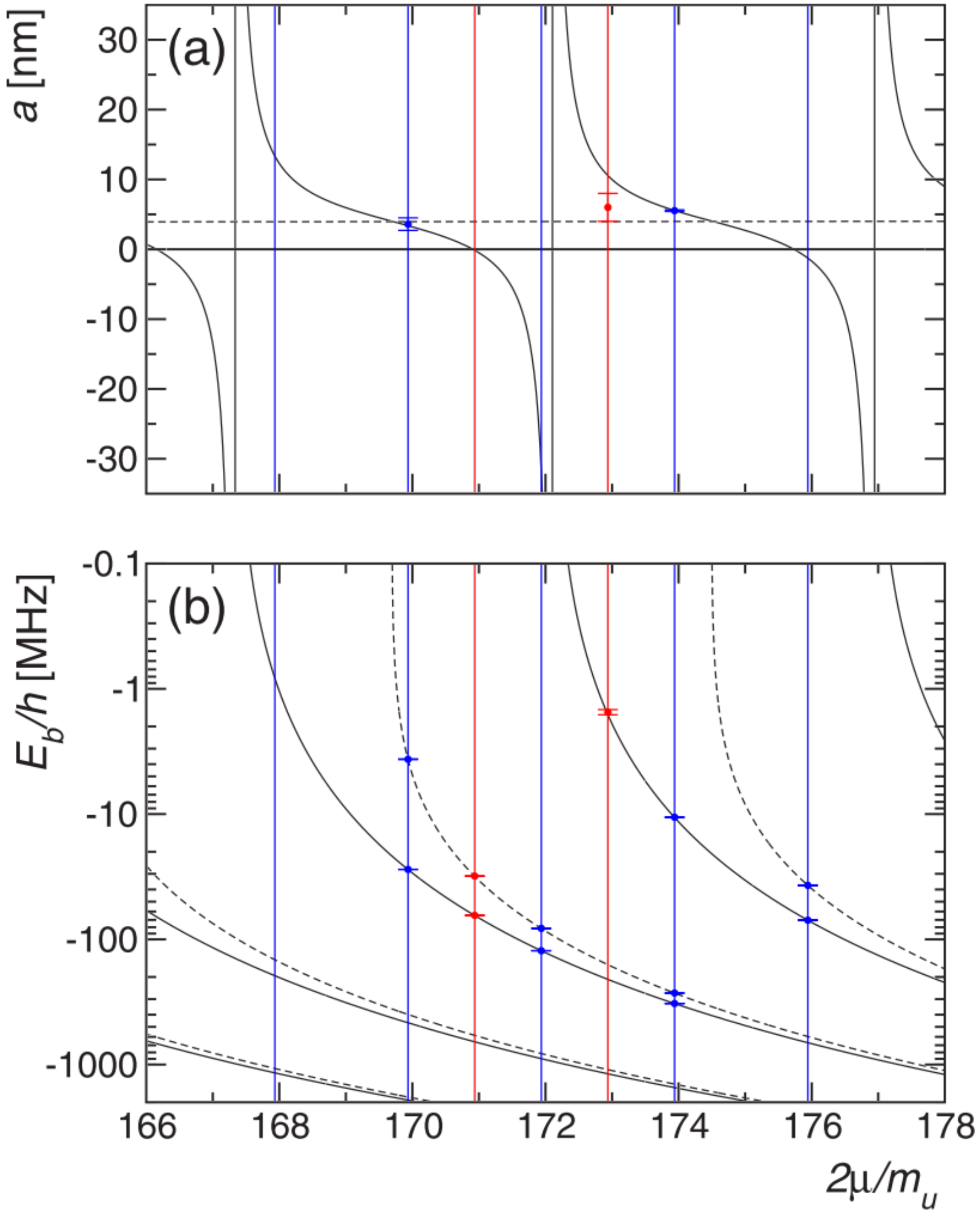
Van der Waals energy scale

$$E_{vdW} = \frac{\hbar^2}{2\mu l_{vdW}^2}$$

Solution of the $-C_6/r^6$ potential at negative energy

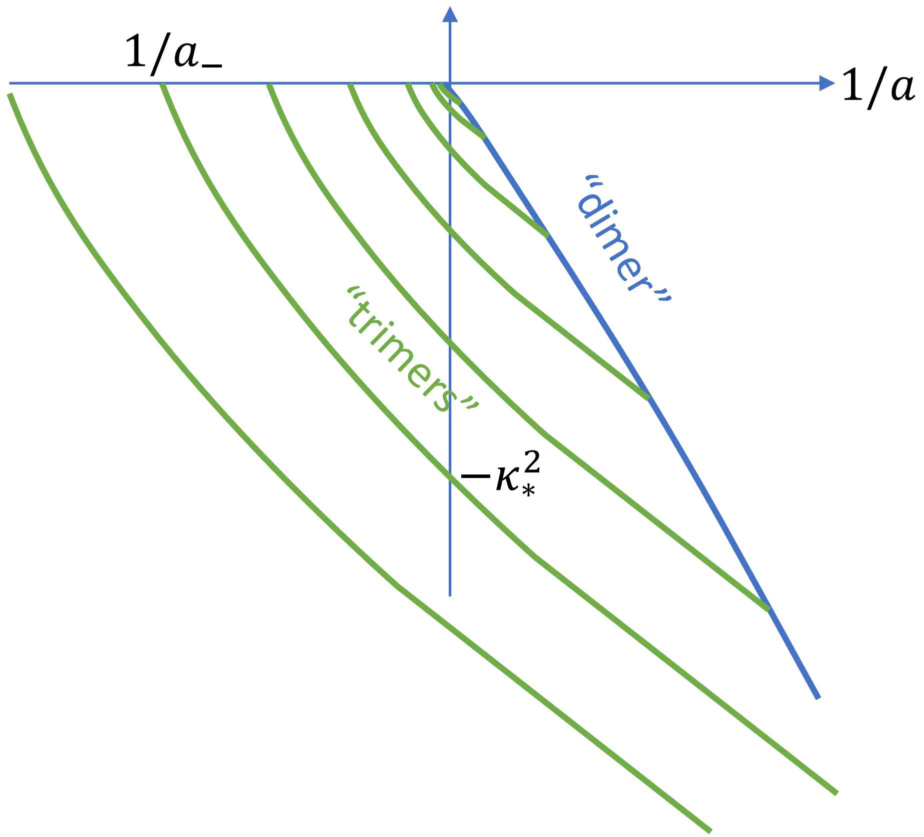
Example: Ytterbium isotopes

Kitagawa et al. Phys Rev A 77, 012719 (2008)



The three-body parameter

In the zero-range theory, the three-body has to be introduced “by hand” as an extra boundary condition to quantise the spectrum:



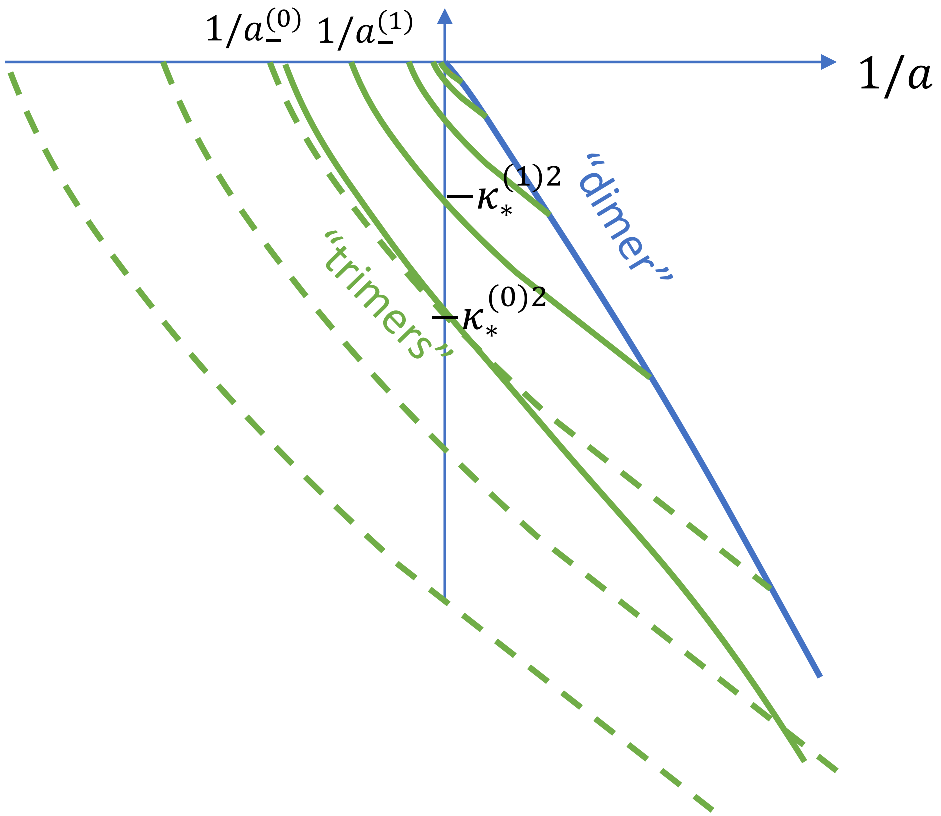
$$a_- = -1.50763/\kappa_*$$

The three-body parameter

For a system with finite-range interactions, the spectrum is bounded from below and the three-body parameter is set from the two-body or three-body interactions

$$\kappa_* = \lim_{n\to\infty} \kappa_*^{(n)} e^{-n\pi/|s_0|}$$

$$a_- = \lim_{n\to\infty} a_-^{(n)} e^{n\pi/|s_0|}$$

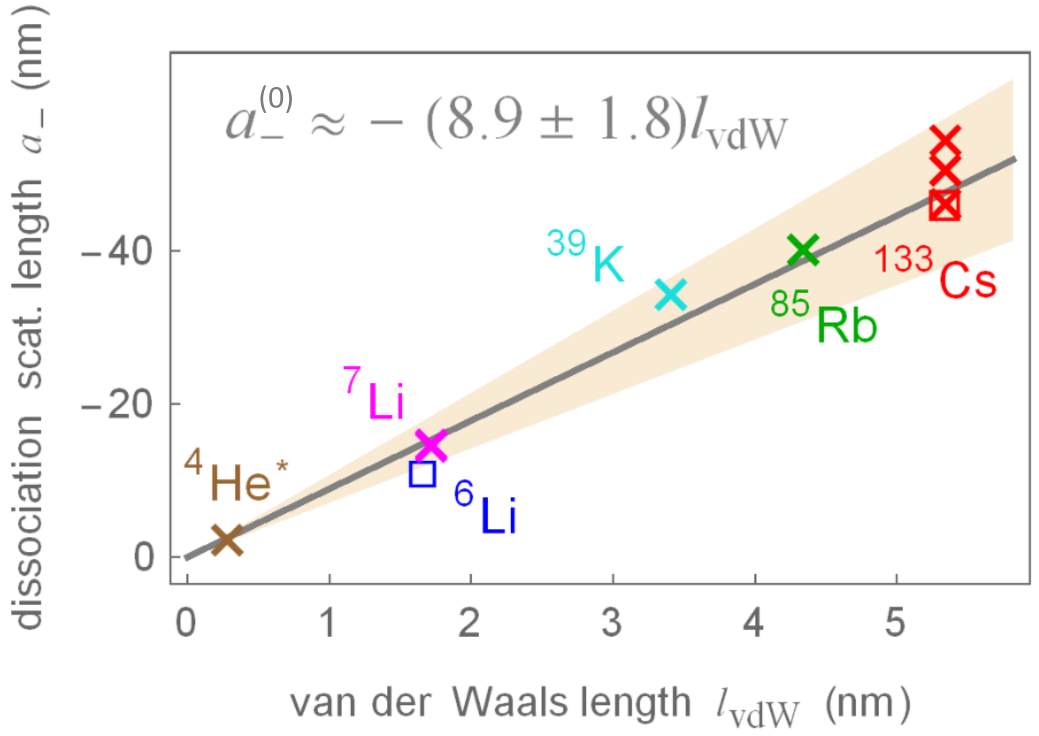


Question: what in the interaction potentials determines the value of the three-body parameter?

Range of two-body forces?
What about three-body forces?

The three-body parameter

Experiments with atoms: “**Van der Waals universality of the three-body parameter**”



The potential between two atoms has a van der Waals tail:

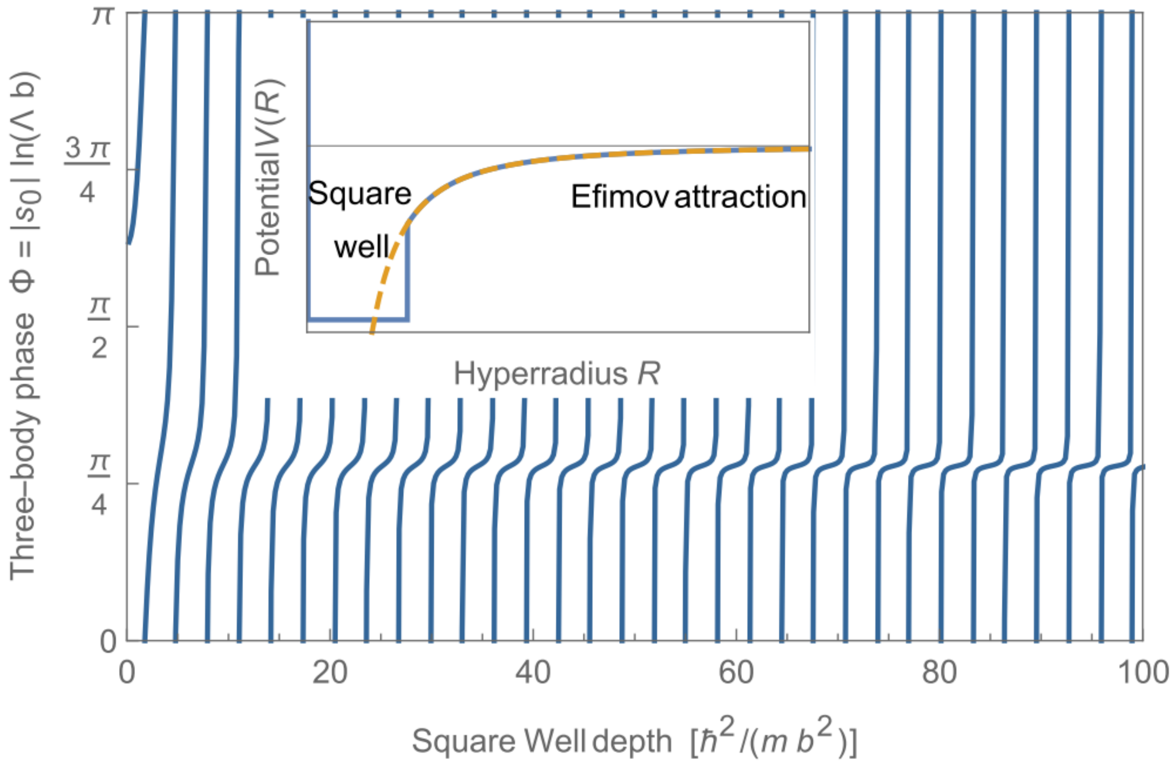
$$V(r) \xrightarrow[r\to\infty]{} -C_6/r^6$$

Van der Waals length

$$l_{vdW} = \frac{1}{2}\left(\frac{mC_6}{\hbar^2}\right)^{1/4}$$

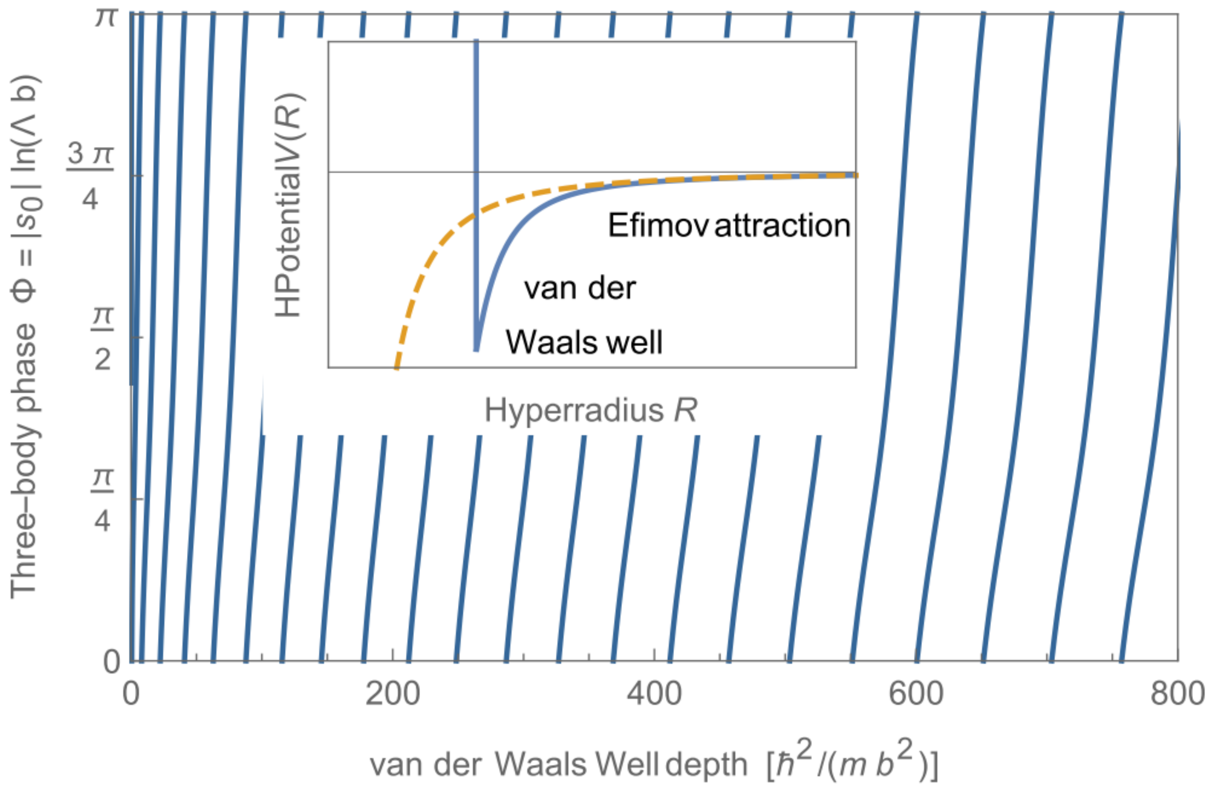
The three-body parameter

Is it quantum reflexion?



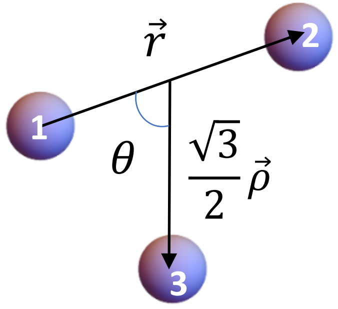
The three-body parameter

Is it quantum reflexion? No.



The three-body parameter

J. Wang, J. D'Incao, B. Esry, and C. Greene, "*Origin of the Three-Body Parameter Universality in Efimov Physics.*" Phys. Rev. Lett., 108, 263001 (2012).



Adiabatic Hyperspherical Representation:

$$\Psi = \sum_n F_n(R)\Phi_n(\theta,\alpha;R)$$

$$\left(-\frac{d^2}{dR^2} + W_n(R) - E\right)F_n(R) + \sum_{n'\neq n} W_{n,n'}(R)F_{n'}(R) = 0$$

Hyper-radius R
Hyper-angle α

$$r = R\sin\alpha$$
$$\rho = R\cos\alpha$$

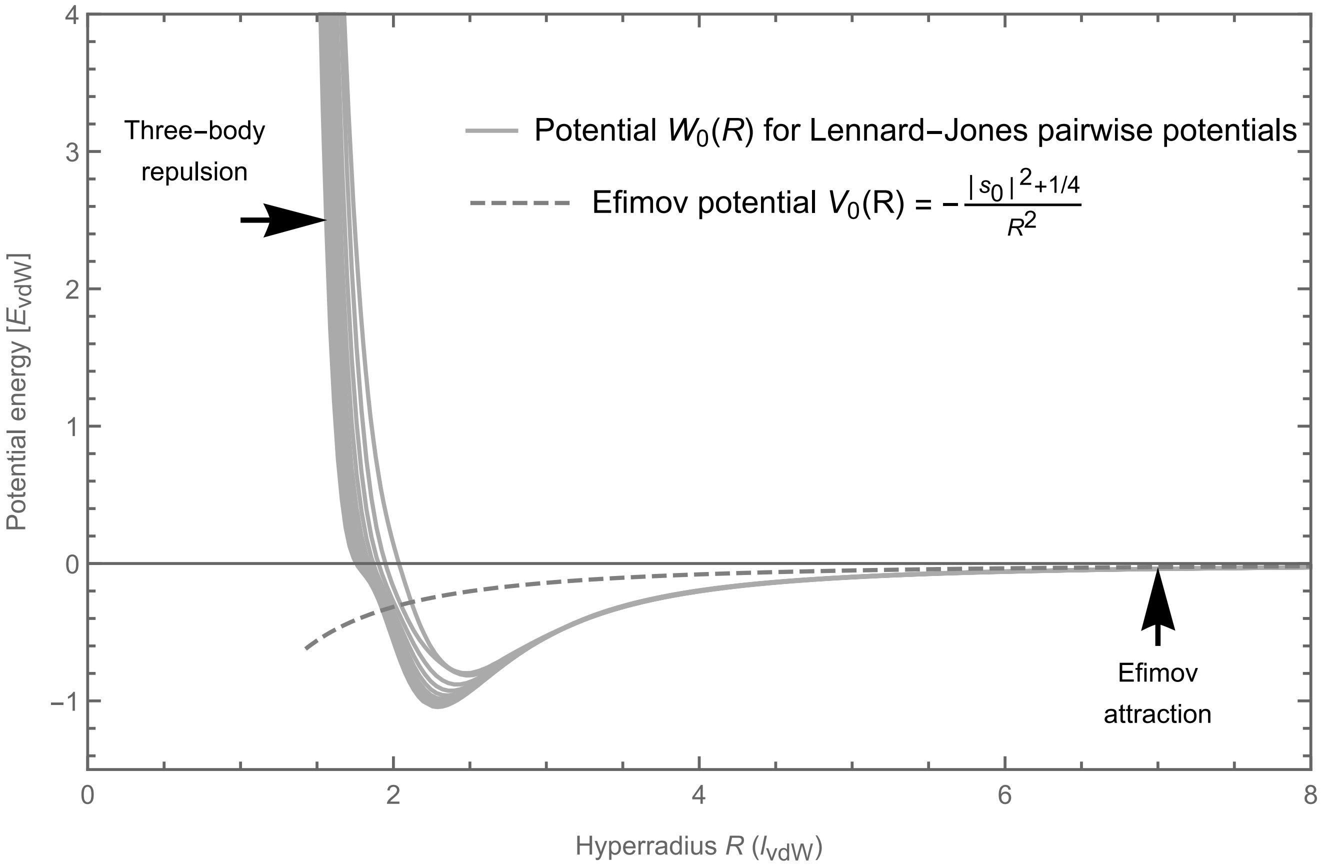
Solved for various two-body interactions with a van der Waals tail.
In the limit of deep van der Waals interactions:

$$\boxed{\kappa_*^{(0)} = (0.21 \pm 0.01)/\ell_{\mathrm{vdW}}}$$

$$\boxed{a_-^{(0)} = -(10.70 \pm 0.35)\,\ell_{\mathrm{vdW}}}$$

The three-body parameter

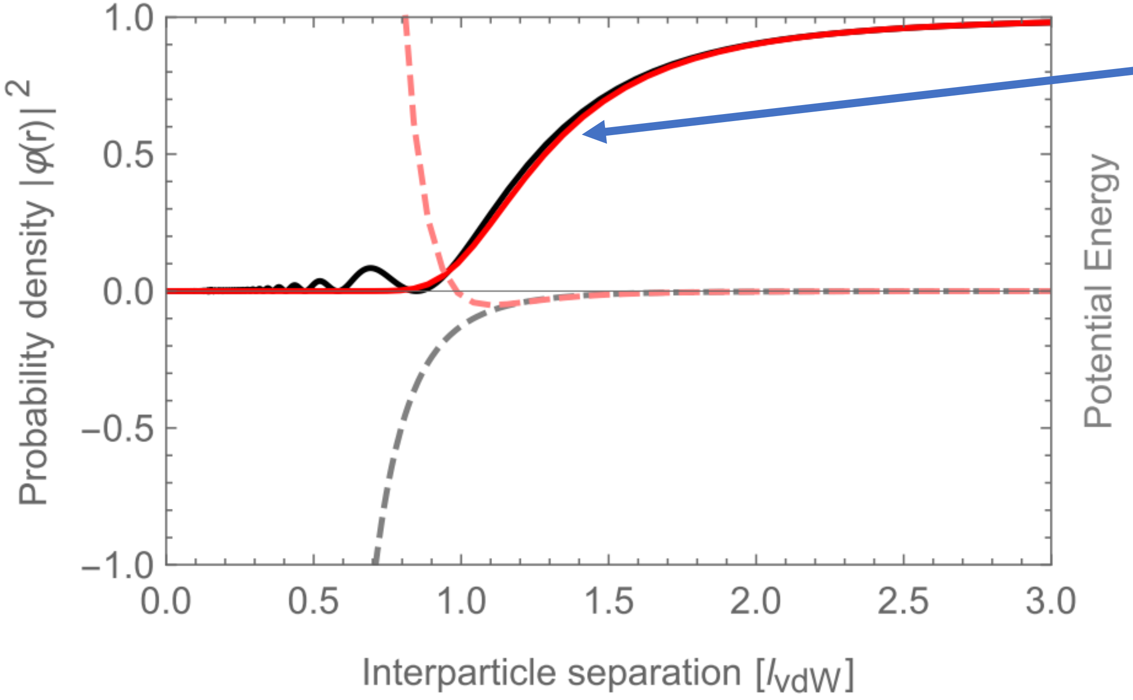
Three-body repulsive barrier



The three-body parameter

What is the origin of the three-body repulsion?
(how can we get repulsion from purely attractive interactions?)

Two-body van der Waals universality



Universal suppression of two-body probability for $r \lesssim l_{vdW}$

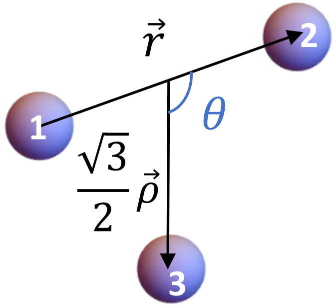
Analytical solution at zero energy

$$\varphi(r) \underset{r \gtrsim \ell_{\mathrm{vdW}}}{=} \Gamma(5/4)\sqrt{x} J_{\frac{1}{4}}(2x^{-2})$$

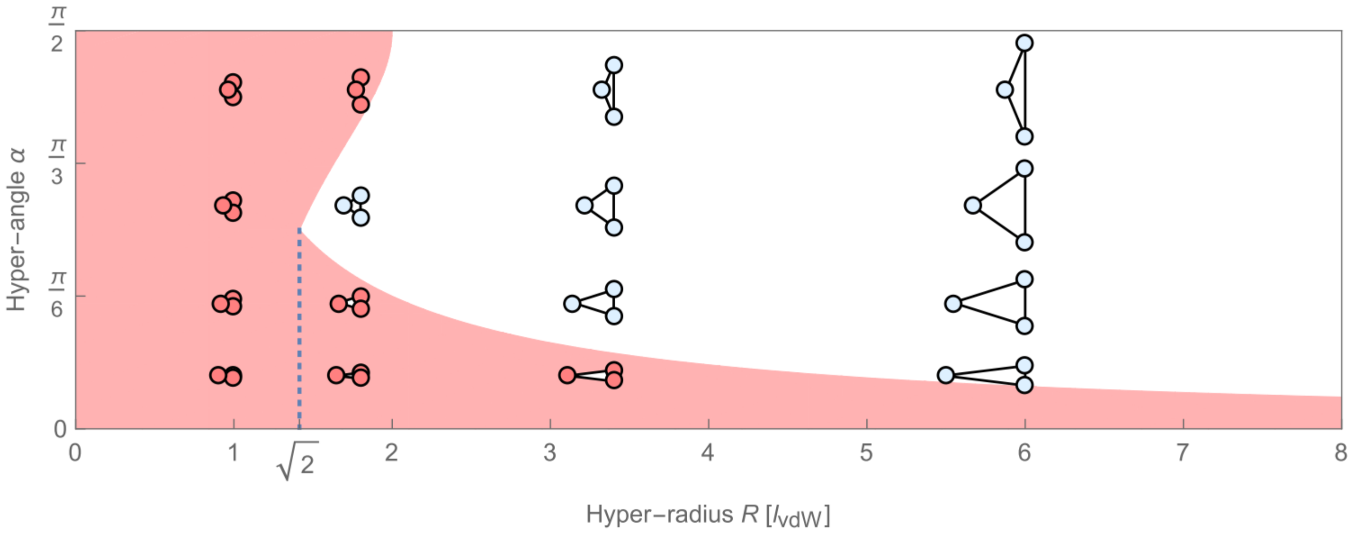
$$x = r/\ell_{\mathrm{vdW}}$$

The three-body parameter

Consequence for three-body:
Suppressed configurations for $r_{ij} \lesssim l_{vdW}$



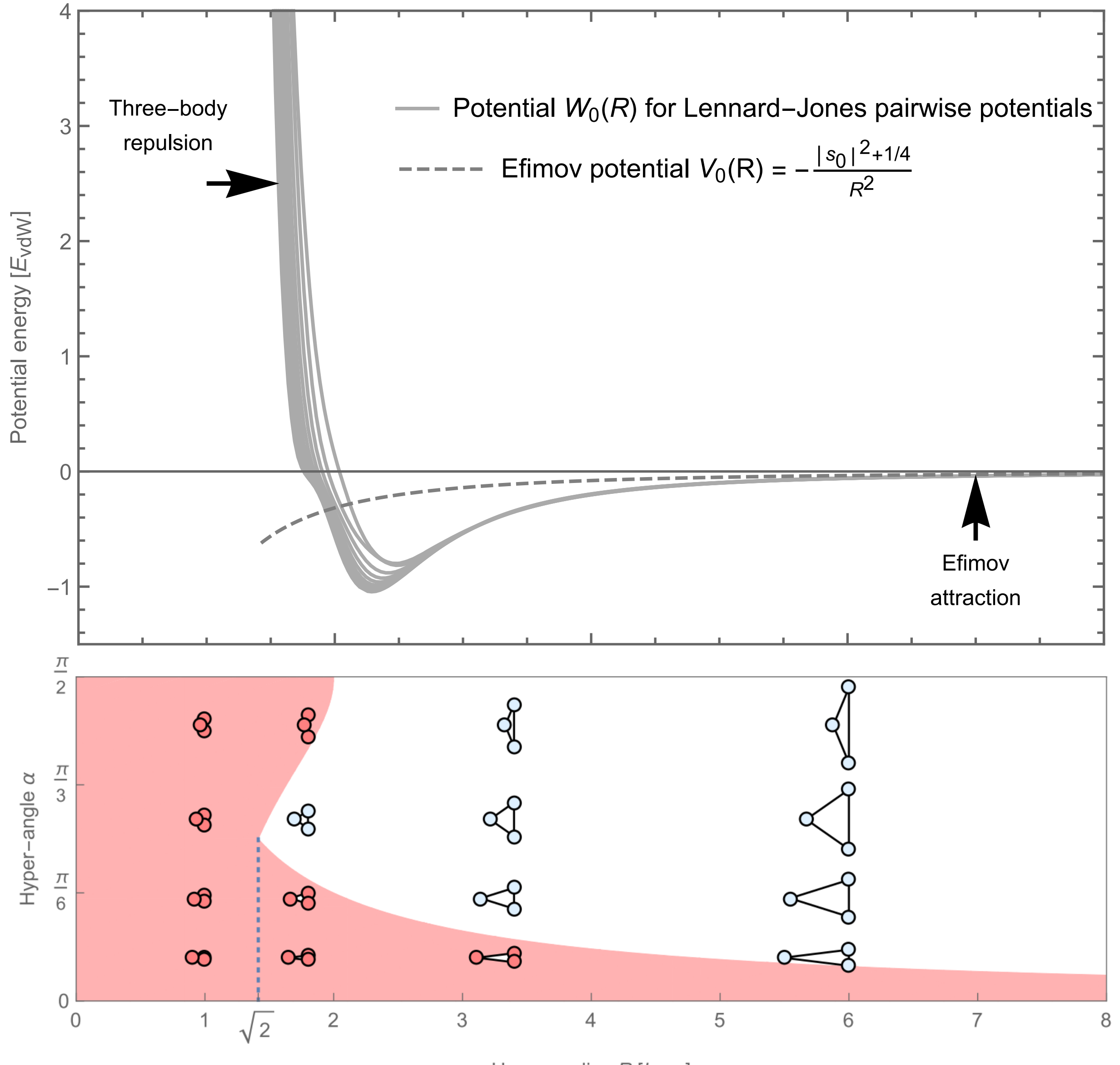
(for $\theta = \pi/2$)



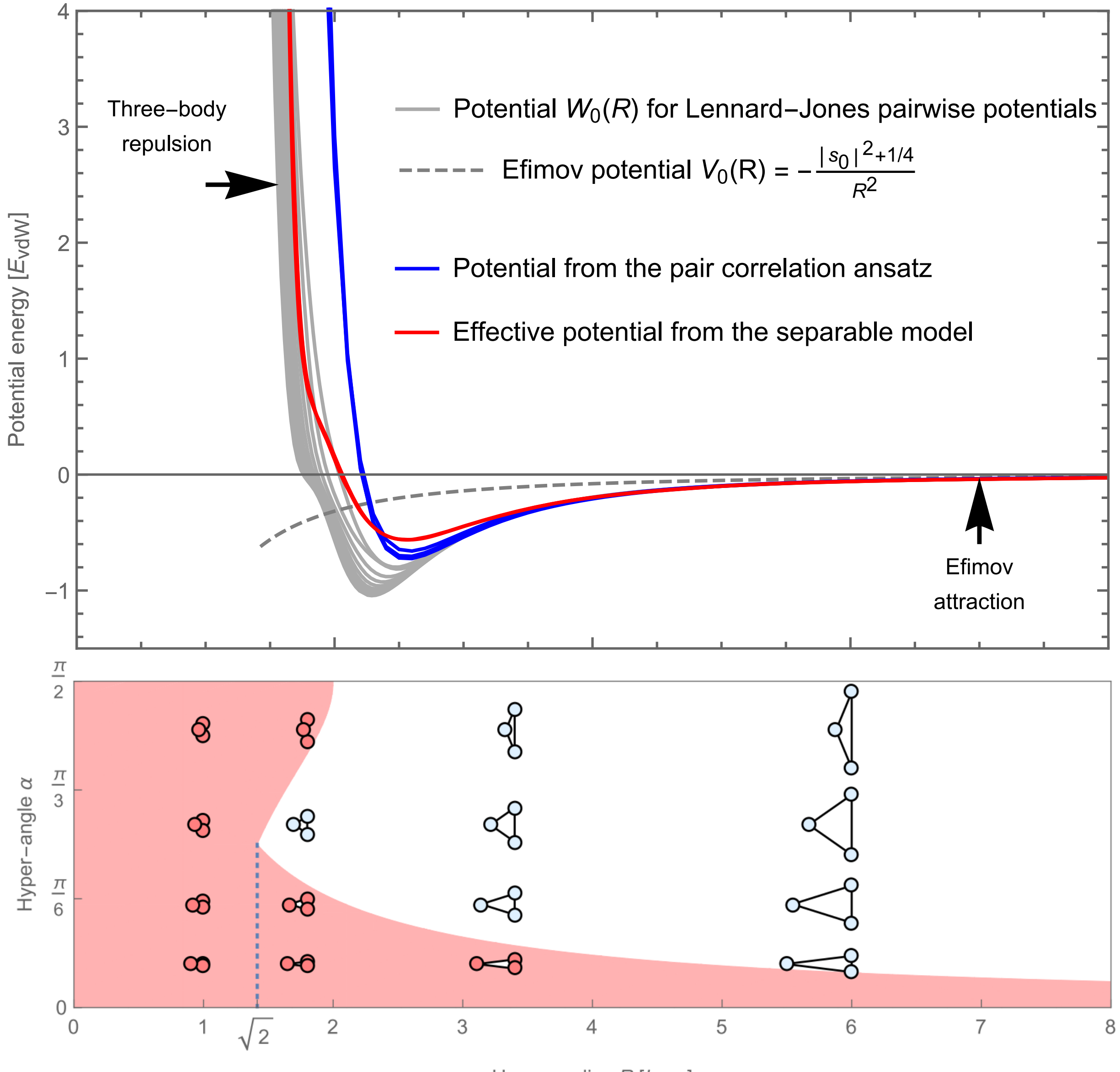
Hyper-radius R
Hyper-angle α

$$r = R\sin\alpha$$
$$\rho = R\cos\alpha$$

The three-body parameter



The three-body parameter



P. Naidon, S. Endo, and M. Ueda, "*Physical origin of the universal three-body parameter in atomic Efimov physics*" Phys. Rev. A **90**, 022106 (2014)

Approximate ansatz to check this interpretation:

1) Pair correlation ansatz

$$\Phi_0(\alpha,\theta;R) = \Phi_0^{(ZR)}(\alpha,\theta) \times \varphi(r_{12})\varphi(r_{23})\varphi(r_{31})$$

2) Separable potential model

$$\hat{V} = \underbrace{\frac{1}{\langle\psi_0|V|\psi_0\rangle}}_{\substack{g \\ \text{Coupling} \\ \text{constant}}} \underbrace{V|\psi_0\rangle}_{|\phi\rangle} \underbrace{\langle\psi_0|V}_{\langle\phi|}$$

Form factor

with

$$V(r) = -C_6/r^6$$
$$\psi_0(\vec{r}) = \varphi(r)/r$$

The three-body parameter

2) Separable potential model

$$\hat{V} = \frac{1}{\langle\psi_0|V|\psi_0\rangle} V|\psi_0\rangle\langle\psi_0|V$$ with

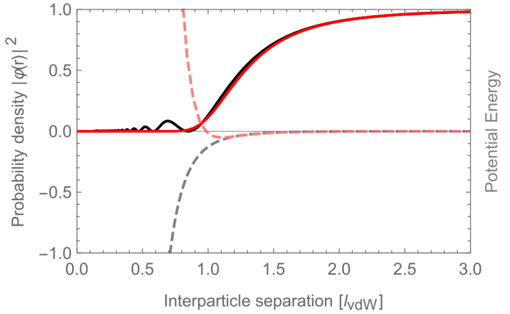
$$V(r) = -C_6/r^6$$
$$\psi_0(\vec{r}) = \varphi(r)/r$$

The solution of the true potential at zero energy: $(\hat{T} + V)|\psi_0\rangle = 0$

is also the solution of the separable potential at zero energy:

$$(\hat{T} + \hat{V})|\psi_0\rangle = \left(\hat{T} + \frac{1}{\langle\psi_0|V|\psi_0\rangle} V|\psi_0\rangle\langle\psi_0|V\right)|\psi_0\rangle$$
$$= \left(\hat{T}|\psi_0\rangle + \frac{1}{\cancel{\langle\psi_0|V|\psi_0\rangle}} V|\psi_0\rangle\cancel{\langle\psi_0|V|\psi_0\rangle}\right)$$
$$= (\hat{T} + V)|\psi_0\rangle = 0$$

This separable potential is an approximation that reproduces exactly the zero-energy two-body wave function of the true potential



The three-body parameter

2) Separable potential model Three-body treatment (for negative energy $E = -\hbar^2\kappa^2/m$)

$$\left(\frac{3}{4}P^2 + p^2 + \kappa^2\right)\tilde{\Psi}(\vec{P},\vec{p}) = \sum_{i=1,2,3} \phi_i(\vec{p}_i)\frac{2\mu}{\hbar^2}\underbrace{g_i\int d^3p_i\phi_i^*(\vec{p}_i)\tilde{\Psi}(\vec{P},\vec{p})}_{4\pi\tilde{F}_i(\vec{P}_i)} \quad \text{with} \quad \frac{4\pi\hbar^2}{m}\frac{1}{g_i}\tilde{F}_i(\vec{P}_i) = \int d^3p_k\phi_i^*(\vec{p}_i)\tilde{\Psi}(\vec{P},\vec{p})$$

2

$$\tilde{\Psi}(\vec{P},\vec{p}) = \frac{4\pi\sum_{i=1,2,3}\phi_i(\vec{p}_i)\tilde{F}_i(\vec{P}_i)}{\frac{3}{4}P^2 + p^2 + \kappa^2}$$

1

Generalised Skorniakov – Ter-Martirosian integral equations:

$$\boxed{\left(\frac{4\pi\hbar^2}{m}\frac{1}{g_k} + \frac{2}{\pi}\int_0^\infty \frac{q^2|\phi_k(\vec{q})|^2}{q^2+\frac{3}{4}P^2+\kappa^2}dq\right)\tilde{F}_k(\vec{P}) + 4\pi\int\frac{d^3Q}{(2\pi)^3}\phi_k^*\left(\vec{Q}+\vec{P}/2\right)\frac{\phi\left(\vec{P}+\frac{\vec{Q}}{2}\right)\tilde{F}_i(\vec{Q})+\phi\left(\vec{P}-\frac{\vec{Q}}{2}\right)\tilde{F}_j(\vec{Q})}{P^2+Q^2+\vec{Q}\cdot\vec{P}+\kappa^2} = 0}$$

with $\{i,j,k\} = \{1,2,3\}$

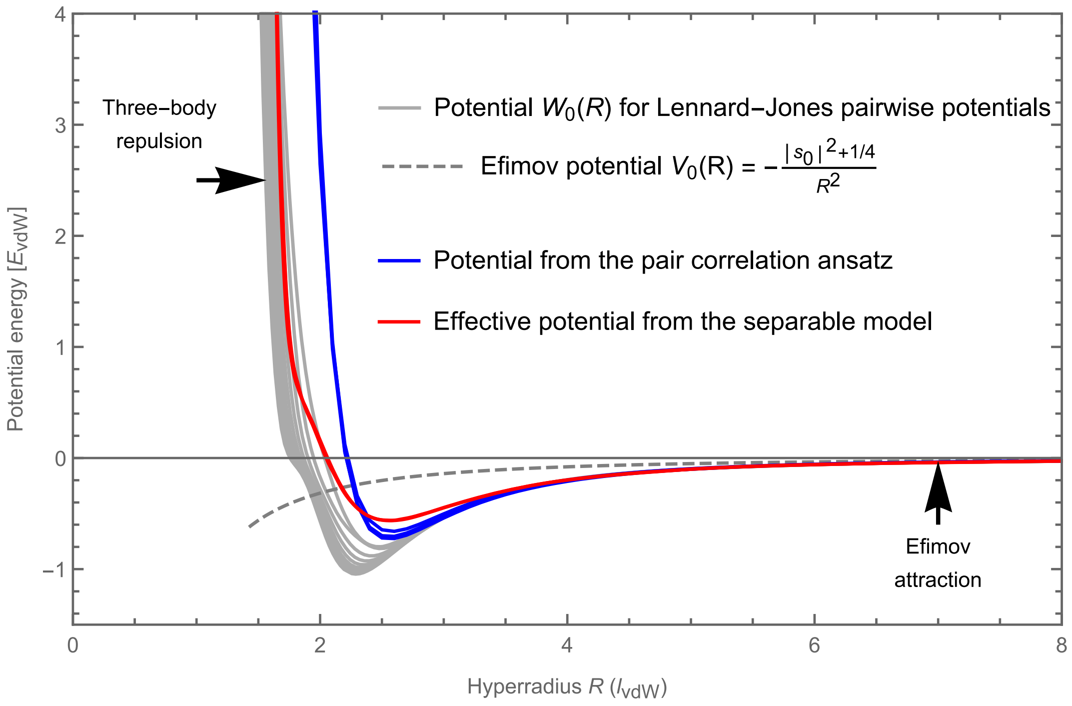
Same benefit as the zero-range theory:
Now, the unknown function has only 1 argument!

$$\tilde{\Psi}(\vec{P},\vec{p}) \longrightarrow \tilde{F}(\vec{P})$$

The three-body parameter

2) Separable potential model

Three-body treatment (for negative energy $E = -\hbar^2\kappa^2/m$)



$$\tilde{\Psi}(\vec{P},\vec{p}) = \frac{4\pi \sum_{i=1,2,3} \phi_i(\vec{p}_i)\tilde{F}_i(\vec{P}_i)}{\frac{3}{4}P^2 + p^2 + \kappa^2} \Rightarrow \tilde{\Psi}(\rho,r,\theta) = \tilde{\Psi}(R,\alpha,\theta)$$

Fourier transform

$$\int d^3\vec{r}d^3\vec{\rho}|\Psi|^2 = \int_0^\infty 4\pi r^2 dr \int_0^\infty 2\pi\rho^2 d\rho \int_0^\pi d(\cos\theta)\,|\Psi|^2$$

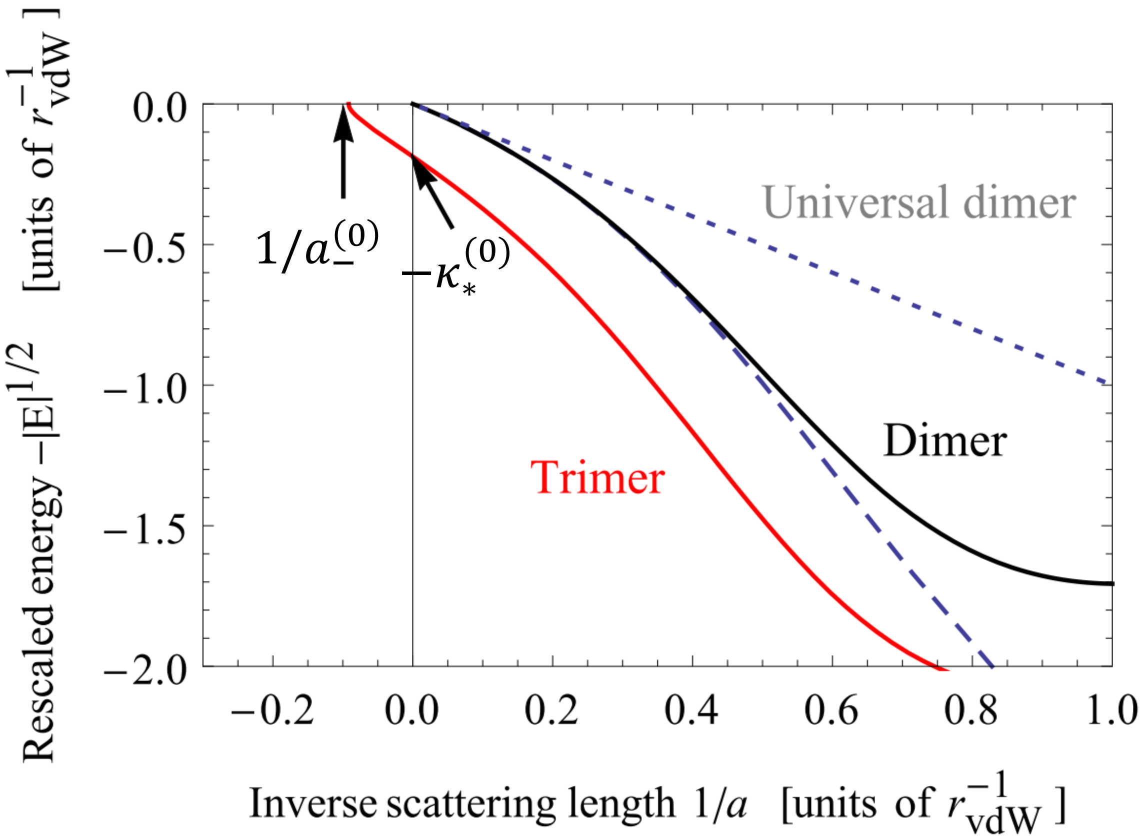
$$= (4\pi)^2 \int_0^\infty dR\, \underbrace{\int_0^{\frac{\pi}{2}} d\alpha (R\sin\alpha)^2 (R\cos\alpha)^2 \int_0^\pi d(\cos\theta)\,|\Psi|^2}_{\text{Hyper-radial probability density } |f(R)|^2}$$

Effective Schrödinger equation: $-\frac{d^2 f}{dR^2} + W(R)f(R) = Ef(R)$

Effective potential: $\boxed{W(R) = E + \frac{f''(R)}{f(R)}}$

The three-body parameter

2) Separable potential model Three-body treatment (for negative energy $E = -\hbar^2\kappa^2/m$)



$$\boxed{\kappa_*^{(0)} = 0.187(1)\ell_{\mathrm{vdW}}^{-1}}$$

$$\boxed{a_-^{(0)} = -10.86(1)\ell_{\mathrm{vdW}}}$$

Close to the exact results:

$$\boxed{\kappa_*^{(0)} = (0.21 \pm 0.01)/\ell_{\mathrm{vdW}}}$$

$$\boxed{a_-^{(0)} = -(10.70 \pm 0.35)\,\ell_{\mathrm{vdW}}}$$

Conclusion

Universal few-body physics has been a developing field of quantum few-body physics, both theoretical and experimental, unveiling a whole collection of universal few-body states with remarkable properties.

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arXiv:1704.02029

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Review

Efimov physics: a review

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