Ergodicity of infinite particle systems and Applications

(Based upon joint works with Zegarlinski, Inglis, Futorny, Bock)

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Outline

Conservative interacting particle systems (CIPS): Preliminaries

- Space of configurations X. Examples: $\mathbb{R}^{\mathbb{Z}^d}$, $\{0,1\}^{\mathbb{Z}^d}$...
- Energy function $E : X \to \mathbb{R}$.
- Equilibrium is described by the family of measures $d\mu_T =$ "$\frac{e^{-\frac{E}{T}}}{N(T)}dx$" enumerated by the temperature $T \in A$.
- Evolution is described by the Markov semigroup $P_t : L^2(X, d\mu_T) \to L^2(X, d\mu_T), t \geq 0$ with infinitesimal generator

$$\mathcal{L} = \sum_{\gamma} D_{\gamma}^2 + Z$$

  and invariant measures μ_T, $T \in A$ ($\mathcal{L}E = 0$).

Question of interest: Rigorous proof that measures are invariant? Ergodicity of the system? Rate of Convergence to equilibrium?

Heat conduction model

N, Zegarlinski, Inglis have considered

$$\mathcal{L} = b^2 \sum_{\mathbf{i}\in\mathbb{Z}^N} \sum_{\mathbf{j}\in\mathbb{Z}^N:|\mathbf{i}-\mathbf{j}|_1=1} \left(x_{\mathbf{i}}\partial_{\mathbf{j}} - x_{\mathbf{j}}\partial_{\mathbf{i}}\right)^2, \tag{1}$$

-infinitesimal generator corresponding to SDE (N=1)

$$dx_k(t) = -x_k dt + x_{k+1} dW_k(t) - x_{k-1} dW_{k-1}(t), k \in \mathbb{Z} \tag{2}$$

Physical interpretation (Giardina, Kurchan, Redig (2007)):
We have set of atoms on the lattice $\mathbb{Z}$ described by their momentums $(x_i)_{i\in\mathbb{Z}}$, x_i^2–energy of the atom i
Operators $\mathbf{X}_{i,j} = \left(x_{\mathbf{i}}\partial_{\mathbf{j}} - x_{\mathbf{j}}\partial_{\mathbf{i}}\right)$ conserve local energy $x_{\mathbf{i}}^2 + x_{\mathbf{j}}^2$;
Total energy $E = \sum_i x_{\mathbf{i}}^2$ is conserved.

Operators $\left(x_{\mathbf{i}}\partial_{\mathbf{j}} - x_{\mathbf{j}}\partial_{\mathbf{i}}\right)^2$ describe the exchange of momentum between atoms $x_{\mathbf{i}}$ and $x_{\mathbf{j}}$.

Invariant measures of the form "$d\mu_T = \frac{1}{N_T} e^{-\frac{E}{T}} dx$" i.e. gaussian product measures parameterized by T – "temperature".
$\mathbf{X}_{i,j}$ – nontrivial Lie bracket, Hörmander condition is violated.

Existence and non-uniqueness of invariant measure

Theorem

Measures δ_0 and $d\mu_T, T > 0$ are invariant measures for semigroup $P_t, t \geq 0$. Furthermore,

$$\int_E fP_t g d\mu_T = \int_E gP_t f d\mu_T, t, T > 0, f, g \in L^2(d\mu_T). \qquad (3)$$

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Sketch of the proof.

Trotter formula and representation of generator as a sum of two operators with "simple" structure. □

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Ergodicity with polynomial decay. Part I

Let $\mathbb{X}$ –Sobolev space of order 1 with homogeneous norm

$$\mathcal{A}(f) \equiv \left(\sum_{\mathbf{i}\in\mathbb{Z}^N} \mu_T |\partial_{\mathbf{i}} f|^2 \right)^{1/2} \sim |f|_{\tilde{\mathbb{X}}}.$$

Lemma (N, Zegarlinski, Inglis)

For any $f \in \mathbb{X}$, $i \in \mathbb{Z}^N$ and $t > 0$,

$$\mu_T(|\partial_i (P_t f)|^2) \leq \frac{A^N}{t^{\frac{N}{2}}} \mathcal{A}^2(f), \tag{4}$$

where $A = \frac{1}{b} \sup_{t>0} \sqrt{t} \int_0^1 e^{-2t(1-\cos(2\pi\beta))} d\beta$.

The order of convergence above is optimal ($f = x_i^2$). If $f_K = \sum_{|i|\leq K} x_i^2$ then

$$\mu_T(P_t f_K - \mu_T f_K)^2 \sim C_2(\delta) \frac{1}{\sqrt{t}^N} \mu_T(f_K - \mu_T f_K)^2 + C_3(\mu_T, \delta) e^{-4\delta t}$$

Proof of Lemma

Denote $f_t = P_t f$ for $t \geq 0$.

$$|\partial_{\mathbf{i}} f_t|^2 - P_t|\partial_{\mathbf{i}} f|^2 = \int_0^t \frac{d}{ds} P_{t-s}|\partial_{\mathbf{i}} f_s|^2 ds$$
$$= \int_0^t P_{t-s}(-\mathcal{L}(|\partial_{\mathbf{i}} f_s|^2) + 2\partial_{\mathbf{i}} f_s \mathcal{L} \partial_{\mathbf{i}} f_s + 2\partial_{\mathbf{i}} f_s [\partial_{\mathbf{i}}, \mathcal{L}] f_s) ds$$

Algebraic structure–

$$[\partial_{\mathbf{i}}, \mathbf{X}_{k,l}] = \delta_{\mathbf{i},\mathbf{k}} \partial_{\mathbf{k}} - \delta_{\mathbf{i},\mathbf{l}} \partial_{\mathbf{l}}$$

Question: what is this structure?
Applying it we can calculate $[\partial_{\mathbf{i}}, \mathcal{L}]$ and (after some cancellations) we get

$$F(t) \leq F(0) + \int_0^t b^2 \triangle F(s)\, ds, \qquad t \in [0, \infty).$$

where $F(\mathbf{i}, t) = \mu_T |\partial_{\mathbf{i}}(P_t f)|^2$ for $t \geq 0, \mathbf{i} \in \mathbb{Z}^N$.

$$F(t) \leq e^{tb^2 \triangle} F(0), t \in [0, \infty)$$

Ergodicity with polynomial decay. Part II

$$\mathcal{A}(f) \equiv \left(\sum_{\mathbf{i}\in\mathbb{Z}^N} \mu_T |\partial_{\mathbf{i}} f|^2 \right)^{1/2}, \mathcal{B}(f) \equiv \left(\sum_{\mathbf{i}\in\mathbb{Z}^N} (\mu_T |\partial_{\mathbf{i}} f|^2)^{\frac{1}{2}} \right).$$

Corollary (N, Zegarlinski, Inglis)

For $f \in \mathbb{X}$ such that $\mathcal{B}(f) < \infty$, we have

$$\sum_{\mathbf{i}\in\mathbb{Z}^N} \mu_T |\partial_{\mathbf{i}} (P_t f)|^2 \leq \frac{A^{\frac{N}{2}}}{t^{\frac{N}{4}}} \mathcal{A}(f)\mathcal{B}(f). \tag{5}$$

Furthermore, there exists a constant $C \in (0, \infty)$ such that

$$\mu_T \left((P_t f)^2 \log \frac{(P_t f)^2}{\mu_T (P_t f)^2} \right) \leq C \frac{A^{\frac{N}{2}}}{t^{\frac{N}{4}}} \mathcal{A}(f)\mathcal{B}(f), \tag{6}$$

and hence

$$\mu_T (P_t f - \mu_T(f))^2 \leq C \frac{A^{\frac{N}{2}}}{t^{\frac{N}{4}}} \mathcal{A}(f)\mathcal{B}(f), \tag{7}$$

i. e. our system is ergodic with polynomial rate of convergence.

Algebraic structure

- $$[\partial_{\mathbf{i}}, \mathbf{X}_{k,l}] = \delta_{\mathbf{i},\mathbf{k}}\partial_{\mathbf{k}} - \delta_{\mathbf{i},\mathbf{l}}\partial_{\mathbf{l}}$$

  The Jordan–Schwinger map D for a matrix $X = (X_{ij})_{i,j=1}^{n}$ (pp. 212–213, Bidernharn-Louck-1981):

  $$D : X \mapsto \sum_{i,j=1}^{n} X_{ij} a_i a_j^*, \tag{8}$$

  $\{a_i, a_j^*\}_{i,j=1}^{n}$– boson creation and annihilation operators. Realization: $a_i = x_i$, $a_j^* = \partial_{x_j}$. Properties:

  - algebraic homomorphism i.e. $D([A, B]) = [D(A), D(B)]$.
  - corresponds to linear vector field.

  Physicists use to construct representations of classical Lie algebras of matrices as an operators on Fock space.

Generalization of Jordan-Schwinger map

Definition

V and W – topological vector spaces(TVS) in duality $< \cdot, \cdot >_{V,W}$, V is separable, biorthogonal system $\{e_k, f_k\}_{k=1}^{\infty}$

$$D(A) := \sum_{\alpha,\beta\in\mathbb{N}} < Ae_\alpha, f_\beta > x_\alpha \frac{\partial}{\partial x_\beta}, A \in \mathcal{L}(V, V),$$

$$\partial(h) := \sum_{\alpha\in\mathbb{N}} < h, f_\alpha > \frac{\partial}{\partial x_\alpha}, h \in V.$$

$$\bar{\partial}(r) := \sum_{\alpha\in\mathbb{N}} < e_\alpha, r > x_\alpha, r \in W.$$

Lemma

$$\begin{gathered}[D(A), D(B)] = D([B, A]),\\ [\partial(f), D(A)] = \partial(Af),\\ [\partial(f), \partial(g)] = 0\\ [D(A), \bar{\partial}(r)] = \bar{\partial}(A^* r), f, g \in V, r \in W, A, B \in \mathcal{L}(V, V).\end{gathered}$$

Notice that $\mathbf{X}_{i,j} = x_i\partial_{x_j} - x_j\partial_{x_i} = D_{ij} = D(E_{ij} - E_{ji})$ (E_{ij} –matrix unit), $\partial_{x_i} = \partial(e_i)$. So our algebraic structure is an example of identity (9).

Representation of Lie algebra with linear vector fields

Theorem

Let $\mathfrak{g}$ be an arbitrary Lie algebra and $\rho : \mathfrak{g} \to End(V)$ a faithful representation of $\mathfrak{g}$. Then $-D \circ \rho$ and $D^* \circ \rho$ embed $\mathfrak{g}$ into $\widehat{\mathcal{A}}_l$, and hence, define representations of $\mathfrak{g}$ by linear vector fields.

$\mathfrak{g}$–Lie algebra with center $Cent(\mathfrak{g})$ $V = \mathfrak{g}$ and restrict the mapping $D = D_V$ on the subspace

$$Z := \{ad(v) := [v, \cdot], v \in V\} \subset \mathcal{L}(V, V)$$

$$\tilde{D} := D \circ ad : \mathfrak{g} \to End(\mathbb{R}[\overline{x}]), \tilde{D}(v) = \sum_{\alpha,\beta \in A} < [v, e_\alpha], f_\beta > x_\alpha \frac{\partial}{\partial x_\beta} \tag{9}$$

Theorem

For any topological locally convex Lie algebra $\mathfrak{g}$ there exists an embedding, given by formula (9)*, of $\mathfrak{g}/Cent(\mathfrak{g})$ into the semidirect product $\tilde{D}(\mathfrak{g}) \ltimes \partial(\mathfrak{g}) \subset End(\mathbb{R}[\overline{x}])$ of linear differential operators.*

Schrödinger-Virasoro Lie algebra

Schrödinger-Virasoro Lie algebras– stat. physics (Henkel-1994).
$s = 0, \frac{1}{2}$, $\rho \in \mathbb{Q}$. $\mathcal{L}[s,\rho]$–Lie algebra $\{L_n, Y_p, n \in \mathbb{Z}, p \in \mathbb{Z}+s\}$

$$[L_m, L_n] = (n-m)L_{n+m}$$
$$[L_m, Y_p] = (p - m\rho)Y_{m+p}$$
$$[Y_p, Y_q] = 0 \ , m,n \in \mathbb{Z}, p,q \in \mathbb{Z}+s.$$

Schrödinger-Virasoro algebra (Roger-2006, Liu-2016, Li-Su-2008).
H– separable Hilbert space,$\mathcal{H} = S'(D,H)$– space of tempered distributions on the unit disk $D = \{z \in \mathbb{C} | |z| \leq 1\}$

$$A_m : \mathcal{H} \mapsto \mathcal{H}, A_m f := e^{-imz}(-m\rho f + i\frac{df}{dz}), z \in D$$

Then

$$[A_m, A_n] = (n-m)A_{n+m} \ , m,n \in \mathbb{Z},$$

i.e. $\{A_m, m \in \mathbb{Z}\}$–representation of Witt algebra. Let $g \in H$, $\{e_p := ge^{-ipz}\}_{p \in \mathbb{Z}+s} \subset \mathcal{H}$. Then

$$[\bar{D}(A_m), \bar{D}(A_n)] = (n-m)\bar{D}(A_{n+m}) \ , m,n \in \mathbb{Z}.$$

$$[\partial(e_p), \partial(e_q)] = 0 \, , p,q \in \mathbb{Z}+s.$$

$$[\bar{D}(A_m), \partial(e_p)] = \partial(A_m e_p) = (p - m\rho)\partial(e_{m+p}), p \in \mathbb{Z}+s, m \in \mathbb{Z}.$$

$\{\partial(e_p), \bar{D}(A_m), p \in \mathbb{Z}+s, m \in \mathbb{Z}\}$– a representation of $\mathcal{L}[s,\rho]$ by linear vector fields.

Cuntz algebras

$\mathcal{O}_n, n \in \mathbb{N}^*$– Cuntz algebras (Cuntz–1977) with generators $\{s_i\}_{i=1,\ldots,n}$

$$s_j^* s_j = 1, s_j^* s_k = 0, j \neq k, j, k = 1 \ldots, n.$$

$$\sum_{i=1}^{n} s_i s_i^* = 1 (\leq 1, n = \infty)$$

Algebras of mutually orthogonal isometries of separable Hilbert space.

Example

$$\mathcal{O}_2 : s_1 e_k := e_{2k}, s_2 e_k := e_{2k+1}$$

Algebraic analogue of Jordan-Schwinger map

V is separable locally convex TVS in duality with W, biorthogonal system $\{e_k\}_{k=1}^{\infty} \subset V$, $\{f_k\}_{k=1}^{\infty} \subset W$.
$V = \overline{sp\{e_k, k \in \mathbb{N}\}}$.

$$D(A) := \sum_{\alpha,\beta \in \mathbb{N}} < Ae_\alpha, f_\beta > s_\alpha s_\beta^*, \quad A \in \mathcal{L}(V, V),$$

$$\partial(h) := \sum_{\alpha \in \mathbb{N}} < h, f_\alpha > s_\alpha^*, \quad h \in V.$$

$$\bar{\partial}(f) := \sum_{\alpha \in I} < e_\alpha, f > s_\alpha, f \in W.$$

Analogue of J.-S. map: Algebraic homomorphism

Lemma

$$D(A)D(B) = D(BA),$$
$$\partial(h)D(A) = \partial(Ah),$$
$$D(A)\bar{\partial}(f) = \bar{\partial}(A^* f),$$
$$\partial(h)\bar{\partial}(g) = < h, g >$$
$$h \in V, f, g \in W, A, B \in \mathcal{L}(V, V),$$

where the adjoint is taken with respect to the duality $< \cdot, \cdot >$.

Invariance of spectrum

$\sigma(A)$–spectrum of an operator A, $\mathcal{R}(A)$–resolvent set.

Corollary (Invariance of the spectrum)

Assume that the generators $\{s_i\}$ of $\mathcal{O}_\infty$ satisfy

$$\sum_i s_i s_i^* = Id.$$

Then $\sigma(D(A)) \subset \sigma(A)$ and, if V is a Frechet space, $\sigma(D(A)) = \sigma(A)$.

Spectral Theorem

Corollary (Spectral Theorem)

Assume that $A \in \mathcal{L}(V, V)$ is symmetric i.e.

$$< Ae_\alpha, f_\beta > = < Ae_\beta, f_\alpha >, \alpha, \beta \in I$$

and $D(A)$ is selfadjoint (symmetry of $D(A)$ immediately follows from symmetry of A). Then we have

$$< Ah, g > = \int\limits_{\sigma(D(A))} \lambda d \left[\partial(h) E_\lambda \bar{\partial}(g)\right], \quad h \in V, g \in W,$$

where E_λ denote the spectral projections for the operator $D(A)$.

Representation of algebras

$V = X$ is a topological algebra in duality with X^*. $l_a : X \to X$, $l_a(x) = ax$. $A, B \in \mathcal{L}(X, X)$, $Ax = l_a x$, $Bx = l_b x$, $a, b, \in X$.

Corollary

Assume that X is a topological separable locally convex Hausdorff algebra. Then we have

$$\begin{gathered} D(l_a)D(l_b) = D(l_{ba}), \\ \partial(h)D(l_a) = \partial(ah), \\ D(l_a)\bar{\partial}(f) = \bar{\partial}(l_a^* f), \\ \partial(h)\bar{\partial}(g) = < g, h > \\ a, b, h \in X, f, g \in X^*. \end{gathered}$$

If X has identity then $X \ni a \mapsto D(l_a) \in \mathcal{O}_\infty$ is injective.

Case of finitedimensional algebra

Definition (Dutkay-2014)

Y be a topological compact space, μ a Borel probability measure on Y, $r : Y \to Y$ an n-to-1 Borel measurable map, i.e. $|r^{-1}(z)| = n$, μ-a. a. $z \in Y$. μ is a strongly invariant measure w.r. . r, i.e.

$$\int f d\mu = \frac{1}{n} \int \sum_{r(\omega)=r(z)} f(\omega) d\mu(z), f \in C(Y)$$

Definition (Dutkay-2014)

A quadrature mirror filter (QMF) for r is $m_0 \in L^\infty(Y, \mu)$

$$\frac{1}{n} \sum_{r(\omega)=z} |m_0(\omega)|^2 = 1, z \in Y.$$

A QMF basis is a set of n QMF's $m_0, m_1, \ldots, m_{n-1}$

$$\frac{1}{n} \sum_{r(\omega)=z} m_i(\omega)\overline{m_j}(\omega) = 1\delta_{ij},\ i, j \in \{0, 1, \ldots, n-1\}, z \in Y.$$

Case of finitedimensional algebra

Proposition (Dutkay-2014)

$\{m_i\}_{i=0}^{i=n-1}$– QMF basis.

$$S_i(f) = m_i(f \circ r), f \in L^2(Y, d\mu), i = 0, \dots, n-1.$$

Then the operators $\{S_i\}_{i=0}^{n-1}$ form a representation of $\mathcal{O}_n$.
Furthermore,

$$S_i^*(f)(z) = \frac{1}{n} \sum_{r(\omega)=z} \overline{m_i}(\omega) f(\omega), i = 0, \dots, n-1, z \in Y.$$

Consequently,

$$D(l_a)f(z) = \frac{1}{n} \sum_{r(\omega)=r(z)} \left(\sum_{i,j=0}^{n-1} < ae_i, f_j > m_i(z)\overline{m_j}(\omega) \right) f(\omega),$$

$z \in Y, a \in \mathcal{A}, f \in L^2(Y, d\mu)$
$D(l_a)$– transfer operator of dynamical system.

Quantization of finite dimensional systems

$(C^\infty(M), \{\cdot,\cdot\})$ be a Poisson manifold with Poisson bracket $\{\cdot,\cdot\}$.

$$Q(h) := D^*(\{h,\cdot\}), R(h) := D^*(h\cdot), h \in C^\infty(M).$$

Definition

Quantization $\widehat{Q} \in \mathcal{L}(C^\infty(M), \mathcal{O}_\infty)$ $\widehat{Q} := R - 2iQ$.

Theorem (Bock,Futorny, N,LMP,2022)

$$\begin{aligned}
\widehat{Q}(1) &= Id, \\
[\widehat{Q}(f), \widehat{Q}(g)] &= -2i\widehat{Q}(\{f,g\}), \\
[\widehat{Q}(q_k), \widehat{Q}(q_j)] &= [\widehat{Q}(p_k), \widehat{Q}(p_j)] = 0, \\
[\widehat{Q}(q_k), \widehat{Q}(p_j)] &= -2i\delta_{kj} Id, k, j = 1, \ldots, \dim M
\end{aligned}$$

Furthermore, if $\phi : \mathbb{R} \to \mathbb{R}$ is an analytic function then

$$\Re\widehat{Q}(\phi(f)) = \phi(\Re\widehat{Q}(f))$$

Properties similar to Van Hove prequantization. Generalization to infinite dimensional case in the framework of White noise calculus.

Summary

Thus we have

- Class of ergodic system with polynomial rate of convergence
- Extensions of Jordan-Schwinger map
- Applications: Quantization, Spectral Theorem...
- Unsolved problems:
  - Conservation of q-bracket.
  - Jordan-Schwinger map for infinite dimensional algebras such as Yangians, vertex algebras,...
  - Jordan-Schwinger map as a "linearisation" tool
  - Connection with dynamical systems

Literature

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