Decoupling theorem
and effective quantum gravity

Ilya L. Shapiro

Universidade Federal de Juiz de Fora, MG, Brazil

Partial support: CNPq, ICTP-SAIFR

Minicourse on perturbative and nonperturbative treatment of quantum gravity problems

ICTP-SAIFR/IFT-UNESP – May 20-24, 2024

Lecture 1.

- **Decoupling at the classical level.**
- **Nonlocal form factors from Feynman diagrams.**
- **High energy (UV) and low-energy (IR) limits.**
  Quantum decoupling.
- **Extracting form factors in semiclassical gravity.**
- **Derivation using heat-kernel solution.**
- **Renormalization group in the physical setting.**
- **On the running of cosmological (CC) and Newton constants.**
- **Implications for trace (conformal) anomaly.**

Main references

Appelquist and Carazzone (AC) decoupling theorem.

T. Appelquist and J. Carazzone, PRD (1975) 2856.

A.V. Manohar, Effective Field Theories, Lectures, hep-ph/9606222.

A.O. Barvinsky and G.A. Vilkovisky, NPB 333 (1990) 471.

Ed. Gorbar & I.Sh. hep-ph/0210388, 0303124; 0311190 (JHEP).
Ed. Gorbar, G. de Berredo-Peixoto & Sh. 2005, and others.

Wagno C. e Silva and I.Sh., arXiv:2301.13291, JHEP

Pedagogical introduction

I.L. Buchbinder, I. Sh., Introduction to Quantum Field Theory with Applications to Quantum Gravity, (Oxford Un. Press, 2021).

Decoupling at the classical level.

Consider propagator of massive field at very low energy (IR)

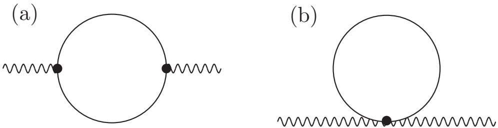
$$\frac{1}{k^2+m^2} = \frac{1}{m^2}\left(1 - \frac{k^2}{m^2} + \frac{k^4}{m^4} + ...\right) .$$

In case of $k^2 \ll m^2$ there is no propagation of a particle.

What about quantum theory, loop corrections?

Formally, in loops integration goes over all values of momenta.

Is it true that the effects of heavy fields always become irrelevant at low energies? E.g., the diagrams with external gravity include



Simplest example using Feynman diagram

Consider the analytic continuation of the Euclidean integral

$$I_4 \;=\; \frac{1}{(2\pi)^4}\int \frac{d^4k}{(k^2+m^2)[(p-k)^2+m^2]}.$$

We continue the dimension $I_4 \to I_{2\omega}$, making it analytic except some points. In the vicinity of $\omega = 2$,

$$I_{2\omega} = \frac{1}{2-\omega}\,\text{pole term} + \text{finite term} + O(2-\omega)\ \text{term}.$$

The purpose is to find the term with the pole at $\omega = 2$ and the finite part. Consider the representation

$$I_{2\omega} \;=\; \frac{1}{(2\pi)^{2\omega}}\int \frac{d^{2\omega}k}{(k^2+m^2)[(k-p)^2+m^2]}$$

$$=\; \int \frac{d^{2\omega}k}{(2\pi)^{2\omega}}\int_0^\infty d\alpha_1\int_0^\infty d\alpha_2\; e^{-\alpha_1(k^2+m^2)-\alpha_2[(k-p)^2+m^2]}.$$

Changing the order of integrations and using Gaussian form of the integral over Euclidean momentum, we get

$$I_{2\omega} \;=\; \int_0^\infty d\alpha_1 \int_0^\infty \frac{d\alpha_2}{(2\pi)^{2\omega}} \Big(\frac{\pi}{\alpha_1+\alpha_2}\Big)^\omega e^{\frac{\alpha_2^2 p^2}{\alpha_1+\alpha_2} - \alpha_2(p^2+m^2) - \alpha_1 m^2}.$$

We will need the gamma function $\Gamma(z) = \int_0^\infty dt\; t^{z-1} e^{-t}$

with $\Gamma(2-\omega) \;=\; \int_0^\infty \frac{e^{-t}\,dt}{t^{1-w}} \;=\; \frac{1}{2-\omega} - \gamma + \mathcal{O}(2-\omega)\,,$

$$\Gamma(1-\omega) = -\frac{1}{2-\omega} - 1 + \gamma + \mathcal{O}(2-\omega),$$

$$\Gamma(-\omega) = \frac{1}{2(2-\omega)} + \frac{3}{4} - \frac{\gamma}{2} + \mathcal{O}(2-\omega)\,,$$

and the volume of the m**-dimensional sphere of radius** R**,**

$$V_m \;=\; \frac{\pi^{\frac{m}{2}}}{\Gamma\big(\frac{m}{2}+1\big)}\,R^m.$$

This relation may be continued to a complex dimension 2ω**.**

Using these formulas, after some efforts we arrive at

$$I_{2\omega} \;=\; \frac{1}{(4\pi)^\omega}\Big[\frac{1}{2-\omega} - \gamma\Big]\int_0^1 d\alpha[m^2 + \alpha(1-\alpha)k^2]^{\omega-2}.$$

Let us denote $\tau = \frac{k^2}{m^2}$ **and transform**

$$[m^2 + \alpha(1-\alpha)k^2]^{\omega-2} \;=\; (m^2)^{\omega-2}\, e^{(\omega-2)\log[1+\alpha(1-\alpha)\tau]}$$
$$=\; (m^2)^{\omega-2}\Big\{1 - (2-\omega)\log\big[1+\alpha(1-\alpha)\tau\big]\Big\} \;+\; \mathcal{O}\big((\omega-2)^2\big).$$

In this way, we arrive at

$$I_{2\omega} \;=\; \frac{(m^2)^{\omega-2}}{(4\pi)^\omega}\Big[\frac{1}{2-\omega} + \gamma - \int_0^1 d\alpha\log\big[1+\alpha(1-\alpha)\tau\big]\Big].$$

The integration gives

$$Y \;=\; -\frac{1}{2}\int_0^1 d\alpha\;\log\big[1+\alpha(1-\alpha)\tau\big] \;=\; 1 - \frac{1}{a}\log\Big|\frac{2+a}{2-a}\Big|,$$

where $\qquad a^2 \;=\; \frac{4k^2}{k^2+4m^2} \;\longrightarrow\; \frac{4\Box}{\Box - 4m^2}.$

Finally, we arrive at the complete expression for the integral

$$I_{2\omega} \quad = \quad \frac{\mu^{2\omega-4}}{(4\pi)^2}\Big[\frac{1}{2-\omega} + \gamma + \log\Big(\frac{4\pi\mu^2}{m^2}\Big) + 2Y\Big].$$

This expression includes the following elements:

1. The divergences and the accompanying part with $\log(\mu/m)$.

The correspondence of the coefficients of these two terms enables one to construct the Minimal Subtraction scheme of renormalization and the corresponding formulation of the renormalization group equations.

2. Nonlocal form factor, i.e., in the simplest case, the expression $Y(\Box, m^2)$**, or** $Y(-k^2, m^2)$ **in the momentum representation.**

Upon subtracting divergences, this part remains and represents the physical result.

Our purpose is to explore what is the behavior of the form factor in the high-energy (UV) and low-energy (IR) regimes. We shall see that there is a qualitative difference between the two limits.

An important detail about the divergent part is its universality.

The coefficient of the leading one-loop divergence does not depend on regularization.

A. Salam, Phys.Rev. 84 (1951) 426.

One can establish the relation between, e.g., dimensional and cut-off regularizations,

$$\log \frac{\Omega^2}{m^2} \;\longleftrightarrow\; -\frac{\mu^{n-4}}{\varepsilon}, \qquad \varepsilon = (4\pi)^2(n-4) = 32\pi^2(\omega - 2),$$

We shall see that this universality can be extended to the leading-log part of the nonlocal form factor, i.e., to the UV limit of the finite part of the quantum correction.

Using the terms of renormalization group, the correspondence between the dependence on μ^2 and on k^2 implies an equivalence between Minimal Subtraction and Momentum Subtraction regularization schemes in the UV.

However, this does not concern the low-energy limit, where the IR version of the form factor has no direct link to the divergence.

Consider the form factor Y in the two limits:

1. UV regime $k^2 \gg m^2$ **.**

$$a^2 = \frac{4k^2}{k^2 + 4m^2} = \frac{4}{1 + \frac{4m^2}{k^2}} = 4\Big(1 - \frac{4m^2}{k^2} + \dots\Big),$$

In this case,

$$I_{2\omega} \;=\; \frac{1}{(4\pi)^2}\Big[-\frac{2}{n-4} + \log\Big(\frac{\mu^2}{k^2}\Big) + \text{constant}\Big].$$

2. IR regime $k^2 \ll m^2$. **Then**

$$Y \;=\; -\frac{1}{12}\,\frac{k^2}{m^2} + \frac{1}{120}\Big(\frac{k^2}{m^2}\Big)^2 + \dots .$$

This is the quadratic decoupling, similar to what was discovered in QED in 1975 by Appelquist and Corrazzone (AC).
There is no $\log(k^2/\mu^2)$ **to correspond** $1/(n-4)$**.**

The derivation of the β-functions in the mass-dependent scheme: one has to subtract the counterterm at the momentum $p^2 = M^2$, where M is the renormalization point. Then,

$$\beta_C \,=\, \lim_{n\to 4}\, M\,\frac{dC}{dM} \,=\, -\lim_{n\to 4}\, p\,\frac{dC}{dp}\,.$$

Example: a fermion loop effect in QED.

The Momentum-Subtraction scheme - based β-function reads

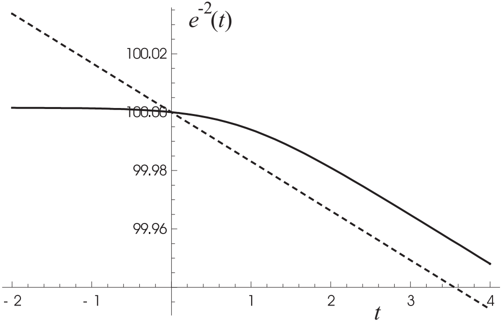
$$\beta_e^1 \,=\, \frac{e^3}{6a^3\,(4\pi)^2}\,\left[20a^3 - 48a + 3(a^2-4)^2\,\ln\left(\frac{2+a}{2-a}\right)\right]\,,$$

UV limit $\qquad p^2 \gg m^2 \;\Longrightarrow\; \beta_e^{1\;UV} \,=\, \frac{4\,e^3}{3\,(4\pi)^2} + \mathcal{O}\left(\frac{m^2}{p^2}\right).$

IR limit $\qquad p^2 \ll m^2 \;\Longrightarrow\; \beta_e^{1\;IR} \,=\, \frac{e^3}{(4\pi)^2}\cdot\frac{4\,p^2}{15\,m^2} \,+\, \mathcal{O}\left(\frac{p^4}{m^4}\right).$

This is the standard form of the Appelquist and Carazzone (AC) decoupling theorem (PRD, 1977).

Using the general mass-dependent expression, interpolating between the UV and IR limits, one can plot the "running" of the electric charge in QED.



Effective electron charge as a function of $\log(\mu/\mu_0)$ **for the MS-scheme and** $\ln(p/\mu_0)$ **for momentum-subtraction scheme.**

In the UV, i.e., in the high-energy limit, the unique difference is a small shift of the initial value of the effective charge. However, in the IR, there is a qualitative difference between the two plots.

In the UV, the mass of quantum fermion is negligible, this simplifies the form factor, and we arrive at

$$-\,\frac{1}{4e^2}\int d^4x\;F^{\mu\nu}\left[1+\tilde{\beta}\ln\left(\frac{\Box}{\mu^2}\right)\right]F_{\mu\nu}\,.$$

Here and always, the coefficient $\tilde{\beta}$ is proportional to the coefficient of the logarithmic divergence.

With the mass-dependent terms, things are more complicated, See details in *B. Gonçalves et al, PRD, arXiv:0906.3837*

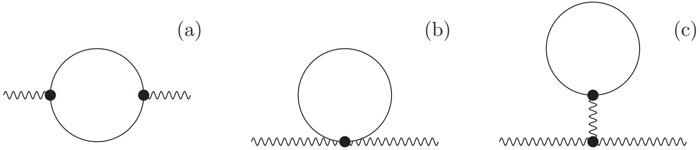
$$\bar{\Gamma}^{(1)}_{\sim A^2} \;=\; -\,\frac{e^2}{2(4\pi)^2}\;\int d^4x\;F_{\mu\nu}\left[\frac{2}{3\,\epsilon}+k_2^{FF}(a)\right]F^{\mu\nu}\;+\;...$$

where $\quad k_2^{FF}(a)=Y\left(1+\frac{4}{3a^2}\right)+\frac{1}{9}\,.$

It is worth noting that there is also an interesting ambiguity called nonlocal multiplicative anomaly, but the last does not qualitatively affect the Appelquist and Carazzone theorem.

Semiclassical gravity

The calculation can be performed using Feynman diagrams and linearized gravity, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. It is sufficient to consider



In the semiclassical gravity (i.e., when matter is quantum and gravity is classical), the third diagram (c) **does not emerge because there is an internal gravity line.**

Both diagrams (a) **and** (b) **contribute to divergences, but only the first one** (a) **to the nonlocal form factors. Anyway, both diagrams are relevant to establish correspondence between divergences and the logarithmic form factors in the UV.**

The main technical difficulty is to distribute the form factor into the four different terms in the effective action.

Covariance and power counting arguments show that the divergences belong to the fourth-derivative action

$$S_{HD} = - \int d^4x \sqrt{-g} \left\{ \frac{1}{2\lambda} C^2 + \frac{1}{\xi} R^2 + \frac{1}{\kappa^2}(R - 2\Lambda) \right\},$$

$+$ **surface terms.**

On the other hand, there are five tensor structures in gravity,

$$H_{\mu\nu,\alpha\beta}(k;\eta) = -\Big[a_1(k^2)\delta_{\mu\nu,\alpha\beta}k^2 + a_2(k^2)\eta_{\mu\nu}\eta_{\alpha\beta}k^2 + a_3(k^2)\big(\eta_{\mu\alpha}k_\beta k_\nu + \eta_{\nu\alpha}k_\beta k_\mu + \eta_{\mu\beta}k_\alpha k_\nu + \eta_{\nu\beta}k_\alpha k_\mu\big)] + a_4(k^2)\big(\eta_{\mu\nu}k_\alpha k_\beta + \eta_{\alpha\beta}k_\mu k_\nu\big) + a_5(k^2)k_\alpha k_\beta k_\mu k_\nu\Big].$$

Omitting the details, it is possible to split nonlocal form factor into the parts corresponding to the terms in the effective action.

Ed. Gorbar & Sh. JHEP (2003) hep-ph/0210388.

A. Codello and O. Zanusso, J.Math.Phys. (2013) arXiv:1203.2034.

The result of these calculations with nonlocal form factors, for the nonminimal real scalar with $\xi R\varphi^2$, has the form

$$\bar{\Gamma}_{vac} = \frac{1}{2(4\pi)^2}\int d^4x\sqrt{-g}\,\Big\{\frac{m^4}{2}\Big(\frac{1}{\epsilon_\mu}+\frac{3}{2}\Big)+\tilde{\xi}m^2R\Big[\frac{1}{\epsilon_\mu}+1\Big]$$

$$+\frac{1}{2}C_{\mu\nu\alpha\beta}\Big[\frac{1}{60\epsilon_\mu}+k_W(a)\Big]C^{\mu\nu\alpha\beta}+R\Big[\frac{1}{2\epsilon_\mu}\tilde{\xi}^2+k_R(a)\Big]R\Big\},$$

$$k_W(a) = \frac{8Y}{15\,a^4}+\frac{2}{45\,a^2}+\frac{1}{150},$$

$$k_R(a) = \Big(\tilde{\xi}^2+\frac{4-a^2}{3a^2}\tilde{\xi}+\frac{16-8a^2+a^4}{144a^4}\Big)Y+\frac{20-7a^2}{2160\,a^2}+\frac{1}{18}\tilde{\xi}^2,$$

where $$\frac{1}{\epsilon_\mu} = \frac{1}{2-w}+\ln\Big(\frac{4\pi\mu^2}{m^2}\Big),\qquad \tilde{\xi}=\xi-\frac{1}{6},$$

$$a^2 = \frac{4\Box}{\Box-4m^2},\qquad Y = 1-\frac{1}{a}\ln\frac{1+a/2}{1-a/2}.$$

In the IR, logarithms disappear and the form factors become local, i.e., we get a gravitational version of decoupling theorem.

Using the heat-kernel solution

Using Feynman diagrams is relatively complicated.

The calculations can be performed by using the heat-kernel solution, i.e., by summing up the Schwinger-DeWitt series,

I. Avramidi, Sov.J.Nucl.Phys. 49 (1989).

A. Barvinsky & G.A. Vilkovisky, Nucl Phys. B333 (1990) 471.

Technically this approach is much simpler compared to using diagrams and also provides higher level of universality.

$$\bar{\Gamma}^{(1)} = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\, \mathrm{Tr}\, K(s)\,,$$

where the functional trace of the heat kernel is

$$\mathrm{Tr}\, K(s) \,=\, \frac{\mu^{4-2\omega}}{(4\pi s)^\omega}\int d^4x\,\sqrt{g}\,e^{-sm^2}\,\mathrm{tr}\Big\{\hat{1} + s\hat{P} \,+\, s^2\,\big[\,R_{\mu\nu}f_1(-s\Box)R^{\mu\nu}$$
$$+\, Rf_2(-s\Box)R + \hat{P}f_3(-s\Box)R + \hat{P}f_4(-s\Box)\hat{P} + \hat{S}_{\mu\nu}f_5(-s\Box)\hat{S}^{\mu\nu}\big]\Big\}\,.$$

It is important that the two approaches are equivalent.

The elements of the heat-kernel solution are as follows:

$$f_1(\tau) = \frac{f(\tau) - 1 + \tau/6}{\tau^2}\,, \qquad f_3 = \frac{f(\tau)}{12} + \frac{f(\tau) - 1}{2\tau}\,, \qquad f_4 = \frac{f(\tau)}{2}\,,$$

$$f_2(\tau) = \frac{f(\tau)}{288} + \frac{f(\tau) - 1}{24\tau} - \frac{f(\tau) - 1 + \tau/6}{8\tau^2}\,, \qquad f_5 = \frac{1 - f(\tau)}{2\tau}\,,$$

where $$f(\tau) = \int_0^1 d\alpha\, e^{\alpha(1-\alpha)\tau}\,, \qquad \tau = -s\Box\,.$$

One can integrate this out for massive theory and the result fits perfectly with the one from the Feynman diagrams approach.

Ed. Gorbar & I.Sh., JHEP 02 (2003); 06 (2003), hep-ph/0303124., ...

S. Franchino-Viñas, T. de Paula Netto, I.Sh., O. Zanusso, PLB (2019) arXiv:1812.00460.

In this way, the calculations were done for self-interacting scalars, fermions, massive vectors, antisymmetric fields, etc.

It also gave many extra outputs, such as nonlocal multiplicative anomaly, possibility to better explore ambiguities in conformal (trace) anomaly for massless fields, etc.

Back to the results: Consider the UV limit for the form factors in the one-loop effective action, the part quadratic in curvatures.

The classical terms plus the logarithmic corrections are

$$\Gamma^{(1),\,UV}_{vac} \;=\; \int_x C_{\alpha\beta\rho\sigma}\Big[a_1 - \beta_1 \log\Big(-\frac{\Box}{\mu^2}\Big)\Big]C^{\alpha\beta\rho\sigma} + R\Big[a_4 - \beta_4 \log\Big(-\frac{\Box}{\mu^2}\Big)\Big]R \;+\; \dots .$$

Here the beta functions are those of the Minimal Subtraction scheme of renormalization, confirming the correspondence with the Momentum Subtraction scheme in the UV limit.

In the IR, we assume $p^2 \ll m^2$ for Euclidean momenta. Asymptotically, the form factors do not have log's, e.g.,

$$k_W \;=\; -\,\frac{1}{840}\,\frac{p^2}{m^2}\Big(1 + \frac{1}{18}\,\frac{p^2}{m^2}\Big) \;+\; \dots .$$

There is no logarithmic "running" and hence no direct relation between the dependence on momenta p and μ in the IR.

In the gravitational sector, we meet Appelquist and Carazzone - like decoupling, but only in the higher derivative sectors.
In the perturbative approach, with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we do not see running for the cosmological and inverse Newton constants.
Why do we get $\beta_\Lambda = \beta_{1/G} = 0$ **?**

Momentum subtraction running corresponds to the insertion of, e.g., $\ln(\Box/\mu^2)$ **formfactors into effective action.**

Say, in QED:
$$-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{e^2}{3(4\pi)^2}\,F_{\mu\nu}\,\ln\Big(-\frac{\Box}{\mu^2}\Big)\,F^{\mu\nu}\,.$$

Similarly, one can insert formfactors into
$$C_{\mu\nu\alpha\beta}\,\ln\Big(-\frac{\Box}{\mu^2}\Big)\,C_{\mu\nu\alpha\beta}\,.$$

However, such insertion is impossible for Λ and for $1/G$, because $\Box\Lambda \equiv 0$ and $\Box R$ is a full derivative.

Further discussion:
Ed. Gorbar & I.Sh., JHEP (2003,2022); J. Solà & I.Sh., PLB (2009).

How can we separate the gravitational effective action (EA) into Λ- term, Einstein term and higher derivative terms?

The problem is nontrivial since finite part of EA is nonlocal and everything looks mixed, from the first sight.

The most obvious prescription is to use a global scaling.

Consider such a scaling for the metric, $\lambda = const$,

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} e^{2\lambda} \qquad R \longrightarrow R e^{-\lambda}, \quad \Box \longrightarrow \Box e^{-\lambda}, \quad etc.$$

Different terms in the classical action scale differently,

$$\int_x = \int d^4x \sqrt{-g} \longrightarrow \int_x e^{4\lambda}, \qquad \int_x R \longrightarrow \int_x R e^{2\lambda},$$
$$\int_x R^2_{\dots} \longrightarrow \int_x R^2_{\dots}, \quad etc.$$

It looks reasonable to use the same scaling rule for "distributing" quantum corrections into different sectors.

Example: the anomaly-induced EA in the covariant form.

$$\begin{aligned}\Gamma_{ind} \;=\; & S_c(g_{\mu\nu}) + \frac{\omega}{4}\int_x\int_y C^2(x)\,G(x,y)\,\Big(E_4 - \frac{2}{3}\Box R\Big)_y \\ & + \frac{b}{8}\int_x\int_y \Big(E_4 - \frac{2}{3}\Box R\Big)_x G(x,y)\,\Big(E_4 - \frac{2}{3}\Box R\Big)_y \\ & - \frac{3c+2b}{36}\int_x R^2(x).\end{aligned}$$

$$C^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\alpha\beta} + (1/3)\,R^2 \quad \textbf{and} \quad E_4 = R^2_{\mu\nu\alpha\beta} - 4R^2_{\alpha\beta} + R^2$$

are the square of the Weyl tensor and the integrand of the Gauss-Bonnet (GB) topological term. $G(x,y)$ **is the Green function of the Paneitz operator**

$$\Delta_4 = \Box^2 + 2\,R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}\,R\Box + \frac{1}{3}\,(\nabla^\mu R)\nabla_\mu .$$

It is easy to see that all the induced action scales exactly as the classical fourth-derivative terms.

The reason for this global scaling is, of course, that the origin of these nonlocal terms are in

$$C_{\mu\nu\alpha\beta} \ln\Big(-\frac{\Box}{\mu^2}\Big) C_{\mu\nu\alpha\beta}$$

and alike. This term is always present in the massless (in particular, in any kind of conformal) theory if there are UV divergences. We note that both Weyl and the GB divergences never cancel in "normal" theories.

In case of the cosmological term and the Einstein-Hilbert term, correspondingly, the non-local structures would be, e.g.,

$$\int_x R_{\mu\nu}\,\frac{m^4}{\Box^2}\,R^{\mu\nu} \quad \text{and} \quad \int_x R_{\mu\nu}\,\frac{m^2}{\Box}\,R^{\mu\nu}.$$

However, these terms are not generated in a normal way with the logarithmic form factor insertions.

Comment on the apparent relation between form factors.

Starting from the full expression with the form factors,

$$\bar{\Gamma}_{vac} = \frac{1}{2(4\pi)^2}\int d^4x\sqrt{-g}\Big\{\frac{m^4}{2}\Big(\frac{1}{\epsilon_\mu}+\frac{3}{2}\Big)+\tilde{\xi}m^2R\Big(\frac{1}{\epsilon_\mu}+1\Big)$$
$$+\frac{1}{2}C_{\mu\nu\alpha\beta}\Big[\frac{1}{60\epsilon_\mu}+k_W(a)\Big]C^{\mu\nu\alpha\beta}+R\Big[\frac{1}{2\epsilon_\mu}\tilde{\xi}^2+k_R(a)\Big]R\Big\},$$

$$k_W = \frac{8Y}{15\,a^4}+\frac{2}{45\,a^2}+\frac{1}{150}, \qquad \textbf{etc.}$$

In the UV (only!) k_W **has a logarithmic factor** $\log\Box$.

Expanding the coefficient of this logarithm into the power series in $m^2/\Box$, in the second order we get our “desired”

$$\int_x R_{\mu\nu}\,\frac{m^4}{\Box^2}\,\log\Big(-\frac{\Box}{m^2}\Big)\,R^{\mu\nu}.$$

However, by no means this is an “IR running” of the Λ- term.

Ed. Gorbar, I.Sh., arXiv:2203.09232 (JHEP).

In the large-mass limit, one can always expand the Green function in the series, like it is done in the classical theory,

$$\frac{1}{k^2 + m^2} = \frac{1}{m^2} \sum_{n=0}^{\infty} (-1)^n \frac{k^2}{m^2},$$

that apparently guarantees the no-running of the cosmological constant term.

However, there is a loophole. There may be a re-summation of the Green functions and EA, as in the anomaly-induced case.

As a result, there may be an infinite product of the terms like

$$R_{..} \frac{1}{\Box^2} R_{..} \frac{1}{\Box} R_{..} \times \; ... \; \times \frac{1}{\Box} R^{..}.$$

These (and other possible) terms have a global scaling typical for the cosmological constant. And then there is its running.

Let us note that it is a nontrivial task to rule out this kind of terms, as there is no calculational technique for doing this.

Another view on the anomaly-induced action

$$\Gamma_{ind} = S_c(g_{\mu\nu}) + \frac{\omega}{4}\int_x\int_y C^2(x)\,G(x,y)\,\Big(E_4 - \frac{2}{3}\Box R\Big)_y$$
$$+ \frac{b}{8}\int_x\int_y \Big(E_4 - \frac{2}{3}\Box R\Big)_x G(x,y)\Big(E_4 - \frac{2}{3}\Box R\Big)_y - \frac{3c+2b}{36}\int_x R^2(x).$$

We note that this action does not have Green functions of the original scalars, fermions or vectors. Instead, there is a Green function of the artificial Paneitz operator

$$\Delta_4 = \Box^2 + 2\,R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}\,R\Box + \frac{1}{3}\,(\nabla^\mu R)\nabla_\mu.$$

Regardless of this issue, the anomaly-induced action possesses full information about the UV behavior of one loop contributions, at least in the fourth-derivative sector of the action.

Thus, we meet an example of a resummation in the nonlocal terms in the effective action.

Additional observations about trace anomaly

Similar situations hold for conformal scalar, fermion and vector, thus we mainly restrict the consideration by the scalar case.

Consider a legitimate nonlocal terms, e.g.,

$$\bar{\Gamma}_W^{(1)} \;=\; -\int d^4x\sqrt{-g}\,\beta_1\,C_{\alpha\beta\rho\sigma}\log\Big(-\frac{\Box}{\mu^2}\Big)C^{\alpha\beta\rho\sigma}\,.$$

Under the conformal transformation

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\,e^{2\sigma(x)}\,, \qquad \sqrt{-g}\,C_{\alpha\beta\rho\sigma}C^{\alpha\beta\rho\sigma} \;=\; \sqrt{-\bar{g}}\,\bar{C}_{\alpha\beta\rho\sigma}\bar{C}^{\alpha\beta\rho\sigma}.$$

For the d'Alembertian operator, the leading part of the transformation is

$$\Box \;=\; e^{-2\sigma}\big(\bar{\Box} + \text{ derivatives of } \sigma\big).$$

There is a useful relation

$$-\frac{2}{\sqrt{-g}}\,g_{\mu\nu}\,\frac{\delta A[g_{\mu\nu}]}{\delta g_{\mu\nu}} \;=\; -\frac{1}{\sqrt{-\bar{g}}}\,e^{-4\sigma}\,\frac{\delta A[\bar{g}_{\mu\nu}\,e^{2\sigma}]}{\delta\sigma}\,\Big|_{\bar{g}_{\mu\nu}\to g_{\mu\nu},\,\sigma\to 0}\,.$$

Let us use the parametrization

$$g_{\mu\nu} = \bar{g}_{\mu\nu}\, e^{2\sigma(x)}\,, \qquad \sqrt{-g}\, C_{\alpha\beta\rho\sigma} C^{\alpha\beta\rho\sigma} \,=\, \sqrt{-\bar{g}}\, \bar{C}_{\alpha\beta\rho\sigma} \bar{C}^{\alpha\beta\rho\sigma}\,.$$

Obviously,

$$-\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta}{\delta\, g_{\mu\nu}} \int d^4x \sqrt{-g}\, C^2_{\alpha\beta\rho\sigma} \,=\, 0.$$

On the other hand, the log. form factor changes the game:

$$-\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta\, \bar{\Gamma}^{(1)}_{W}}{\delta\, g_{\mu\nu}} \,=\, \beta_1\, C^2_{\alpha\beta\rho\sigma}.$$

In a qualitatively similar way, we get for the Gauss-Bonnet term $E_4 = R^2_{\mu\nu\alpha\beta} - 4R^2_{\alpha\beta} + R^2$,

$$-\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\, \frac{\delta\, \bar{\Gamma}^{(1)}_{E}}{\delta\, g_{\mu\nu}} \,=\, \beta_2\, E_4.$$

We are close to arrive at the standard form

$$\langle T^{\mu}_{\mu} \rangle = \frac{1}{360(4\pi)^2}\, \big(3C^2 - E_4 + 2\Box R\big)\,.$$

Local term and ambiguity

M. Asorey, E. Gorbar, I. Sh., hep-th/0307187, Class. Quant. Grav.

Consider a scalar field

$$S = \int_x \frac{1}{2} \Big\{ (\nabla\varphi)^2 + m^2\varphi^2 + \xi R\varphi^2 \Big\}, \qquad \int_x = \int d^4x\sqrt{-g}.$$

Typically, there are two logarithmic form factors in the UV, but with an important exception of $\xi = 1/6$. In this particular case, in the UV limit $m \to 0$, we obtain in the R^2- sector,

$$-\frac{1}{12\cdot 180(4\pi)^2}\int_x R^2 .$$

Owing to the identity

$$-\frac{2}{\sqrt{-g}}\, g_{\mu\nu}\,\frac{\delta}{\delta g_{\mu\nu}}\int_x R^2 = 12\,\Box R .$$

this provides a perfect fit with $\Box R$ in the conformal anomaly in the point-splitting, ζ-regularization and most of other methods.

Regularization ambiguity in the local term

Consider the Pauli-Villars regularization that was developed for the YM theory.

A.A. Slavnov, Theor. Math. Phys. **33** *(1977) 210.*

M. Asorey and F. Falceto, Nucl. Phys. **B327** *(1989) 427.*

We introduce the set of scalar regulators with masses m_i, nonminimal parameters ξ_i, and indefinite Grassmann parities.

$$S_{reg}^{(i)} = \int_x \Big\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi_i \partial_\nu \varphi_i + \frac{1}{2}\Big(\tilde{\xi}_i + \frac{1}{6}\Big) R\, \varphi_i^2 - \frac{m_i^2}{2}\, \varphi_i^2 \Big\},$$

The Pauli-Villars regularized effective action is defined as

$$\bar{\Gamma}^{(1)}_{\rm reg} = \sum_{i=0}^{N} s_i \bar{\Gamma}^{(1)}_i (m_i, \,\widetilde{\xi}_i, \,n).$$

where Λ is an auxiliary cut-off, $i=0$ is the original scalar contribution. $m_i = \mu_i M$, M is the dimensional parameter of regularization and μ_i are chosen to cancel all divergences.

The use of PV regularization for constructing the second (after dimensional) example of ambiguity in anomaly:

M. Asorey, E.V. Gorbar, I.Sh., hep-th/0307187, Class. Quant. Grav.

M. Asorey, W. C. Silva, I.Sh., P. Vale, arXiv:2202.00154, EPhJC.

The Pauli-Villars conditions for all (including the quadratic and quartic) divergences, have the form

$$\sum_{i=1}^{N} s_i = -s_0 = -1; \qquad \sum_{i=1}^{N} s_i \mu_i^2 = 0, \qquad \sum_{i=1}^{N} s_i \,\widetilde{\xi}_i = 0;$$

$$\sum_{i=1}^{N} s_i \mu_i^4 = 0, \qquad \sum_{i=1}^{N} s_i \,\widetilde{\xi}_i^2 = 0.$$

The possible solutions to these conditions are

$$s_1 = 1, \qquad s_2 = 4, \qquad s_3 = s_4 = s_5 = -2; \qquad \mu_1^2 = \mu_5^2 = 4,$$

$$\mu_2^2 = \mu_4^2 = 3, \qquad \mu_3^2 = 1; \qquad \widetilde{\xi}_i = \mu_i^2 \quad \textbf{or} \quad \widetilde{\xi}_i \equiv 0.$$

The anomaly derived in this way has the form

$$T = -\frac{1}{(4\pi)^2}\left[\frac{1}{120}C^2 - \frac{1}{180}E + \left(\frac{1}{180} - 12\delta\right)\right]\Box R + \,...\,,$$

with $$\delta = \sum_{i=1}^{N} s_i \left(\xi_i - \frac{1}{6}\right)^2 \ln \mu_i^2\,.$$

It is easy to see that δ vanishes if all Pauli-Villars field regulators have conformal couplings $\xi_i = \frac{1}{6}, \quad i = 1, \dots N$.

However, if the regulators are scalar fields with non-conformal couplings, we meet arbitrariness in the $\Box R$- sector of anomaly and in the $\int_x R^2$ induced finite term in the effective action.

In this way, we can use the IR limit of the form factors, in the massive case, to get the second example of the ambiguity of the local term in the anomaly-induced effective action.

First Conclusions

- **The renormalization program in curved spacetime is a full success if we are interested just in removing divergences.**

However, what we really want is to extract the finite part of the effective action, Green functions, etc. This is what we need for the applications, e.g., in cosmology.

- **The derivation of nonlocal form factors in the fourth-derivative sector of the theory shows a perfect agreement with the divergences in the UV and the quadratic decoupling in the IR.**

- **The derivation of nonlocal form factors in the R and Λ sectors meet serious difficulties and require qualitatively new methods of calculations which are not available right now. The question of whether CC can be variable or not is open.**

- **All this concerns the theory of quantum matter on the classical gravity background. The decoupling in quantum gravity is much more complicated and less elaborated.**