Phase and frequency synchronization of non-locally coupled oscillators

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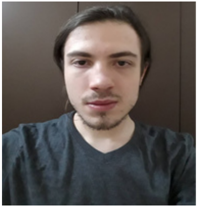

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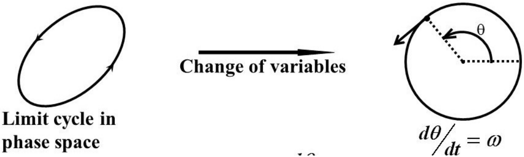
Introduction

- phase oscillators are simple models for many dynamical systems of physical and biological interest
- assemblies of phase oscillators can present collective behavior, like phase and frequency synchronization
- synchronization is often caused by interactions among phase oscillators, even when they are slightly different
- the interaction among phase oscillators can be mediated by a chemical which diffuses along "cells" (pointlike systems)
- the coupling is non-local, and takes into account the relative distances among oscillators
- **this work:** how are the synchronization properties influenced by coupling parameters characteristic of a diffusion-mediated interaction?
- our answer involves the numerical solution of a system of integro-differential equations, containing the Green's functions related to the boundary conditions and the geometrical details

Phase oscillators

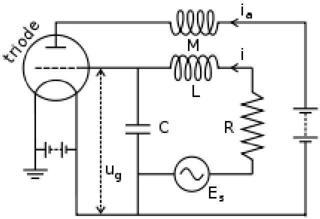


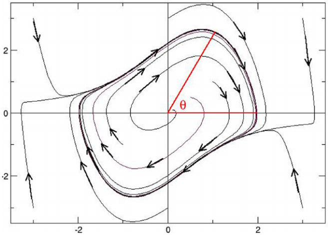
- one-dimensional dynamical systems defined on a topological circle S^1
- characterized by a geometrical phase θ which varies with time according to a given frequency ω

$$\frac{d\theta}{dt} = \omega, \qquad 0 \leq \theta < 2\pi$$

- often appear from a stable limit-cycle in phase space, after a suitable change of variables
- simple mathematical models of periodic phenomena of physical and biological interest

Van der Pol system



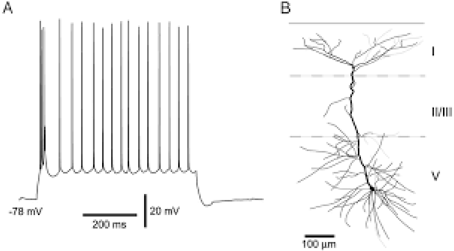


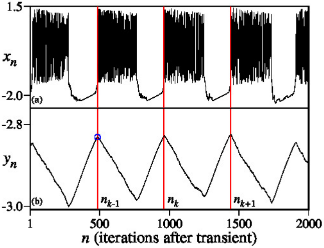
- electronic circuit with nonlinear element (triode, semiconductor, ...)
- x: current through the capacitor (time-rate y)

$$\dot{x} = y, \qquad \dot{y} = \mu(1 - x^2)y - x$$

- $\mu \neq 0$: stable limit-cycle in the phase plane $x - y$
- relaxation oscillations
- geometrical phase: $\theta(t) = \arctan[y(t)/x(t)]$

Bursting neurons



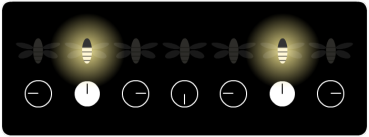


- bursting: rapid sequence of spikes (membrane potential), after a quiescent period
- bursting phase: defined in terms of the (discrete) times at which at which a kth burst begins (n_k) and ends (n_{k+1})

$$\theta(n) = 2\pi k + 2\pi \frac{n - n_k}{n_{k+1} - n_k}, \qquad (n_k \leq n \leq n_{k+1})$$

- bursting frequency: $\omega = (\theta(n) - \theta(0))/n$

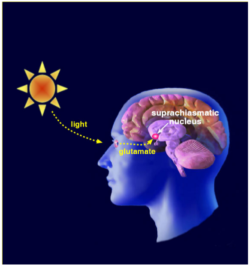
Synchronization of phase oscillators

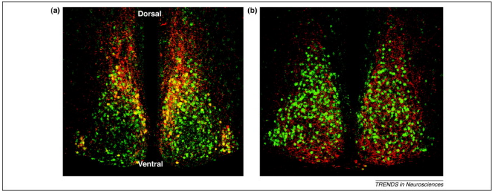


- each firefly flashes periodically: an individual phase oscillator
- fireflies synchronize their flashing rhythms through their visual interaction
- since the velocity of light is large the coupling is instantaneous (mean-field effect)
- "classical" Kuramoto model (global coupling)

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_j \sin(\theta_j - \theta_i)$$

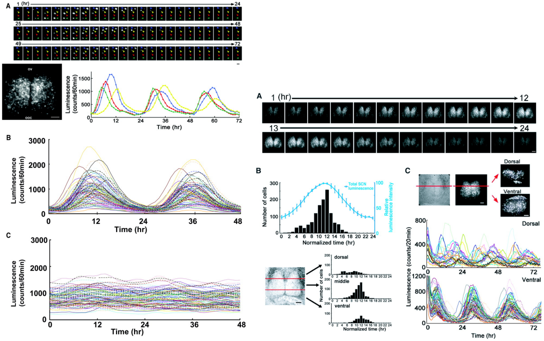
Clock cells in SCN





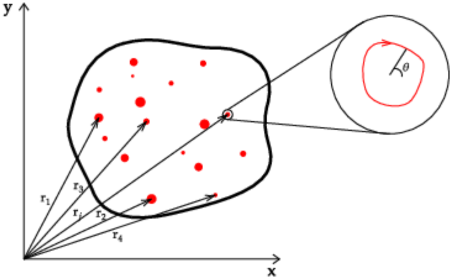
- suprachiasmatic nucleus (SCN): small region in the brain hypothalamus whose function is to control circadian rhythms (photic stimulation)
- it contains *circa* 10^4 clock cells with a natural variety of individual frequencies (~ 24 h cycle)
- their coupling is mediated by a neurotransmitter (GABA) which diffuses through the spatial medium in which the SCN cells are embedded

Synchronization of clock cells



- S. Yamaguchi *et al.*, Science **302**, 1408 (2003)
- since the SCN acts as a pacemaker, in order to generate a collective single rhythm each clock cell must synchronize its own frequency
- synchronization as a coupling-induced collective phenomenon
- coupling is related to the diffusion of GABA in the intercell medium

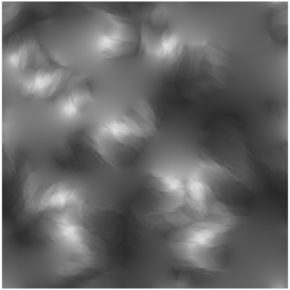
Uncoupled oscillators



- phase oscillators are pointlike and occupy fixed positions in a spatial domain $\mathcal{R}$ (bounded or unbounded) in d dimensions
- θ_j: phase of the jth oscillator $(j = 1, 2, \ldots N)$
- $\mathbf{r}_j$: position vector of the jth oscillator
- ω_j: natural frequency of the jth oscillator $(\dot{\theta}_j = \omega_j)$
- randomly chosen from a unimodal normalized probability distribution $g(\omega)$ (with unit variance)

$$\int_{-\infty}^{\infty} d\omega\, g(\omega) = 1$$

Oscillator coupling induced by a mediating substance

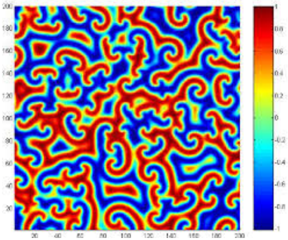


- in the presence of coupling the oscillator dynamics is linearly proportional to the local concentration of the mediating substance A at the oscillator position

$$\dot{\theta}_j = \omega_j + KA(\mathbf{r}_j, t), \qquad (j = 1, 2, \dots N)$$

- $K > 0$: coupling intensity
- all quantities are non-dimensional

Diffusion with pointlike sources



- the substance is produced by all the pointlike oscillators and diffuses through the spatial region

$$\frac{\partial A}{\partial t} = -\eta\, A + D\, \nabla^2 A + \sum_{k=1}^{N} h(\theta_k)\, \delta(\mathbf{r} - \mathbf{r}_k)$$

- D: diffusion coefficient, η: coefficient of chemical degradation
- the source term for the diffusion equation depends on the oscillator phases by a (generally nonlinear) function $h(.)$
- suitable initial and boundary conditions have to be specified

General formulation

- diffusion characteristic time is arbitrary with respect to the oscillator periods ("slow" diffusion)
- we have to solve simultaneously the following system of ordinary/partial differential equations

$$\frac{d\theta_j}{dt} = \omega_j + KA(\mathbf{r}_j, t) \qquad (j = 1, 2, \ldots N),$$

$$\frac{\partial A}{\partial t} + \eta A - D\nabla^2 A = \sum_{k=1}^{N} h(\theta_k)\delta(\mathbf{r} - \mathbf{r}_k),$$

- for appropriate boundary conditions at some limiting surface $\partial\mathcal{R}$, as well as an initial condition profile $A(\mathbf{r}, t = 0)$
- the Green function $G(\mathbf{r}, t; \mathbf{r}', t')$, satisfies

$$\frac{\partial G}{\partial t} + \eta\, G - D\, \nabla^2 G = \delta(\mathbf{r} - \mathbf{r}')\, \delta(t - t'),$$

- homogeneous Dirichlet boundary conditions: $G(\mathbf{r}, t; \mathbf{r}', t')$ for $\mathbf{r} \in \partial\mathcal{R}$, and the initial condition $G(\mathbf{r}, t = 0; \mathbf{r}', t') = 0$

General formulation

- solution of the inhomogeneous equation for absorbing boundary conditions on $\partial\mathcal{R}$: $A(\mathbf{r} \in \partial\mathcal{R}, t) = 0$, and initial profile $A(\mathbf{r}, t = 0) = 0$,

$$A(\mathbf{r}, t) = \sum_{k=1}^{N} \int_0^t dt' \, G(\mathbf{r}, t | \mathbf{r}_k, t') \, h(\theta_k(t')).$$

- system of integro-differential equations $(j = 1, 2, \ldots N)$

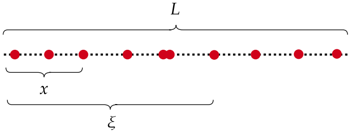
$$\frac{d\theta_j}{dt} = \omega_j + K \sum_{k=1}^{N} \int_0^t dt' G(\mathbf{r}_j, t; \mathbf{r}_k, t') \, h(\theta_k(t')),$$

- we choose: $h(\theta_k) = (1/N) \sin(\theta_k - \theta_j)$

$$\frac{d\theta_j}{dt} = \omega_j + \frac{K}{N} \sum_{k=1}^{N} \int_0^t dt' \sin[\theta_k(t') - \theta_j(t')] G(\mathbf{r}_j, t; \mathbf{r}_k, t').$$

- main difficulty: the coupling term takes into account all previous history $\theta(t')$ for $0 \leq t' \leq t$

One-dimensional bounded domain



- finite domain $0 \le x \le L$ with absorbing boundary conditions $(A(0,t) = A(L,t) = 0)$ and initial condition $A(x, t = 0) = 0$
- Green function as a superposition of eigenfunctions

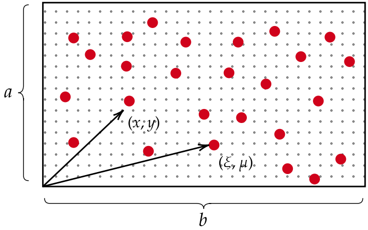
$$G(x, t; x', t') = \frac{2H(t-t')}{L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x'}{L}\right) \times$$

$$\exp\left\{-\left[D\left(\frac{n\pi}{L}\right)^2 + \eta\right](t-t')\right\},$$

- randomly chosen positions $\{x_j\}_{j=1}^{N}$ in $0 < x < L$

$$\frac{d\theta_j}{dt} = \omega_j + \frac{K}{N} \sum_{k=1}^{N} \int_0^t dt' \sin[\theta_k(t') - \theta_j(t')] G(x_j, t; x_k, t'),$$

Two-dimensional rectangular domain



- rectangular domain: $0 < x < a,\ 0 \le y \le b$

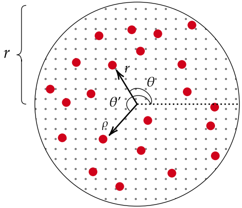
$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{4H(t-t')}{L}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)\times$$

$$\sin\left(\frac{n\pi x'}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)\times$$

$$\exp\left\{-\left[D\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)\pi^2+\eta\right](t-t')\right\}.$$

- oscillators have randomly chosen positions $\{x_j, y_j\}_{j=1}^{N}$

Two-dimensional circular domain

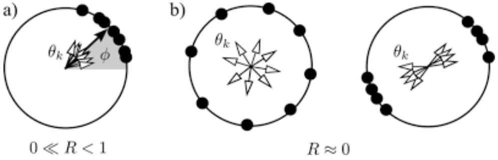


- circular domain of radius $r = a$

$$G(r,\vartheta,t;r',\vartheta',t') = \frac{1}{\pi D}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{1}{[J'_m(x_{mn})]^2}\times$$
$$J_m\left(x_{mn}\frac{r}{a}\right)J_m\left(x_{mn}\frac{r'}{a}\right)\cos[m(\vartheta-\vartheta')]\times$$
$$\exp\left\{-\left(\eta+\frac{Dx_{mn}^2}{a^2}\right)(t-t')\right\},$$

- x_{mn}: nth positive root of the Bessel function J_m
- randomly chosen positions $\{r_j,\vartheta_j\}_{j=1}^N$, with $0 \le r_j < a$

Phase synchronization

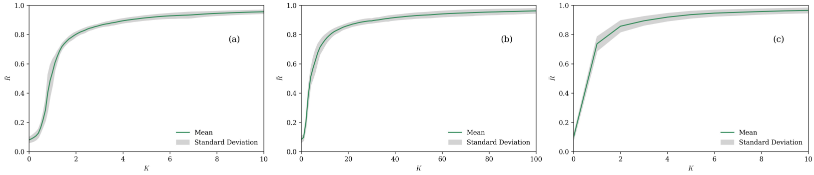


- Kuramoto complex order parameter

$$z(t) = R(t)\, e^{\phi(t)} = \frac{1}{N} \sum_{k=1}^{N} e^{i\theta_k(t)}$$

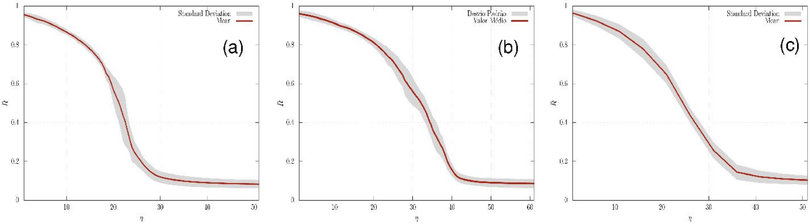
- order parameter magnitude $R(t) = \sqrt{z^* z}$. After some transient we take its mean $\bar{R}$ over a time interval
- $\bar{R} \approx 0$: the oscillator phases are uniformly distributed and the resultant phasor vanishes
- $\bar{R} \approx 1$: all the oscillators are phase-synchronized since their phasors in the unit circle add coherently
- numerically $\bar{R} = 0.95$ as a threshold for complete phase synchronization, lower values characterizing partial sync

Order parameter magnitude *vs* K



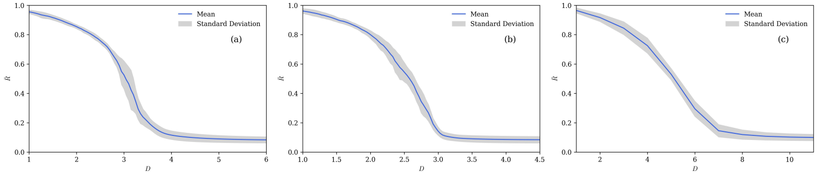
- $N = 100$ oscillators randomly distributed in (a) linear, (b) rectangular, (c) circular domains
- $\bar{R}$ as a function of the coupling strength K, for $D = \eta = 1$,
- monotonic increase of $\overline{R}$ with the coupling strength K, signaling a synchronization transition roughly at $K = 1$
- rectangular domain [of sides $a = b = 1$]: the range of K is ten times higher than for the linear domain
- circular domain (radius $a = 1$): similar range as for one-dimensional domain (radial symmetry)

Order parameter magnitude *vs* η



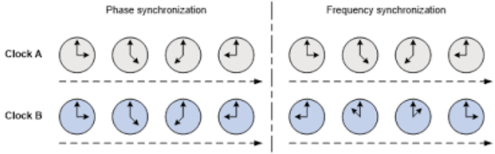
- $N = 100$ oscillators randomly distributed in (a) linear, (b) rectangular, (c) circular domains
- $\bar{R}$ as a function of the degradation parameter η, for $K = 10$ and $D = 1$.
- decrease of $\bar{R}$ as η increases
- since η measures the loss of the substance mediating the coupling, the basic effect of its increase is the decrease in the amount of phase synchronization

Order parameter magnitude *vs* D



- $N = 100$ oscillators randomly distributed in (a) linear, (b) rectangular, (c) circular domains
- $\bar{R}$ as a function of the diffusion coefficient D, for $K = 10$ and $\eta = 1.0$.
- $\overline{R}$ decrease monotonically as D increases: a large D actually desynchronizes the oscillators
- the coupling effect is more effective the longer the mediating substance remains in the spatial medium in which the oscillators are embedded
- for large D the permanence time of the substance is small, reducing the coupling effect on synchronization (for absorbing boundary conditions)

Frequency synchronization



- perturbed oscillator frequencies

$$\Omega_j(T) = \lim_{t\to\infty} \frac{1}{t} \{\theta_j(t+T) - \theta_j(T)\}$$

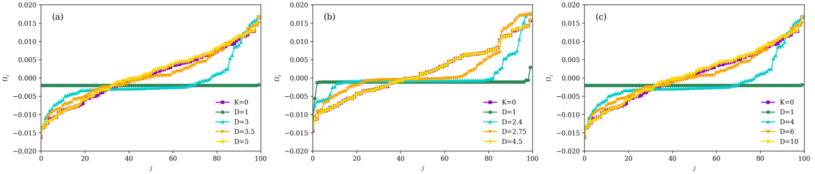
- if uncoupled $(K = 0)$ then $\Omega_j = \omega_j$
- frequency synchronization: $\Omega_1 = \Omega_2 = \dots$ (up to a given tolerance)
- numerically we found that the above limit always exists and is independent on T

Characterizing frequency synchronization

- frequency order parameter: we divide the frequency interval $[\min \Omega_j, \max \Omega_j]$ into sub-intervals of size δ
- consider the fraction of oscillators belonging to the sub-interval with the largest number of oscillators
- if the largest number of oscillators in a given interval is N_{max}, we define a frequency order parameter by $P = N_{max}/N$
- if all the oscillators have the same frequency, then $P = 1$ (frequency synchronization)
- if there is no frequency synchronization $N_{max} \approx 1$ and $P \sim 1/N \to 0$ for $N \gg 1$
- in the numerical simulations we have chosen $\delta = \sigma/2001$, where $\sigma = \sqrt{\pi/8}$ is the standard deviation of the uncoupled frequency (Gaussian) distribution

$$g(\omega) = \frac{2}{\pi} e^{-4\omega^2/\pi}.$$

Frequency synchronization *vs* D



- perturbed oscillator frequencies Ω_j in increasing order of their values
- no coupling: $\Omega_j = \omega_j$
- (a) $K = \eta = 1.0$ and different values of D (linear domain)
- (b) rectangular domain and (c) circular domain
- larger values of D produce frequency desynchronization just like they do for oscillator phases (for absorbing boundary conditions)

Adiabatic limit

- Y. Kuramoto, Prog. Theor. Phys. **94**, 321 (1995); Y. Kuramoto and H. Nakao, Physica D **103**, 294 (1997)
- if the diffusion characteristic time is much smaller than any of the oscillator periods $2\pi/\omega_j$, then $\partial A_E/\partial t \approx 0$
- the concentration of the substance undergoes a fast relaxation and converge very rapidly to its stationary limit A_E

$$\eta A_E - D\nabla^2 A_E = \sum_{k=1}^{N} h(\theta_k)\delta(\mathbf{r}-\mathbf{r}_k)$$

- local equilibrium concentration of the mediating chemical

$$A_E(\mathbf{r}) = \sum_{k=1}^{N} h(\theta_k) G_E(\mathbf{r},\mathbf{r}_k)$$

- $G_E(\mathbf{r},\mathbf{r}')$: Green function for Dirichlet boundary conditions at the boundary $\partial\mathcal{R}$ of the spatial domain

$$\eta G_E(\mathbf{r},\mathbf{r}') - D\nabla^2 G_E(\mathbf{r},\mathbf{r}') = \sum_{k=1}^{N} \delta(\mathbf{r}-\mathbf{r}')$$

Adiabatic limit

- coupled oscillator equations (in the adiabatic limit)

$$\dot{\theta}_j = \omega_j + K \sum_{k=1}^{N} h(\theta_k) G_E(\mathbf{r}_j, \mathbf{r}_k), \qquad (j = 1, 2, \dots N).$$

- choosing the nonlinear response function

$$h(\theta_k) = \frac{1}{N} \sin(\theta_k - \theta_j),$$

- we have a Kuramoto-like model of coupled phase oscillators

$$\dot{\theta}_j = \omega_j + \frac{K}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j) G_E(\mathbf{r}_j, \mathbf{r}_k), \qquad (j = 1, 2, \dots N),$$

- for simplicity we choose free boundary conditions: $\lim_{|\mathbf{r}| \to \infty} G_E(\mathbf{r}, \mathbf{r}') = 0$.

Adiabatic limit: one dimension

- equilibrium Green function (free space)

$$G_E(x,t;x_k,t') = \frac{H(t-t')e^{-\eta(t-t')}}{\sqrt{4\pi D(t-t')}} \exp\left\{-\frac{(x-x_k)^2}{4D(t-t')}\right\}$$

- interaction kernel

$$\sigma(\mathbf{r}_j,\mathbf{r}_k,t) = \int_0^t dt'\, G_E(\mathbf{r}_k,t;\mathbf{r}_k,t')$$

- the adiabatic limit is equivalent to take the $t \to \infty$ limit in the interaction kernel

$$\sigma(x_j,x_k) = \lim_{t\to\infty} \sigma(x_j,x_k,t) = \frac{\gamma}{2\eta}\, e^{-\gamma(x_j-x_k)}$$

- which is the result previously derived by Kuramoto and Nakao [Chaos **9**, 902 (1999)]

Adiabatic limit: two and three dimensions

- equilibrium Green functions (free space)

$$G_E(\mathbf{r},t;\mathbf{r}',t) = \frac{H(t-t')e^{-\eta(t-t')}}{[4\pi D(t-t')]^{d/2}} \exp\left\{-\frac{|\mathbf{r}-\mathbf{r_k}|^2}{4D(t-t')}\right\}$$

- interaction kernel in two dimensions $(d=2)$

$$\sigma(\mathbf{r}_j,\mathbf{r}_k,t) = \frac{1}{4\pi D}\int_{u_1}^{\infty}\frac{du}{u}\exp\left(-u-\frac{a_2}{u}\right)$$

$$a_2 = \frac{\gamma^2|\mathbf{r}_j-\mathbf{r}_k|^2}{4}, \qquad u_1 = \frac{|\mathbf{r}_j-\mathbf{r}_k|^2}{4Dt}$$

- taking the $t\to\infty$ limit

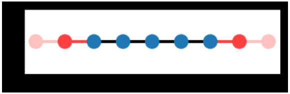
$$\sigma(\mathbf{r}_j,\mathbf{r}_k) = \frac{1}{2\pi D}K_0(\gamma|\mathbf{r}_j-\mathbf{r}_k|)$$

- for the free three-dimensional case

$$\sigma(\mathbf{r}_j,\mathbf{r}_k) = \frac{1}{4\pi D}\frac{1}{|\mathbf{r}_j-\mathbf{r}_k|}e^{-\gamma|\mathbf{r}_j-\mathbf{r}_k|}$$

- both results agree with those of Nakao [Chaos **9**, 902 (1999)]

One-dimensional case



- infinite one-dimensional chain of oscillators

$$\dot{\theta}_j = \omega_j + K C_{1,j}(\gamma, N) \sum_{k=1}^{N} e^{-\gamma |x_k - x_j|} \sin(\theta_k - \theta_j),$$

- coupling length: $\gamma = \sqrt{\eta / D}$,
- regular lattices: the oscillator positions are separated by a fixed distance Δ
- periodic boundary conditions

$$|x_k - x_j| = \Delta \times \min\left\{\Psi_k^j, N - \Psi_k^j\right\},$$

- Ψ_k^j is the remainder of the integer division of $|k - j|$ by N.

One-dimensional case

- normalization condition for the Green functions in d dimensions
$$\int d^d r\, G(\mathbf{r}, \mathbf{r}_k) = 1,$$
- normalization factor
$$C_{1,j}(\gamma, N)^{-1} = \begin{cases} \sum_{k=1}^{N} e^{-\gamma|x_k - x_j|} - 1, & N \text{ even} \\ 2\sum_{k=1}^{(N-1)/2} e^{-\Delta\gamma k}. & N \text{ odd} \end{cases}$$
- for N odd this can be put into a symmetrical form
$$\dot{\theta}_j = \omega_j +$$
$$K C_{1,j}(\gamma, N) \sum_{k=1}^{(N-1)/2} e^{-\Delta\gamma k} \{\sin(\theta_{j-k} - \theta_j) + \sin(\theta_{j+k} - \theta_j)\}$$
- initial conditions $\theta_k(t=0)$ are randomly chosen from a uniform probability distribution in $[0, 2\pi)$

One-dimensional case: limits

- vanishing coupling length: $\gamma = \sqrt{\eta/D} \to 0$
- normalization factor

$$C_{1,j}(\gamma = 0, N) = 1/(N-1)$$

- rearranging the summations we have

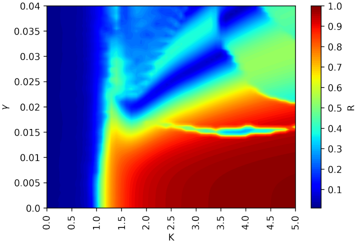
$$\dot{\theta}_j = \omega_j \; + \frac{1}{N-1} \sum_{\ell=1}^{N-1} \sin(\theta_\ell - \theta_j)$$

- which is the classical Kuramoto model of global coupling (all-to-all): each oscillator is influenced by the mean field caused by all other oscillators
- infinitely large coupling length: $\gamma = \sqrt{\eta/D} \gg 1$: only the $k = 1$ terms contribute significantly in the summations
- the coupling term is proportional to

$$\sin(\theta_{j-1} - \theta_j) + \sin(\theta_{j+1} - \theta_j)$$

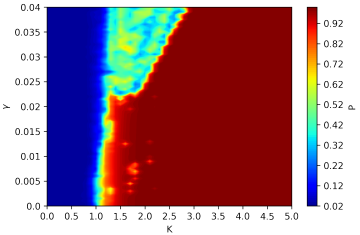
- which is the nearest-neighbor (or diffusive) local coupling

Phase order parameter $\bar{R}$ and coupling parameters



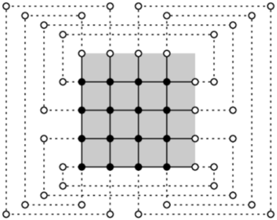
- γ: coupling length, K: strength
- small γ: transition from non-synchronized to completely synchronized states, for $K > K_c$.
- global coupling limit ($\gamma = 0$): for $N \to \infty$: $K_{c,\infty} = 2/\pi g(0) = 1$; for finite N: $K_c \gtrsim K_{c,\infty}$
- $\gamma \lesssim 0.015$: increase of K_c, with a narrow "valley" of non-synchronized behavior in between (chimera states)
- larger γ: synchronization cannot be achieved for $0 \leq K \leq 5.0$ (local coupling limit)

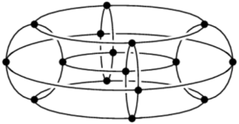
Frequency order parameter P and coupling parameters



- $\gamma < 0.022$: similar behavior in comparison with the phase order parameter
- phase synchronized oscillators are always frequency synchronized but the converse is not always true
- K_c for frequency synchronization should be slightly smaller than for phase synchronization
- $\gamma > 0.02$: large frequency order parameter, with phase order parameter between 0.4 and 0.8 (no valley)

Two-dimensional case





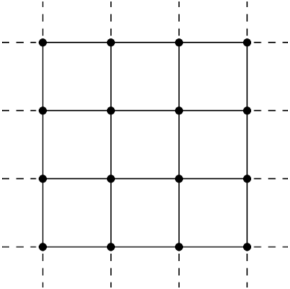
- no boundary surfaces: the Green function is proportional to the modified Bessel function of the second kind $\mathcal{K}_0(\gamma|\mathbf{r}-\mathbf{r}_k|)$,

$$\dot{\theta}_j = \omega_j + K C_{2,j}(\gamma, N) \sum_{k=1, k\neq j}^{N} \mathcal{K}_0(\gamma|\mathbf{r}_k - \mathbf{r}_j|) \sin(\theta_k - \theta_j),$$

- normalization factor

$$C_{2,j}(\gamma, N)^{-1} = \sum_{k=1, k\neq j}^{N} \mathcal{K}_0(\gamma|\mathbf{r}_k - \mathbf{r}_j|).$$

Rectangular lattice



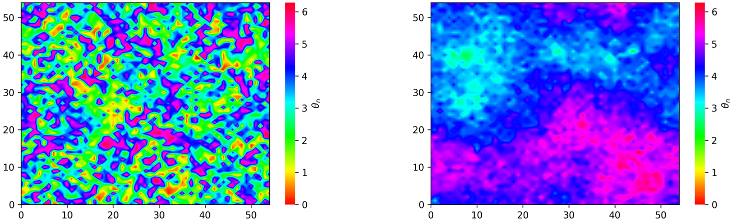
- rectangular lattice with $n \times n$ sites with uniform spacing L_x in the x-axis direction and L_y in the y-axis direction
- periodic boundary conditions: distance between two oscillators located at $\mathbf{r}_k = (i_1 L_x, j_1 L_y)$ and $\mathbf{r}_j = (i_2 L_x, j_2 L_y)$

$$|\mathbf{r}_k - \mathbf{r}_j| = \sqrt{(\Delta x)^2_{k,j} + (\Delta y)^2_{k,j}},$$

$$(\Delta x)_{k,j} = L_x \min\left\{\Psi^{i_2}_{i_1}, n - \Psi^{i_2}_{i_1}\right\},$$

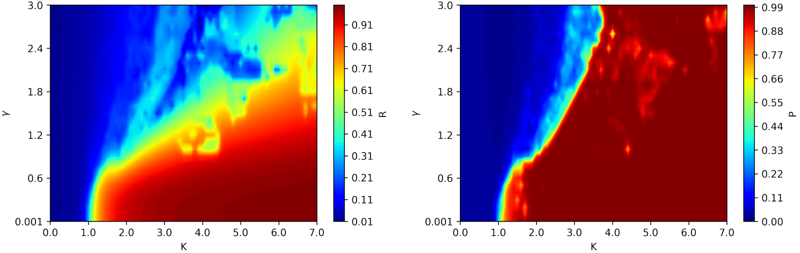
$$(\Delta y)_{k,j} = L_y \min\left\{\Psi^{j_2}_{j_1}, n - \Psi^{j_2}_{j_1}\right\}.$$

Snapshots of the two-dimensional case



- $\Delta = L_x = L_y = 1$ and $N^2 = 55^2 = 3025$
- LEFT ($\gamma = 0.8$ and $K = 0.9$): coexistence of spatial domains with phase-correlated oscillators
- RIGHT ($\gamma = 0.8$ and $K = 2.5$): emergence of spatially phase-coherent regions

Order parameters of the two-dimensional case

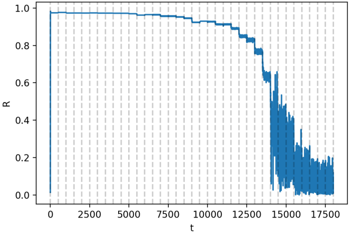


- LEFT: phase order parameter magnitude, RIGHT: frequency order parameter magnitude
- γ small: transition to synchronized behavior for $K > K_c \approx 1.0$
- difference with the one-dimensional case: range of γ has a nearly ten-fold increase: a diffusive process in two dimensions involves a larger range for the same value of γ

Lesions: removal of oscillators

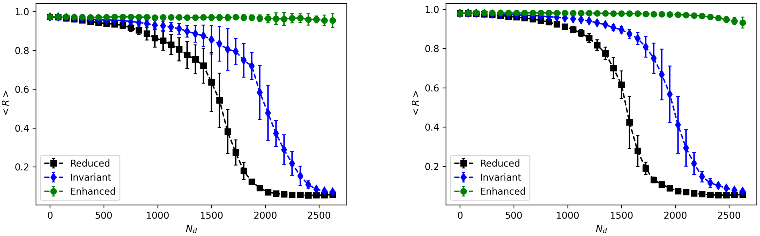
- neurodegenerative conditions (e.g. Alzheimer disease) related to impairment of neurons and/or their synaptical connections
- lesion protocols: given a network initially with N oscillators and a coupling strength K_i, a number N_d of them is removed
- active oscillators: the remaining $N_a = N - N_d$ ones
- the network is supposed to adapt to these alterations, such that the coupling strength will vary with the number of removed oscillators in three possible ways
  1. enhanced coupling: $K(N_d) = K_i N/(N - N_d)$;
  2. invariant coupling: $K(N_d) = K_i$;
  3. reduced coupling: $K(N_d) = K_i N/(N + N_d)$
- we choose values of (K_i, γ) which lead to completely phase-synchronized states and solve the coupled oscillator equations for a long time, until the transients have died out
- then we remove a small quantity of sites and resume integration, such that the order parameter is recomputed.
- the process is repeated until the number of removed oscillators reaches a specified value of N_d.

Removal of oscillators (invariant coupling)



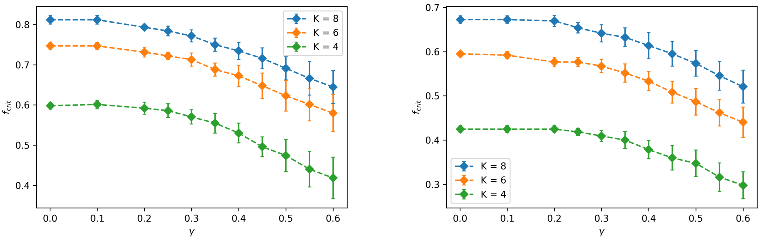
- Phase order parameter magnitude as a function of time for a one-dimensional chain with $\gamma = 0.01$ and $K_i = 4$
- dashed vertical bars indicate the times at which a small quantity of sites is removed until we get $N = N_d$
- the order parameter decreases with time in a similar fashion to the directed percolation scenario
- the oscillators are kept synchronized $(R > 0.95)$ until, after a time $\sim 10^4$, they progressively lose phase synchronization and eventually become completely non-synchronized.

Removal of oscillators (one and two dimensions)



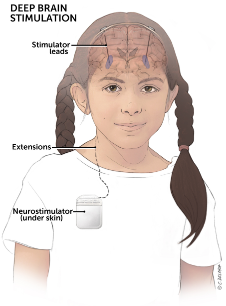
- average phase order parameter $\langle R \rangle$ taken over those time intervals for which the number of active oscillators $N_a = N - N_d$ is constant
- we have made for each protocol eight simulations that differ only in the order of removal of the oscillators
  1. invariant coupling: the chain becomes non-synchronized as N_d is increased
  2. enhanced coupling: the chain remains synchronized even if N_d is as large as 2500, with $N_i = 3000$
  3. reduced coupling: transition to non-synchronized behavior occurs even before the case for which K does not vary with N_d

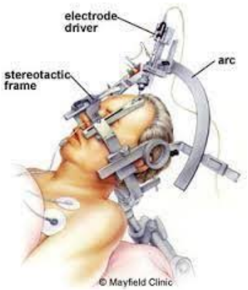
Removal of oscillators (two-dimensional)



- the value of N_d for which the lattice start losing synchronization is practically not affected by the order by which each oscillator is removed
- N_{crit}: critical value of N_d, for given K and γ, such that, if $N_d \geq N_{crit}$, the lattice cannot synchronize
- we estimate the critical fraction of removed oscillators, $f_{crit} = N_{crit}/N$, by taking the minimum value of N_d yielding $R(N_d) \leq 0.9R(N_d = 0)$
- critical fraction of removed oscillators as a function of γ for different values of K_i with (a) invariant, (b) reduced coupling

Suppression of synchronization in neuronal systems





- synchronization in neuronal networks is often related to pathological rhythms (Parkinson´s disease, essential tremor, epilepsy, etc.)
- deep brain stimulation: suppression of phase synchronization by an external electrical signal
- computer simulation: time-delayed feedback control signal depending on the local mean field

Time-delayed feedback control

- coupled oscillator equations

$$\dot{\theta}_j = \omega_j + KY_j(t)$$

- coupling term

$$Y_j(t) = \sum_k G(\mathbf{r}_j, \mathbf{r}_k, \gamma) \sin(\theta_k(t) - \theta_j(t)).$$

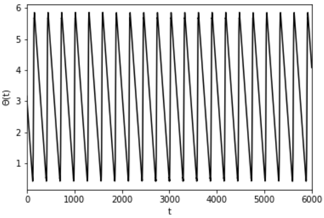
- the values of K and γ are chosen to yield complete synchronization
- external feedback control with time delay τ and amplitude ε

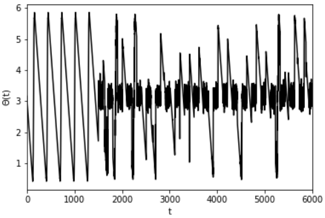
$$\dot{\theta}_j = \omega_j + KY_j(t) + \varepsilon Y_j(t-\tau)H(t-\alpha),$$

- $H(t-\alpha)$ is the Heaviside unit-step function
- α is the time after which the control is continuously applied (chosen after transients have died out)
- mean field of the network at a given time

$$\Theta(t) = \frac{1}{N}\sum_{j=1}^{N} \theta_j(t),$$

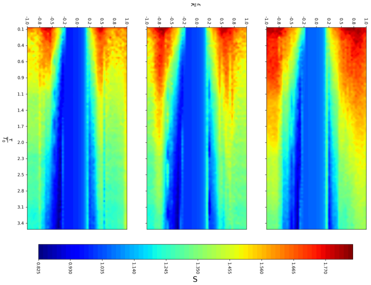
Mean field as diagnostic of synchronization





- complete phase synchronization: the mean field will have the same variation in time as any of the oscillator themselves, with a finite variance $\mathrm{Var}(\Theta)$
- no synchronization: for large N the phases are more or less uniformly distributed over the interval $[0, 2\pi)$, and the mean field have fluctuations of low amplitude, and a corresponding small variance
- LEFT (no control): synchronized behavior
- RIGHT: delayed feedback control applied at $\alpha = 1500$, $\varepsilon = 4$, and $\tau = 100$: partial suppression of synchronization

Quantifying suppression of synchronization



- suppression coefficient (Pikowsky and Rosenblum)

$$S = \sqrt{\mathrm{Var}(\Theta)/\mathrm{Var}(\Theta_f)}$$

- $\Theta_f(t)$: mean field after the control signal has been applied
  1. good suppression of synchronization: $S > 1$
  2. no suppression: $S = 1$
  3. enhanced synchronization: $0 < S < 1$
- S versus τ and ε for a two-dimensional lattice for $K = 6.0$ and (a) $\gamma = 0.02$, (b) 0.4 and (c) 0.6.

Conclusions

- a system of nonlinear integro-differential equations was obtained to model the coupling among phase oscillators mediated by a diffusing substance
- the coupling term is nonlocal and depends on the previous history of the oscillators dynamical behavior
- in the fast relaxation (adiabatic) limit the local concentration of the mediating substance achieves instantaneously its equilibrium value: system of coupled differential equations (no memory effects)
- the corresponding Green function depends on the geometry and the (absorbing) boundary conditions
- three geometries have been investigated: bounded linear, rectangular and circular domains
- in the adiabatic limit (diffusion occurs instantaneously) the expressions reduce to those previously obtained

Conclusions

- collective behavior: phase and frequency synchronization (order parameters)
- increase with coupling strength: transition between non-synchronized and synchronized states
- increasing degradation coefficient reduces synchronization (less mediating substance at local level)
- similar effect for increasing diffusion coefficient: reduces permanence time in the diffusion region (absorbing boundary conditions)
- lesions (removal of oscillators): three protocols. Coupling strength has to increase to keep synchronization
- external time-delayed feedback control: suppression of synchronization

Future works and perspectives

- coupling equations can be extended for nonlinear dynamical systems (flows and maps)
- reflecting boundary conditions can be introduced (but Green´s functions are more complicated!)
- coupling can be mediated by the emission and absorption of waves (finite propagation speed): "retarded potentials"
- it is possible to include advection effects in the diffusion equation (asymmetric coupling)
- chemotaxis: include motion of the pointlike oscillators, according to a chemotactic force $\mathbf{F} = K\nabla A$. The chemical coupling equations must be coupled to Newtonian equations of motion $\mathbf{F}_j = m\ddot{\mathbf{r}}_j$ for each oscillator
- a wealth of cases of potential interest in microbiology/cell biology (*Dictyostelium sp.*)

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- R. P. Aristides and R. L. Viana, *An integro-differential equation for dynamical systems with diffusion-mediated coupling*, Nonlinear Dynamics **100**, 3759 (2020).
- P. Haerter and R. L. Viana, *Synchronization of phase oscillators due to nonlocal coupling mediated by the slow diffusion of a substance*, Braz. J. Phys. **53**, 114 (2023).