Opinion dynamics in complex networks: threshold q-voter

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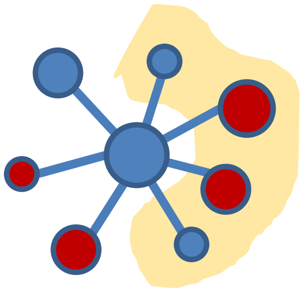
SUMMARY

- Opinion dynamics
  - Usually pairwise interactions
  - interactions with group of influence
  - noisy version (independence)

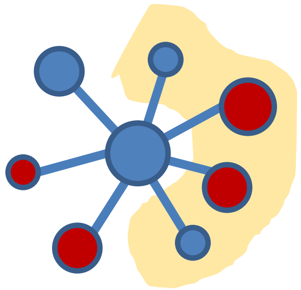


- How order/disorder transitions are affected by
  - composition of the group
  - network structure

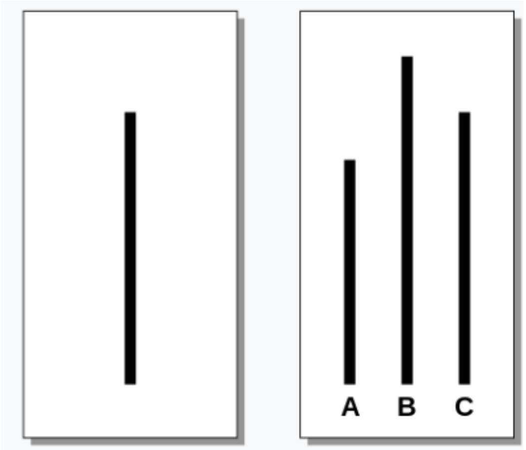
- Approach
  - Agent-based simulations in networks
  - Theoreticallly: pair-approximation

MOTIVATIONS

Social pressure: Individuals interacting with other individuals may change their own opinions and beliefs (even against rationality)



Asch conformity experiments



→ Strong tendency to go along with the rest

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(NOISY) VOTER MODELS

State variable: opinion $s_i \in \{-1, +1\}$

Opinions change through mechanisms that act stochastically:

- with probability 1-p

social influence (by interaction in conformity with other agents) occurs according to a social rule

- with probability p

idiosyncratic changes (independently of other agents)

a random opinion is adopted

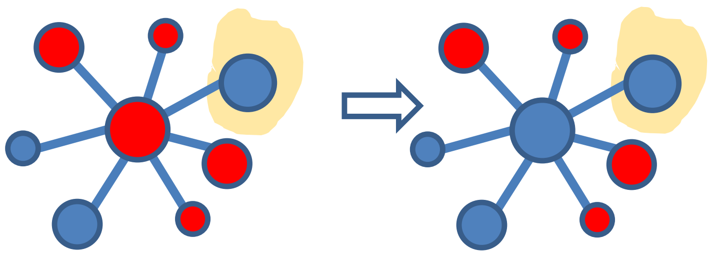
(NOISY) VOTER MODELS: social rules

VOTER:

a neighbor *j* is randomly selected

If *j* has opposite opinion, then $s_i \rightarrow -s_i$.

Liggett, Thomas M. *Annals of Probability* 25 (1997)



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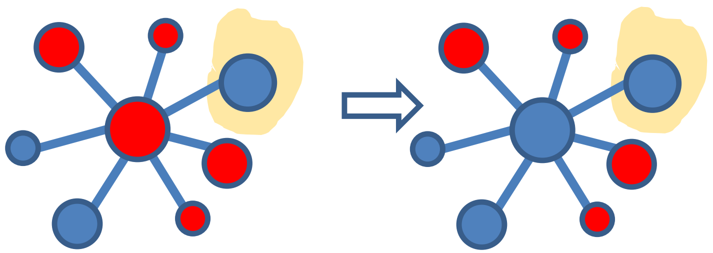
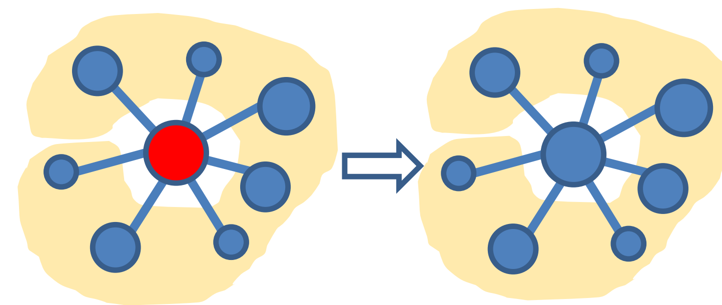


q-VOTER:

q neighbors of agent *i* are analyzed

If all *q* share the opposite opinion: $s_i \rightarrow -s_i$.

Castellano, Muñoz, Pastor-Satorras, PRE 80, 041129 (2009)



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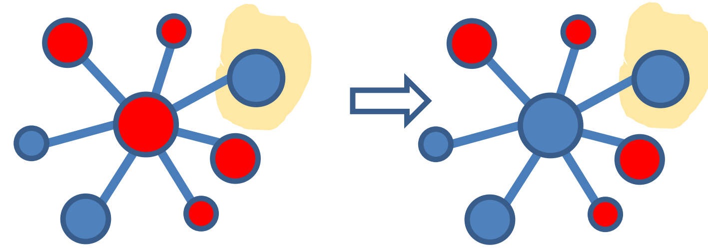

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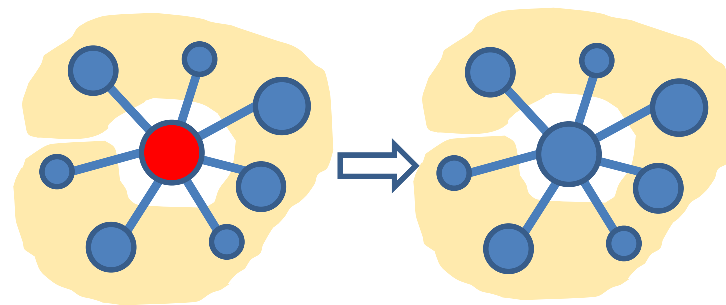

Threshold q-VOTER:

q neighbors of agent *i* are analyzed)

If at least $q_0 \leq q$ share the

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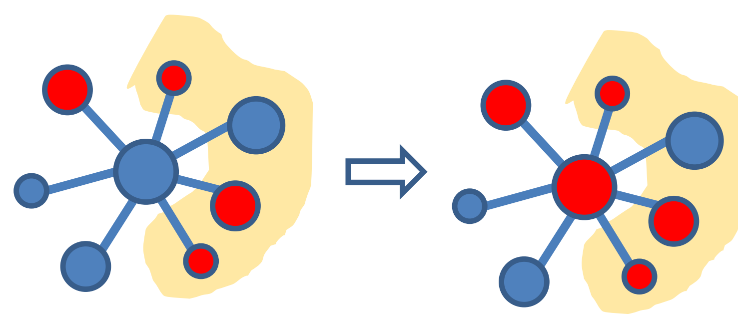
Vieira, Anteneodo, PRE 97, 052106 (2018)
Nyczka, Sznajd-Weron, *J Stat Phys* 151, 174 (2013)



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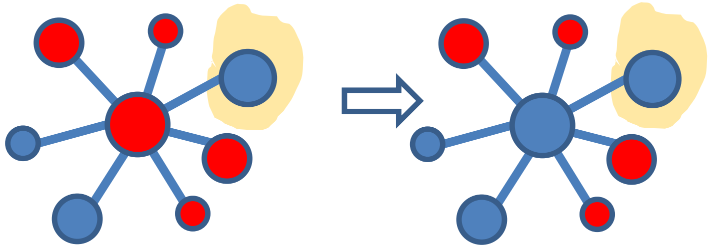

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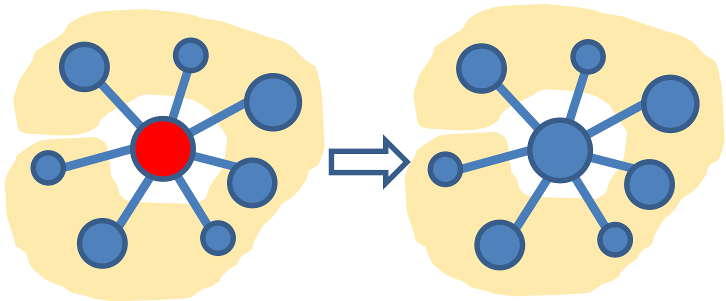

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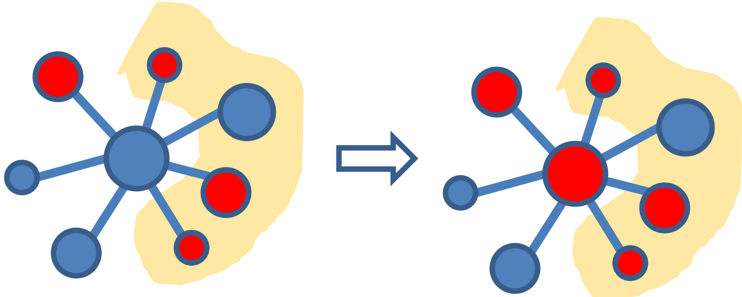
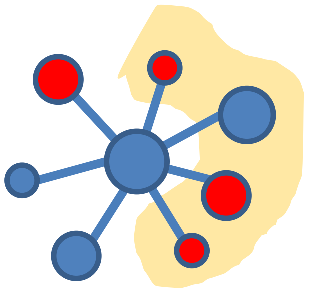


$q_0 = q = 1$: standard voter
$q_0 = q$: *q*-voter, unanimity voter

THRESHOLD q-VOTER

ALL-TO-ALL (MEAN-FIELD)



Transition rates, using $n = n+$, $x=n/N$

$$w(n \to n-1) = n\,G(1-x),$$

$$w(n \to n+1) = (N-n)\,G(x),$$

Conditional probability that a sorted agent with opinion -1 flips its opinion.

$$G(x;q,q_0,p) = (1-p)g_1(x;q,q_0) + p/2\,,$$

$$g_1(x;q,q_0) = \sum_{j=q_0}^{q} \binom{q}{j} x^j (1-x)^{q-j}$$

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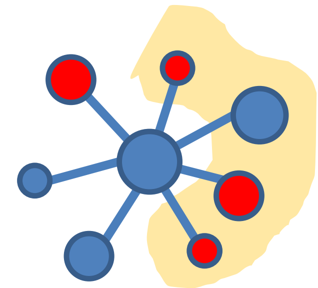

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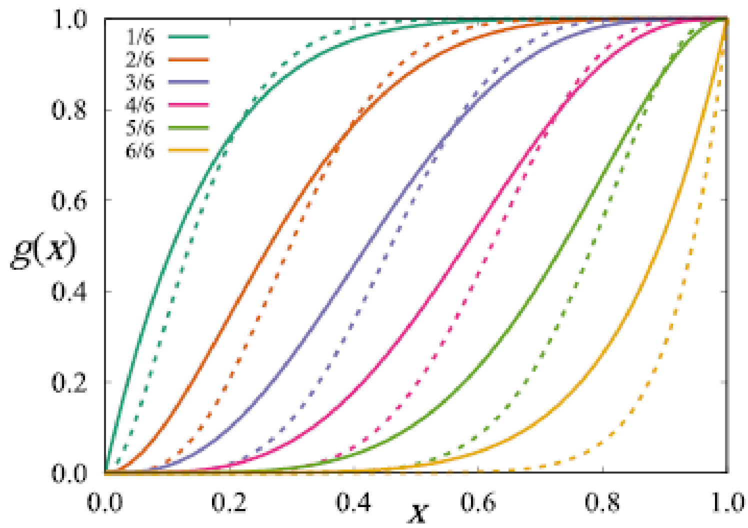
if $q_0 = q$: x^q

$$\frac{dx}{dt} = (1-x)G(x) - xG(1-x)$$

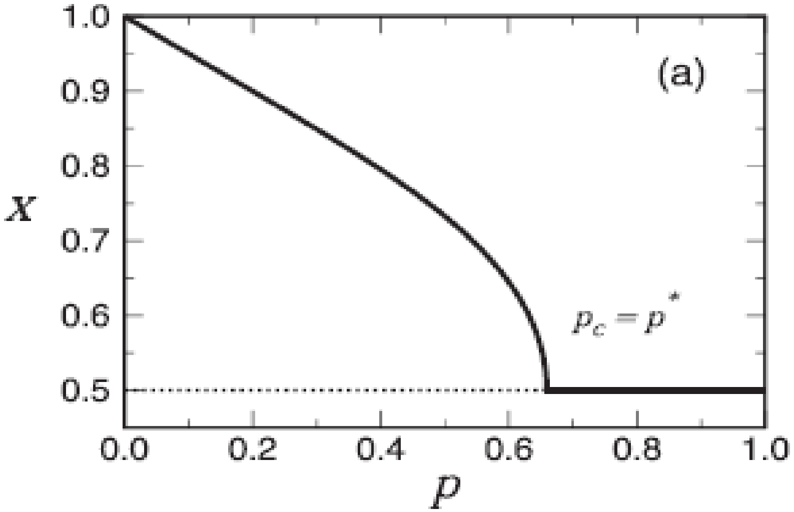
q_0/q, q = 6 (solid) q = 12 (dashed)



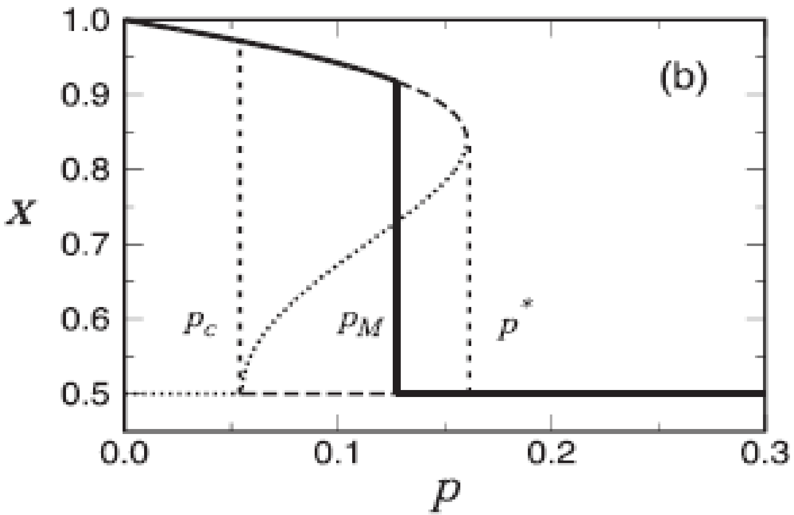
THRESHOLD q-VOTER

ALL-TO-ALL (MEAN-FIELD)

$$\frac{dx}{dt} = (1 - x)G(x) - xG(1 - x) \equiv \upsilon(x; p, q, q_0)$$



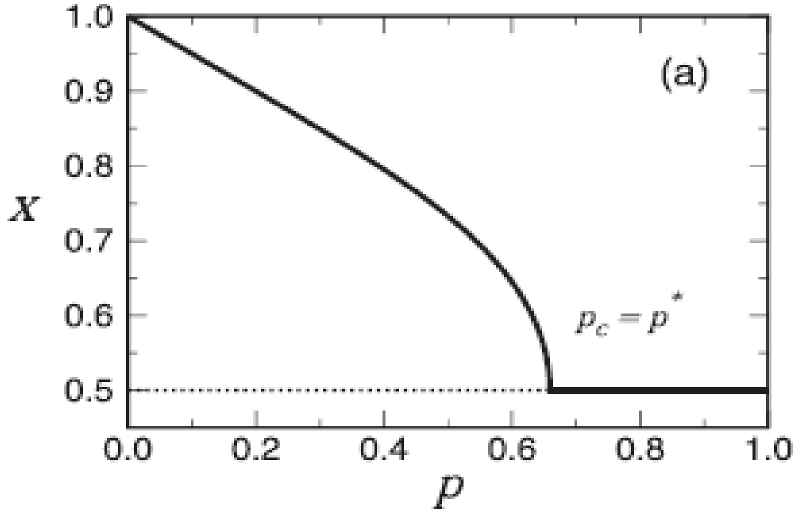
Continuous $q_0^- < q_0 < q_0^+$



First-order $q_0 < q_0^-$ and $q_0 > q_0^+$

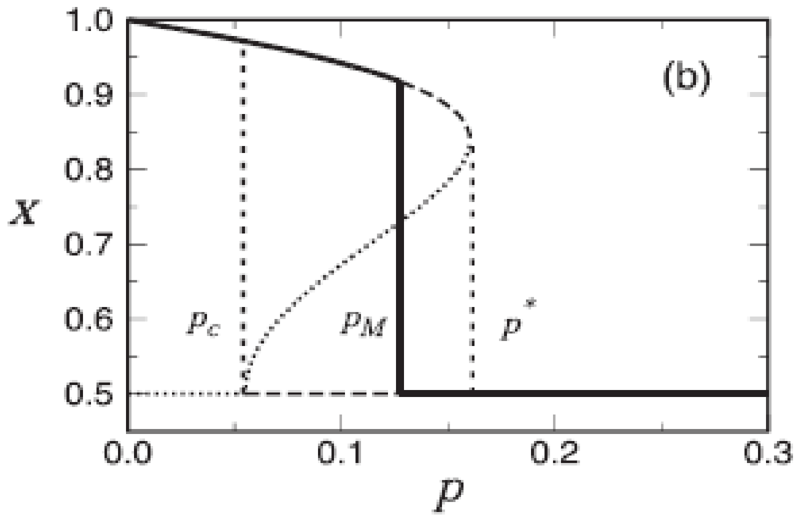
ALL-TO-ALL (MEAN-FIELD)

$$\frac{dx}{dt} = (1-x)G(x) - xG(1-x) \equiv \upsilon(x;p,q,q_0)$$



Continuous $q_0^- < q_0 < q_0^+$

$$p_c^{-1} = 1 + \frac{2^{q-1}\Gamma(q_0+1)\Gamma(q-q_0+1)}{\Gamma(q+1)[q_0 - {}_2F_1(1, q_0-q, q_0+1, -1)]}.$$

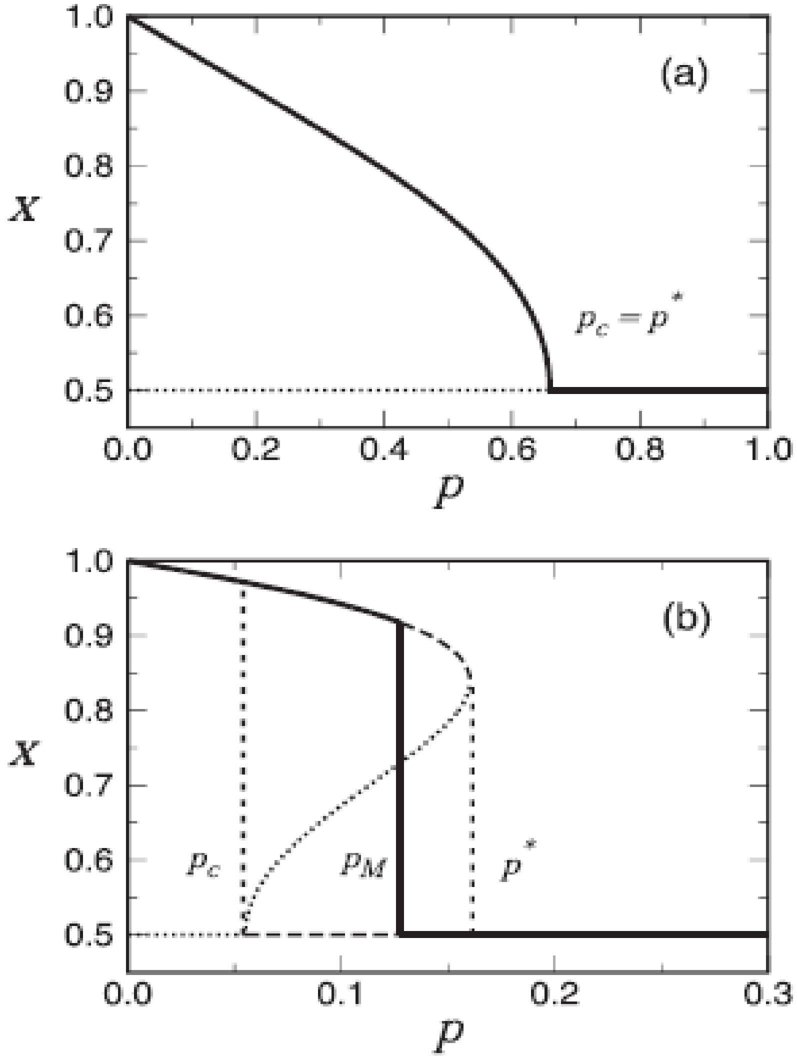


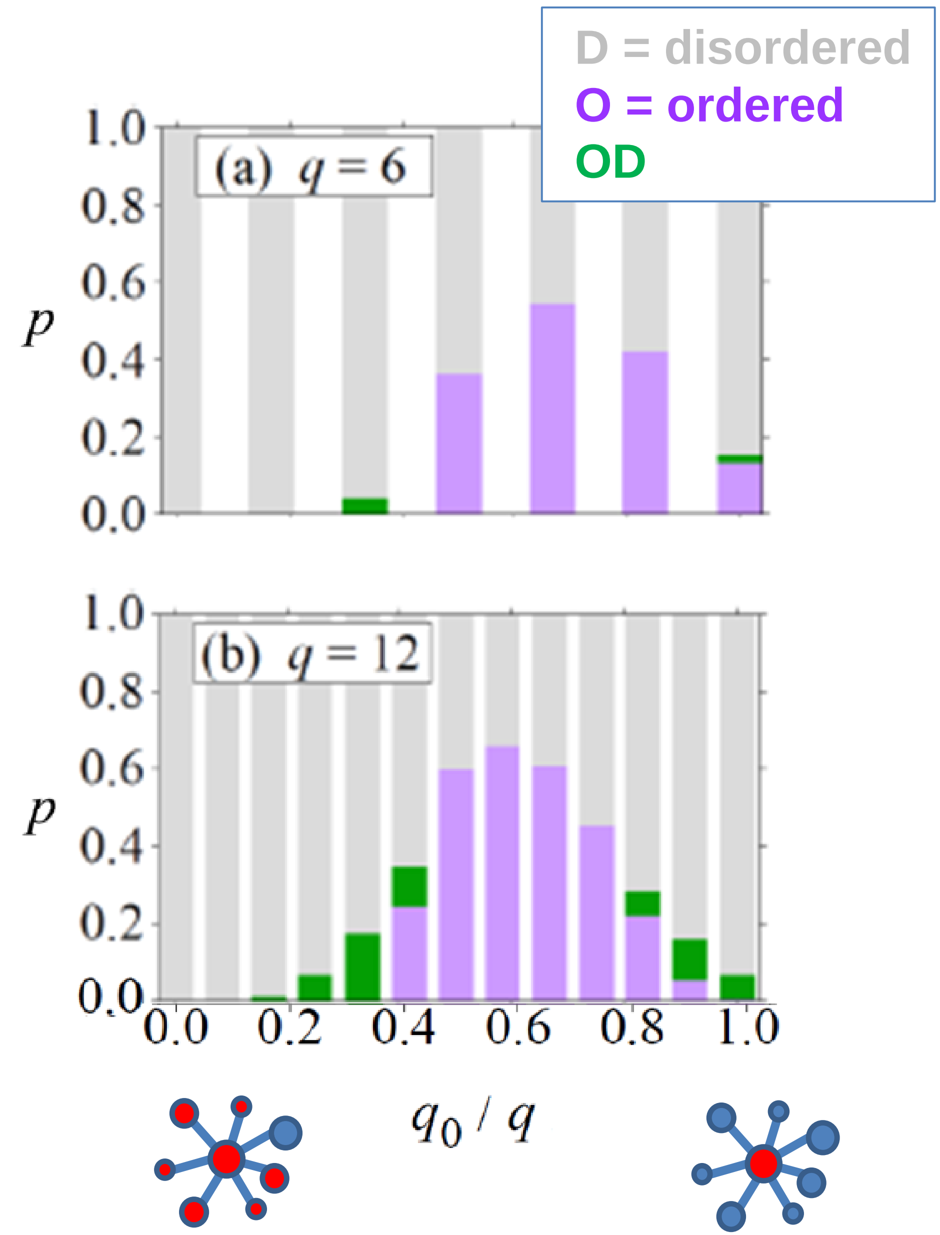
First-order $q_0 < q_0^-$ and $q_0 > q_0^+$

$$p_c = p^* \Rightarrow \quad q_0^{\pm}(q) = \tfrac{1}{4}(5 + 2q \pm \sqrt{5+4q}).$$

THRESHOLD q-VOTER

ALL-TO-ALL (MEAN-FIELD)





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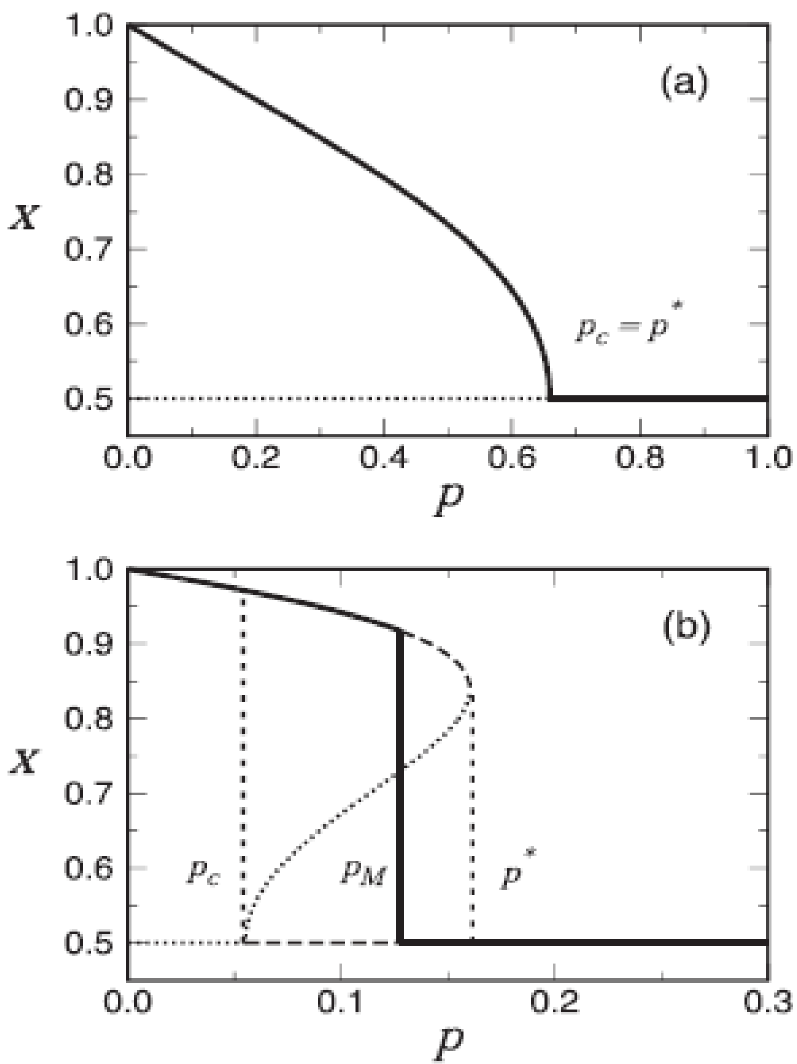
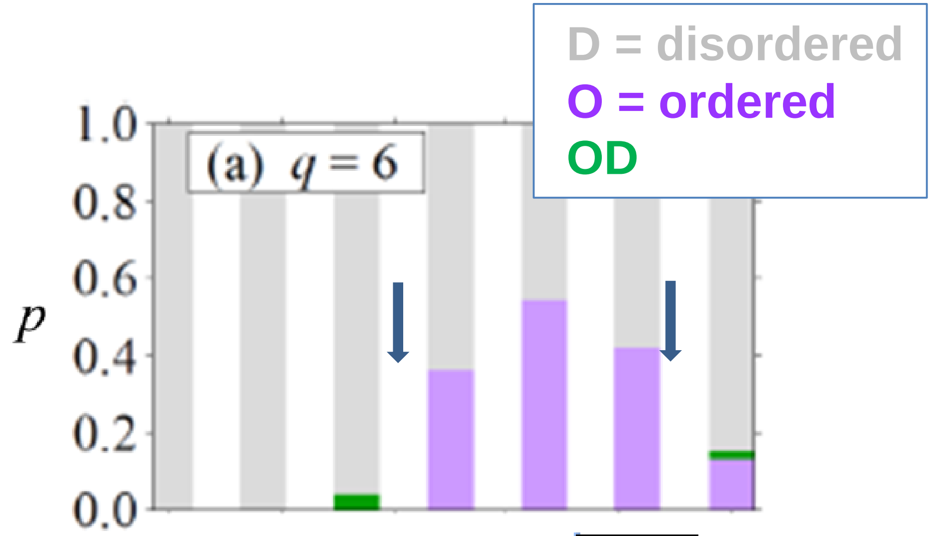
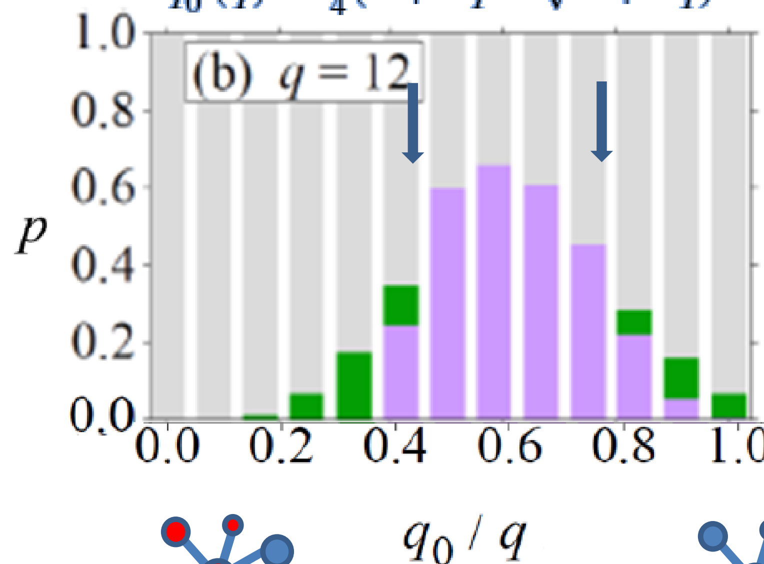
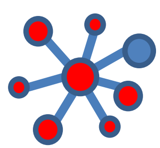
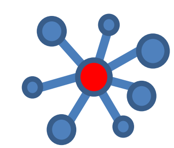




$$q_0^{\pm}(q) = \frac{1}{4}(5 + 2q \pm \sqrt{5 + 4q}).$$



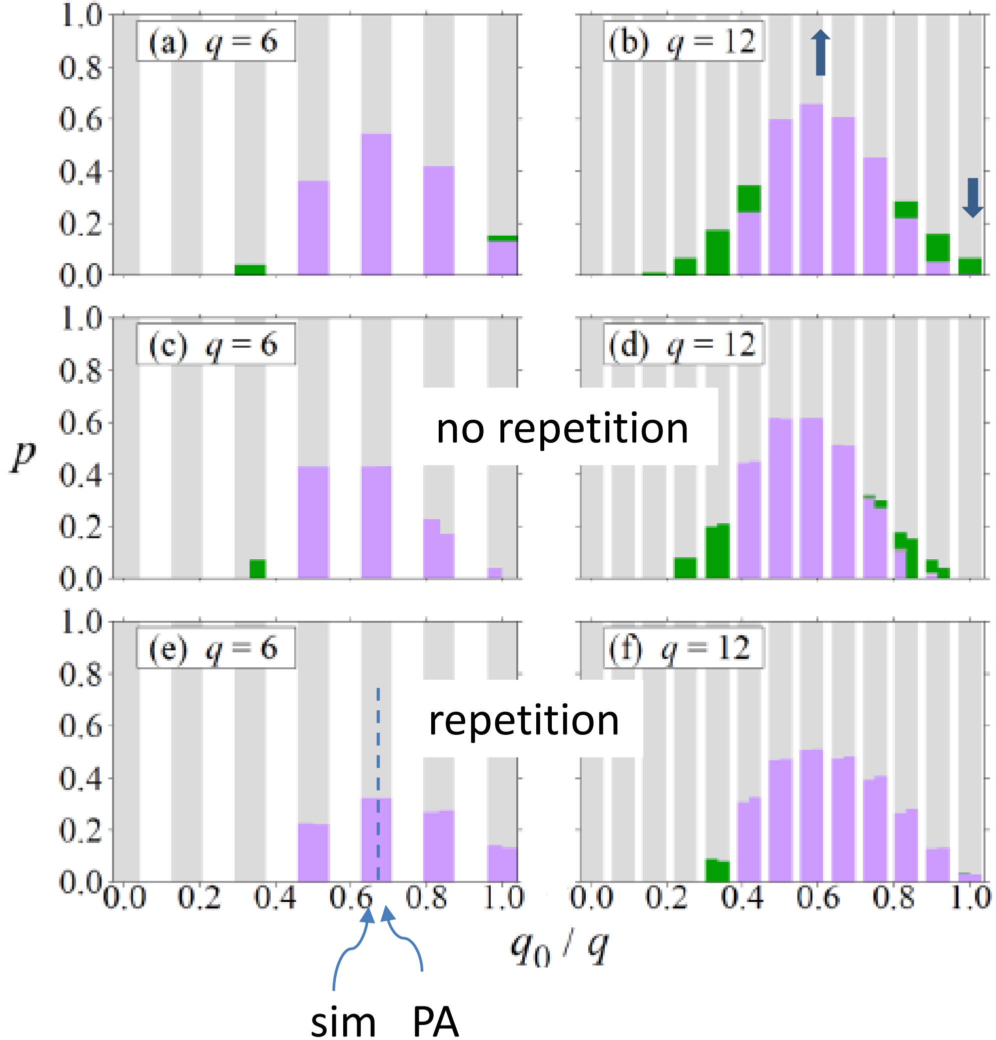




THRESHOLD q-VOTER

In networks (random regular)

simulations vs theory



MF

all to all

D = disordered

O = ordered

OD

random regular

$\mu = q$

$N = 10^4$

Pair approximation (PA)

We look at the fraction of **active links** (joining two nodes in different states)

In all-to-all connected network the fraction of active links ρ
is related to the density x of nodes in the +1 state by $\rho = 2x(1-x)$

In a general network, we must look at $\rho(t)$ and $x(t)$ independently

THRESHOLD q-VOTER

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In a general network, we must look at $\rho(t)$ and $x(t)$ independently

For a chosen node that has k links amongst which $0 \leq \ell \leq k$ are active

$$F(\ell;k,q,q_0,p) = (1-p)f(\ell;k,q,q_0) + p/2$$

$$f(\ell;k,q,q_0) \equiv \begin{cases} \sum_{j=q_0}^{q} \binom{q}{j}\binom{k-q}{\ell-j} \Big/ \binom{k}{\ell}, & \textbf{forbidden} \\ \sum_{j=q_0}^{q} \binom{q}{j}\left(\frac{\ell}{k}\right)^{j}\left(1-\frac{\ell}{k}\right)^{q-j} & \textbf{allowed} \end{cases}$$

repetition

Pair approximation (PA)

Version of the PA based on the single degree distribution *P*(*k*),
expected to work well for random networks not highly clustered or correlated
(otherwise a heterogeneous version of the PA)

$$\frac{d\rho}{dt} = \frac{2}{\mu} \sum_k \sum_{i=\oplus,\ominus} P(k) P_{i,k} \langle (k - 2\ell) F(\ell; k, q, q_0, p) \rangle_{\rho_i},$$

$$\frac{dx_k}{dt} = - \sum_{i=\oplus,\ominus} S_i P(k) P_{i,k} \langle F(\ell; k, q, q_0, p) \rangle_{\rho_i}$$

$S_\oplus = 1,\ S_\ominus = -1$ $\mu \equiv \sum_k P(k) k$ $\langle \cdots \rangle_{\rho_i}$ average over $\binom{k}{\ell} \rho_i^\ell (1 - \rho_i)^{k-\ell}$

$$P_{\oplus,k} = x_k,\ P_{\ominus,k} = 1 - x_k,$$

$$\left.\begin{aligned} \rho_\oplus &= P(\ominus|\oplus) = \rho/(2x_L) \\ \rho_\ominus &= P(\oplus|\ominus) = \rho/[2(1 - x_L)] \end{aligned}\right\} \quad x_L = \sum_k P(k) k x_k / \mu$$

$P(i'|i)$ conditional probability of selecting a neighbor with opinion *i'*

THRESHOLD q-VOTER

Pair approximation (PA)

$$\frac{d\rho}{dt} = \frac{2}{\mu} \sum_k \sum_{i=\oplus,\ominus} P(k)\, P_i \langle (k-2\ell) F(\ell; k, q, q_0, p) \rangle_{\rho_i},$$

$$\frac{dx}{dt} = -\sum_k \sum_{i=\oplus,\ominus} \frac{k}{\mu} P(k)\, S_i\, P_i \langle F(\ell; k, q, q_0, p) \rangle_{\rho_i},$$

$S_\oplus = 1,\ S_\ominus = -1$ $\mu \equiv \sum_k P(k)k$ $\langle \cdots \rangle_{\rho_i}$ average over $\binom{k}{\ell} \rho_i^\ell (1-\rho_i)^{k-\ell}$

$$P_\oplus = x, \quad P_\ominus = 1 - x,$$

$$\rho_\oplus = \rho/(2x), \quad \text{and} \quad \rho_\ominus = \rho/[2(1-x)]$$

THRESHOLD q-VOTER

No repetition

$$\frac{d\rho}{dt} = p(1-2\rho) + \frac{2(1-p)}{\mu}[(1-x)G_2(\rho_{\ominus}; q, q_0, \mu) + xG_2(\rho_{\oplus}; q, q_0, \mu)]$$

$$\frac{dx}{dt} = (1-x)G(\rho_{\ominus}; q, q_0, p) - xG(\rho_{\oplus}; q, q_0, p)$$

$$\Rightarrow \quad p_c^{-1} = 1 + \frac{2^{q-1}\left(\frac{\mu}{\mu-2}\right)^{q_0}\left(\frac{\mu-1}{\mu}\right)^{q}\Gamma(q_0+1)\Gamma(q-q_0+1)}{\Gamma(q+1)[q_0 - {}_2F_1(1, q_0-q, q_0+1, 2/\mu-1)]},$$

$$q_0^{\pm}(q,\mu) \approx \frac{1}{2}\left\{6 + (2q-7)\rho_c \pm \sqrt{16 + \rho_c[-44 + 4q(1-\rho_c) + 29\rho_c]}\right\}$$

$\mu/q \to \infty,\ p_c$ → mean-field values

THRESHOLD q-VOTER

With repetitions

For this modality, we did not manage to find general closed expressions for the averages $\langle \cdots \rangle_{\rho_i}$ over the binomial.

Besides the mean degree μ, also $\langle k^{-m} \rangle$, with $1 \leqslant m \leqslant q-1$, contribute.

For more details

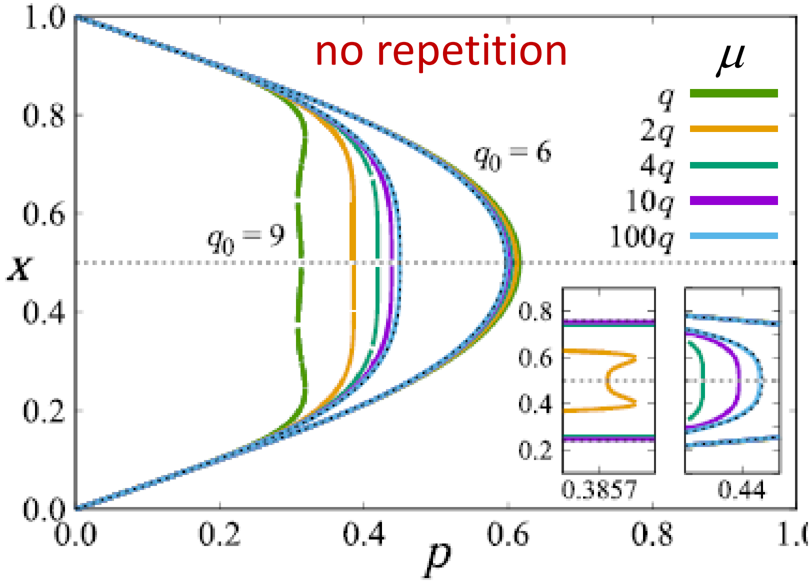
Vieira, Peralta, Toral, Anteneodo, *Phys Rev E* 101, 052131 (2020)

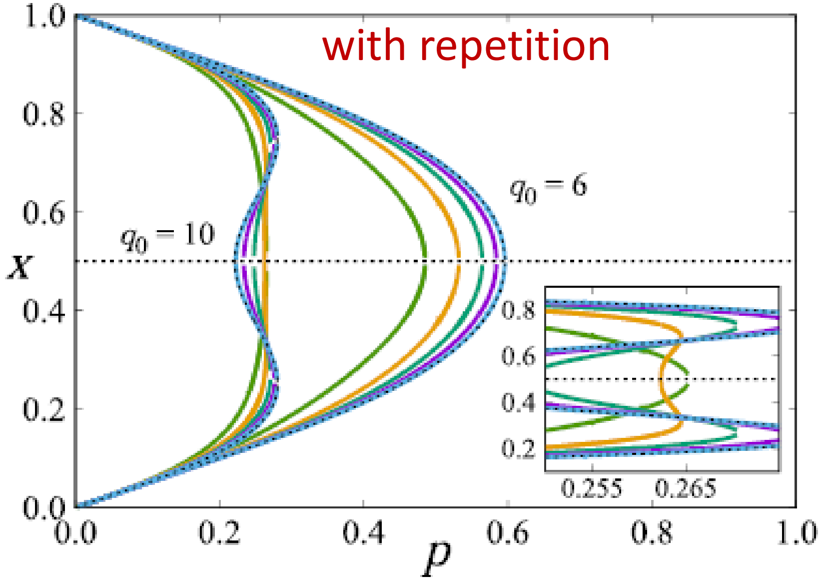
Recent work in q-voter:
Weron, Nyczka, Szwabinski, *Entropy* 26, 132 (2024)

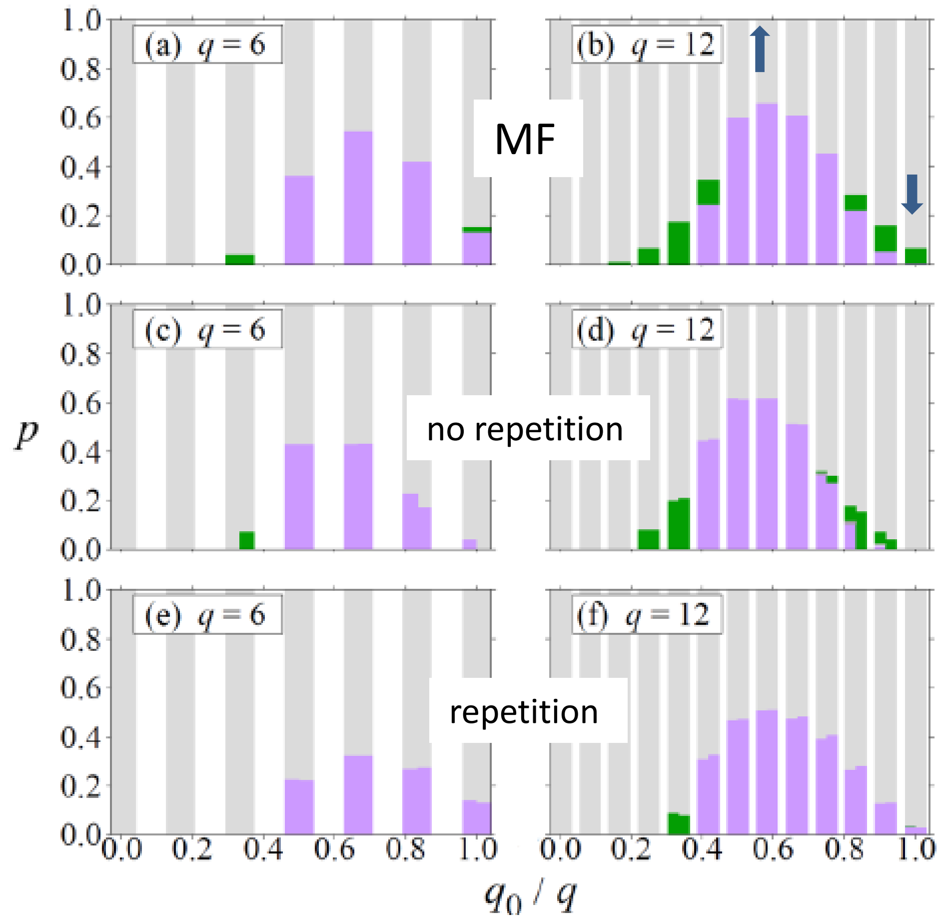
- PA the most accurate vs (network-aware and heuristic)
- hMFA: without repetition

Comparisons

random regular $P(k) = \delta(k - \mu)$ $\mu = q = 12$





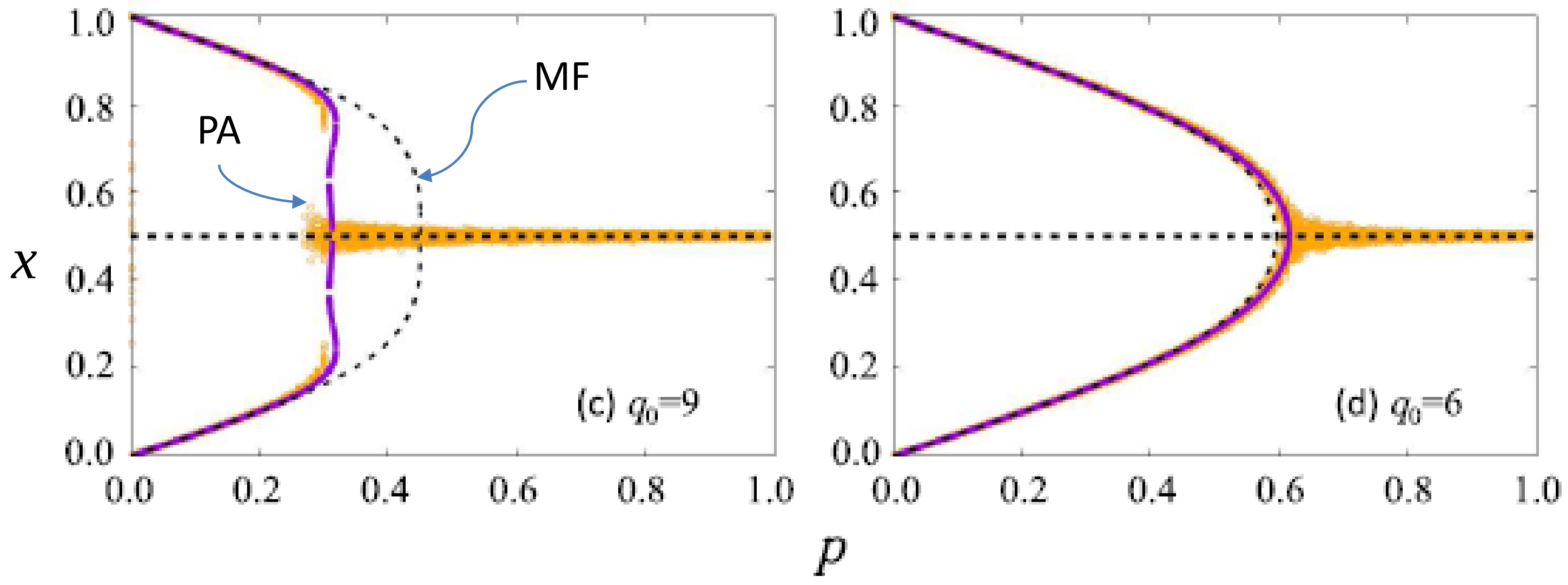


NETWORKS (simulations vs theory)

RR, no-repetitions

$P(k) = \delta(k - \mu)$

$\mu = q = 12$



In the numerical simulations, we did not detect any significant discrepancy between random regular, Erdös-Rényi, and power-law networks, with the same μ, in agreement with PA predictions.

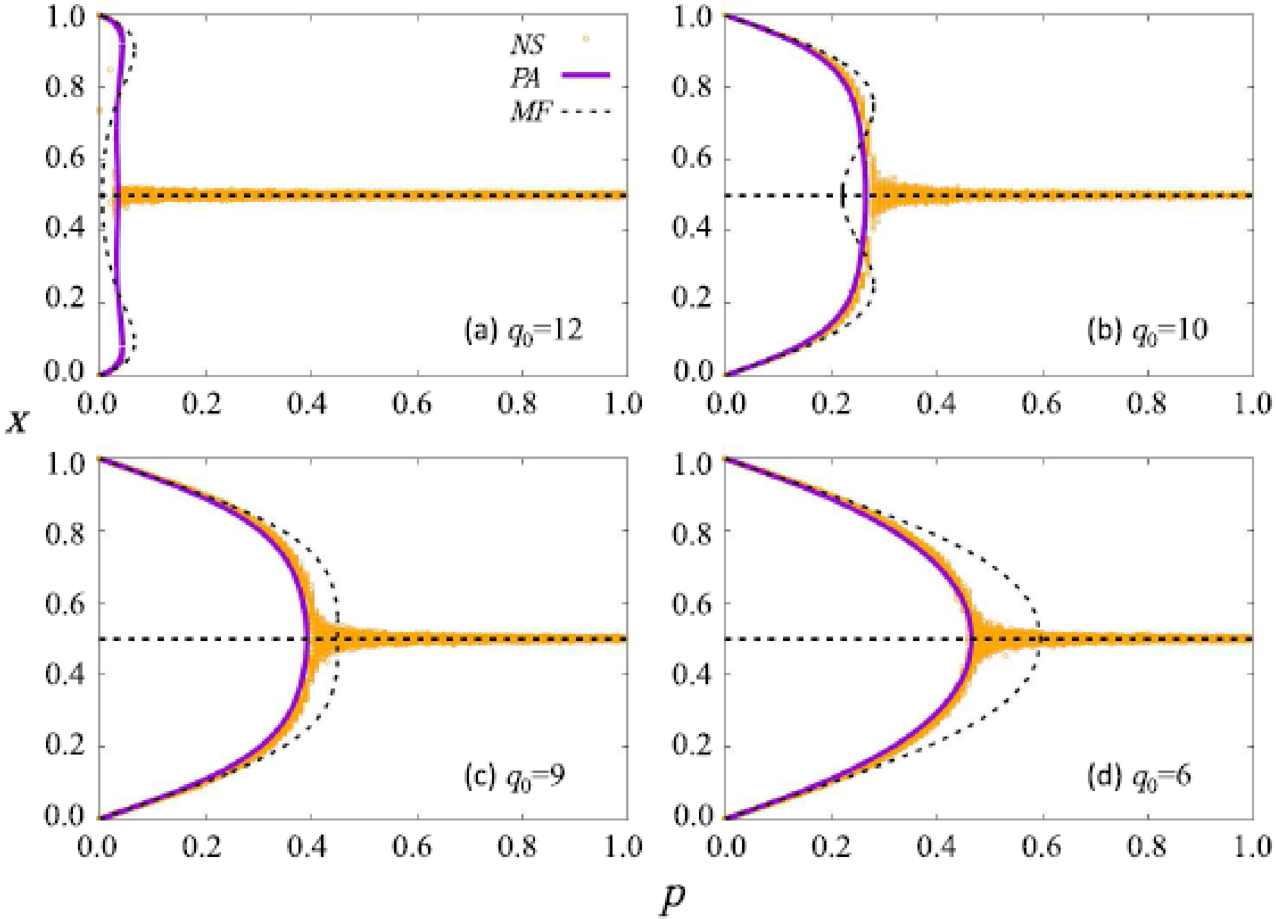
NETWORKS (simulations vs theory)

RR, repetitions

$P(k) = \delta(k - \mu)$

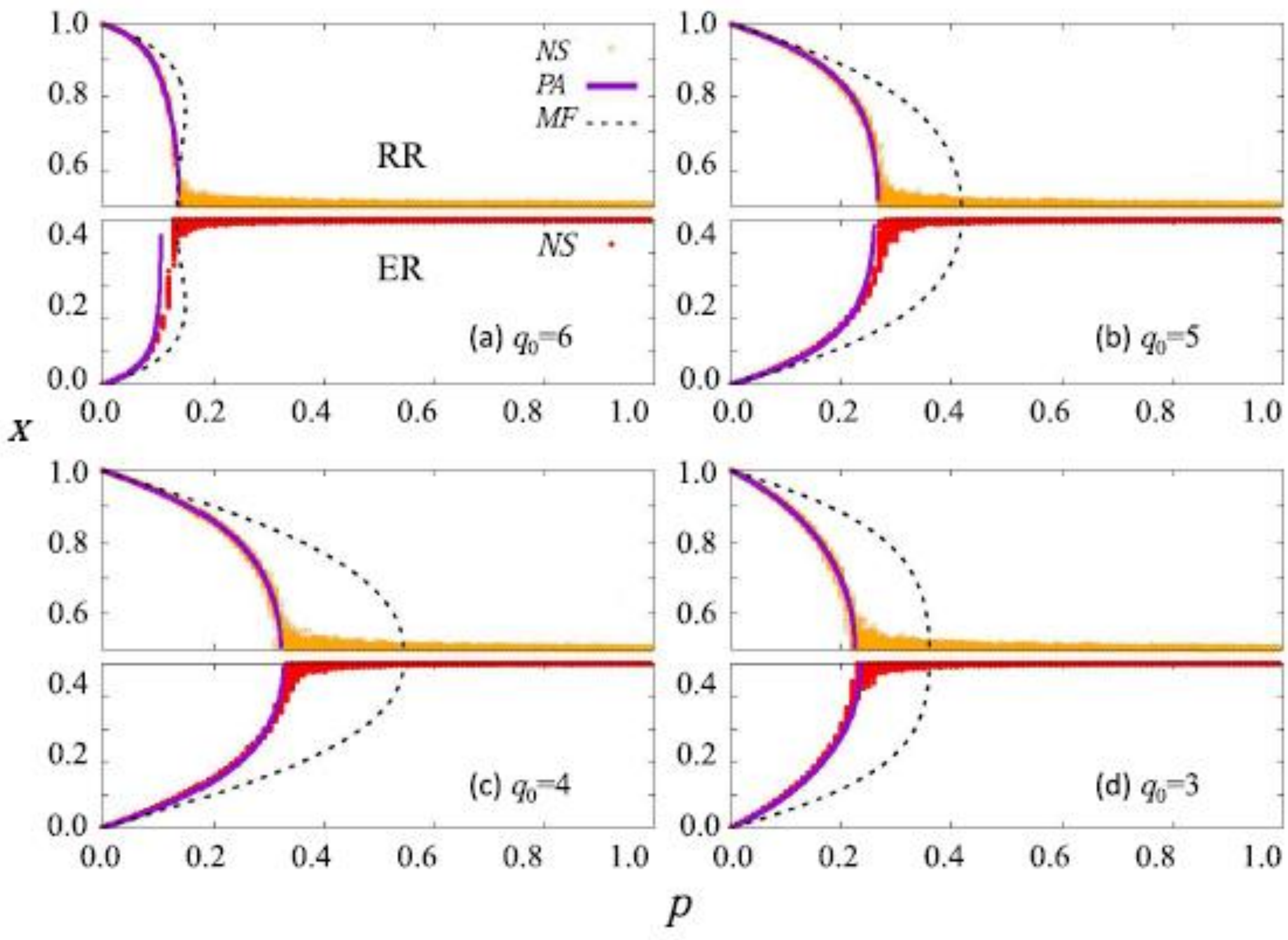
$\mu = q = 12$

$C \sim (\mu - 1)^2/(N\mu)$



NETWORKS (simulations vs theory)

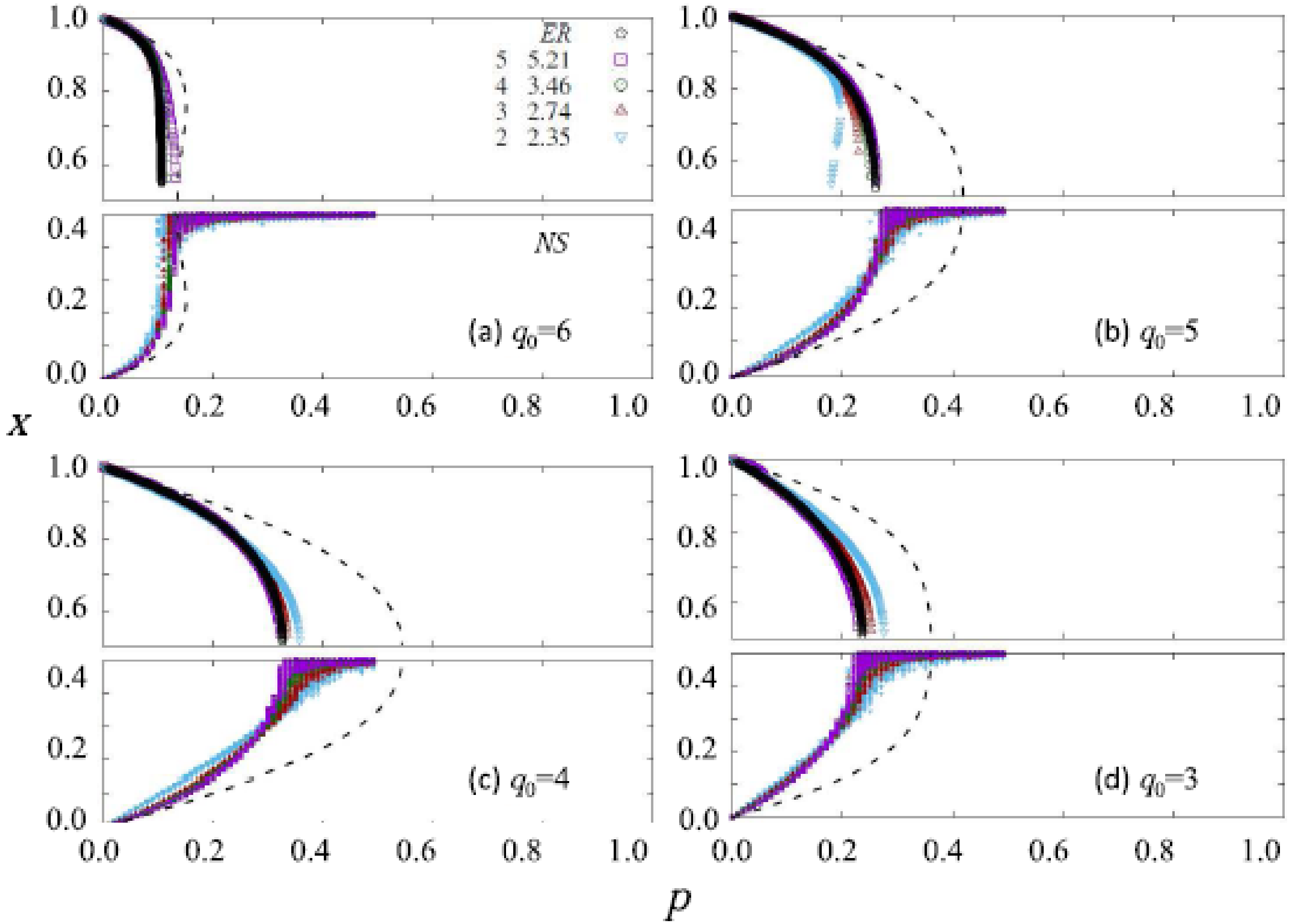
RR and ER, repetitions $\mu = q = 6$



NETWORKS (simulations vs theory)

ER and power-law, repetitions $\mu = q = 6$

$$P(k) = \mathcal{N}/k^a, \text{ for } k_{\min} \leqslant k \leqslant k_{\max} = N/5$$



CONCLUSIONS

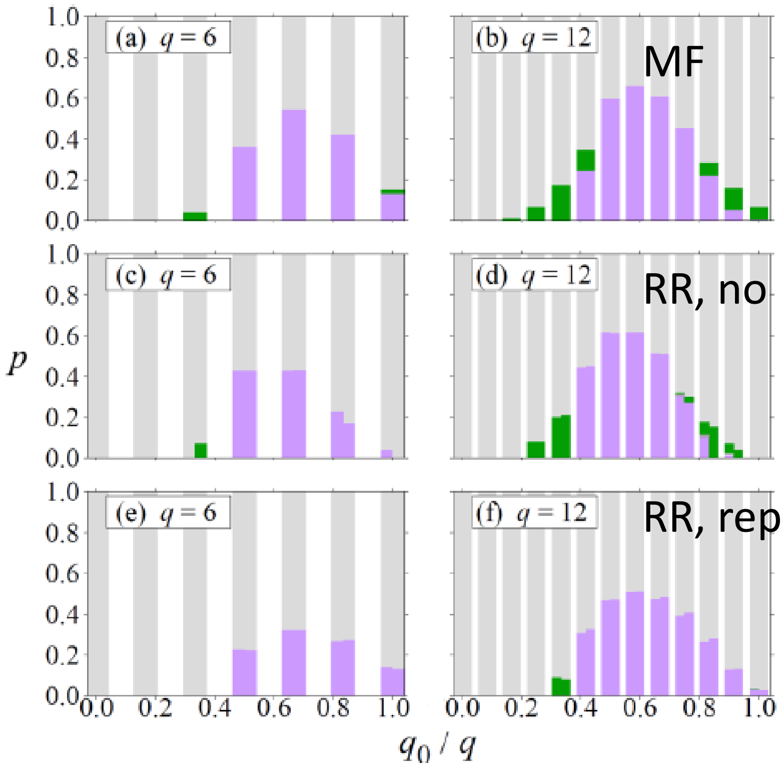
For noisy threshold *q*-voter model, possibility of repetitions or not in the selection of *q* amongst *k* neighbors.

The threshold q_0 influences the nature of transitions. Optimal value for consensus.

The structure has a stronger influence in the case with repetition, where the discontinuous transitions are less common than in fully connected networks.

Differently to the case without repetitions, with repetitions results depend not only on *m* (→ stronger effects for long-tails)

Analytical results were derived using the pair approximation, for random networks with arbitrary degree distribution *P*(*k*).



D = disordered

O = ordered

OD

CONCLUSIONS

With regard to the performance of the **pair approximation**:

The critical points estimated through the PA are in good agreement with simulations in **random regular** networks.

This is especially true when μ increases approaching the exact MF result where repetition and other issues related to structure become irrelevant.

Without repetitions, spurious results, such as multistability beyond three states, are observed in cases with $q0 \approx q \approx \mu$.

For **ER and power-law** networks, good predictions, specially *without repetitions*, but limited structures can be visited.

With repetitions, the PA predicts dependency on network structure beyond the average degree μ, in accord with simulations. These effects are weak in ER networks, but stronger in networks with power-law degree distribution where long tails are concomitant with high probability of poorly connected nodes to realize small values of μ, far away from the MF. Also in this case $q0/q$ plays a crucial role.

Deviations are expected in networks with higher correlations (large clustering coefficient.)

PERSPECTIVES

- Connection with experiments (compare, suggest)
- Times to consensus
- Theory for no-repetition (heuristic-MFA)
- Aging effects

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