Exercises

1. We will start by looking into the seesaw mechanism. In a single-generation scenario, the neutrino mass matrix would look like

$$\mathcal{L} \supset -\frac{1}{2}\begin{pmatrix}\nu & \nu^c\end{pmatrix}\begin{pmatrix}0 & m_D \\ m_D & \mu\end{pmatrix}\begin{pmatrix}\nu \\ \nu^c\end{pmatrix} + \text{h.c.} \tag{0.1}$$

   (a) Find the eigenvalues of this matrix. Expand them in $m_D/\mu \ll 1$.

   (b) What is the rotation angle that diagonalizes this symmetric matrix? Call the physical basis ν_ℓ and ν_h, where ν_h is the heavy neutrino, sometimes called a heavy neutral lepton (HNL). Which will interact more strongly with the rest of the SM particles?

   (c) Identify ν_ℓ state with the neutrinos we observe in the laboratory and take $m_{\nu_\ell} \sim 0.05$ eV. Recalling that m_D comes from the Higgs mechanism, $m_D = yv/\sqrt{2}$ with $v = 246$ GeV and y a coupling constant, we will take m_D to be in the rage $[m_e, m_t] = [511\text{ keV}, 176\text{ GeV}]$, corresponding to $y \sim [3\times 10^{-6}, 1]$. What are the values of μ that you need in order to obtain the assumed value for m_{ν_ℓ}? What values of the mixing angle do these correspond to?

2. This exercise will explore low-scale variants of the seesaw mechanism. The idea will be to turn the seesaw on its head and explore symmetries to make neutrino masses small.

   Low-scale seesaws (sometimes referred to as double, extended, linear, or inverse seesaws) introduce a pair of neutral lepton per neutrino family (call them ν^c and n, for example). In a single-generation scenario, the neutrino mass matrix becomes

$$\mathcal{L} \supset -\frac{1}{2}\begin{pmatrix}\nu & \nu^c & n\end{pmatrix}\begin{pmatrix}0 & m_D & \varepsilon \\ m_D & \mu' & M \\ \varepsilon & M & \mu\end{pmatrix}\begin{pmatrix}\nu \\ \nu^c \\ n\end{pmatrix} + \text{h.c.} \tag{0.2}$$

   (a) The quantum numbers of the neutral leptons are $L(\nu) = 1$, $L(\nu^c) = -1$, and $L(n) = 1$. Out of all the parameters in the matrix above, which ones violate lepton number?

   (b) Calculate the determinant of this matrix. What happens when all the lepton-number-violating parameters go to zero? What does this imply for light neutrino masses?

   (c) Keeping all these lepton-number-violating parameters zero and assuming the Dirac mass m_D is much smaller than all other parameters, diagonalize the matrix above and analyze what happens with the pair ν^c and n. What kind of particle are they?

   (d) Turn on each LNV parameter at a time and diagonalize the matrix assuming $m_D \ll$ LNV parameters $\ll$ anything else. Mathematica, Wolfram Alpha, or

Python are your friends here. You can assume all parameters are real and positive. What happens to the heavy neutral lepton masses? What kind of particle are they?

(e) Estimate the mixing between the weak-interaction eigenstate ν and the heavy neutral leptons. What is the relationship between light neutrino masses and the mixing?

3. This exercise will show you how to estimate the signal event rate for heavy neutrinos produced by pion decay-at-rest sources and decaying inside a large-volume experiment. Consider a spallation source with $\mathcal{O}(\text{GeV})$ protons hitting a dense target. The number of pions produced at this source is related to the number of protons on target by $N_\pi \sim 10\% \times N_{\text{POT}}$.

   (a) In terms of the mixing angle of the heavy neutrino with the muon neutrinos, $|U_{\mu 4}|^2$ [1], what is the flux of HNLs at a distance L from the target?

   (b) Knowing that the muon lifetime is determined by the decay rate $\Gamma = G_F^2 m_\mu^5/192\pi^3$, estimate the analogous decay rate for $\nu_h \to \nu e^+ e^+$.

   (c) What is the probability for ν_h to survive the distance L without decaying? What is then the total probability for it to decay within a region of volume $V = \lambda^3$ located at a distance L from the source? You can assume $\lambda \ll L$.

   (d) Putting all of this together, write down a formula for the number of HNL decays within this detector of size V. You can assume the HNL decays only via $N \to \nu e^+ e^-$ with a rate Γ_N. Expand this formula assuming $\Gamma_N L$ and $\Gamma_N \lambda$ are very small. How does the signal event rate depend on the decay rate Γ_N? What if we add another decay channel for the HNL, like $N \to \nu\nu\bar{\nu}$?

   (e) In your preferred coding language, calculate this signal rate for fixed values of $|U_{\mu 4}|^2$ and m_4. Do this for a range of $|U_{\mu 4}|^2 \in [10^{-8}, 0.1]$ and $2 \text{ MeV} < m_4 < 30 \text{ MeV}$. Try drawing contours of constant event rate in this parameter space.

   (f) Suppose the HNL decays via a stronger $G_X \gg G_F$ four-fermion interaction with SM particles. What happens to the contours in that case? Try, for instance, $G_X \sim 10^3 G_F$.

[1] Note this is analogous to θ^2 in exercise 1, except that it is specific to the muon neutrino flavor