Exercise 1 - Characteristic dark matter (DM) speeds

Estimate the characteristic speed of dark matter particles bound to the dark halos of several structures in the Universe using the Virial Theorem

$$m v^2 \simeq \frac{G M_{halo} m}{R_{halo}}$$

m: DM particle mass ; v: characteristic speed ; M_{halo}, R_{halo} = mass and radius of the structure (including DM)

Estimate the speed v for 1) a dwarf galaxy (typical masses and radius of DG are $M_{DG} \simeq 2 \times 10^6 M_\odot$, $R_{DG} \simeq 1$ kp) 2) the Milky Way (or other galaxies, $M_{MW} \simeq 10^{12} M_\odot$, $R_{MW} \simeq 100$ kp) and 3) galaxy cluster (with, say, $M_c \simeq 3 \times 10^{14} M_\odot$, $R_c \simeq 2$ Mpc). Those just quoted are approximate values for the dark halos of the mentioned structures.

Help : you will find very useful to write first Newton's constant G in units of $\frac{\text{Mpc}\ \text{km}^2}{M_\odot\ \text{s}^2}$. Here

$M_\odot$ is a solar mass, $M_\odot = 2.99 \times 10^{30}$ kg $\simeq 1.7 \times 10^{57}$ GeV, and pc is a parsec, 1 pc = $3.086\ 10^{16}$ m = 3.26 ly.

Notice that in all equations in my lectures I use "Natural Units" $c = 1$, $\hbar = 1$ in which

$$1\,\text{GeV} = 1.8 \times 10^{-27}\,\text{kg} = \frac{1}{0.197\ 10^{-15}\,\text{m}} = \frac{1}{6.58\ 10^{-25}\,\text{s}}$$

Exercise 2 - Self Interacting Dark Matter (SIDM)

SIDM must have a cross section very close to its upper limit in a particular type of structure to be substantially different from the usual collisionless CDM. The ratio $\left(\frac{\sigma_{max}}{m} \Big/ 6 \times 10^{-25} \frac{cm^2}{GeV} \right)$ is between a few-to 100 to have an effective core creation in dwarf galaxies, but is $\simeq 1$ in clusters (here σ_{max} is the maximum value of the self-interaction cross section). A constant cross section $\sigma_{self} = \sigma_{max}$ which would be effective at dwarf galaxy scales would then be forbidden by the limits coming from galaxy clusters. But this may not be a problem if the cross section depends on the DM speed.

Using the results you obtained in Excercise 1 say

2.1 Which dependence of the self scattering cross-section on the DM particles speed could give σ_{self} in dwarf galaxies $\simeq 100 \times \sigma_{self}$ in galaxy clusters?

2.2 Probably you know that in Rutherford scattering the cross section is inversely proportional to the square of the kinetic energy, i.e. $\sigma \sim 1/v^4$. This is characteristic of scattering mediated by a very light gauge boson. In SIDM models one has a "dark photon" (or "hidden-photon") instead of the usual photon but still if the mediator mass $m_\phi \ll m_{DM} v \Rightarrow \sigma_{self} \sim v^{-4}$. Prove that if $\sigma_{self} \simeq \sigma_{max}$ in dwarfs, then σ_{self} would be $\ll \sigma_{max}$ in clusters (so SIDM becomes just CDM at large scales).

Exercise 3 - Lower limit on the dark matter (DM) particle mass due to phase space arguments

3.1 Assume the DM particle is a boson. Bosons tend to occupy the same lowest energy state. The occupation number is so high, that the DM behaves as a classical field obeying a wave equation (see Hu, Barkana and Gruzinov, astro-ph/0003365 P. R. L. 85 (2000) 1158). The problem at hand becomes formally the same as that of a particle of mass equal to the DM mass, m, in a potential well of the size of the dark halo. So we can use Pauli's uncertainty principle $\Delta x \, \Delta p \gtrsim 1$ with $\Delta x \simeq 2 R_{halo}$ and $\Delta p \simeq m v$. Here v is the characteristic speed you estimated in Excercise 1. The most stringent lower limit on m from this relation comes from dwarf galaxies (do you see why?). Using the data and results of Excercise 1 prove that the limit obtained from dwarf galaxies is $m \gtrsim 10^{-22}$ eV.

[A DM particle with $m \simeq 10^{-22}$ eV was called "fuzzy DM" by Hu, Barkana and Gruzinov in the paper mentioned above. It is DM in a cold Bose-Einstein condensate, similar to axion DM.]

Exercise 3 (continuation)

3.2 Now let us assume that the DM particle is a fermion. The argument is different than for a boson due to Pauli's Exclusion Principle. For a fermion the phase space density is always $f(x,p) < 1$, so

$$M_{halo} = m \int f(x,p)\, d^3x\, d^3p \lesssim m\left(\frac{4}{3}\pi R_{halo}^3\right) \int d^3p$$

and for our estimate we can use $\int d^3p \simeq \Delta p^3$ and $\Delta p \simeq m v$ (you may see the original paper of Tremaine and Gunn PRL 42 (1979) 407 to get a more complete explanation. You may see also later papers by Madsen PRD 44 (1991) 999 and Horiuchi et al. PRD 89 (2014) 025017, 1311.0283).

Again dwarf galaxies provide the best lower limit. Prove that this limit is $m >$ few eV.

(this is the so called "Tremaine and Gunn limit").

Exercise 4 - Flux of dark matter (DM) particles on Earth

4.1 Define flux, i.e. number of particles traversing a surface per unit area per unit time, in terms of n = number density = number of particles per unit volume, and v = average speed particles.

In the following you may use for v either the characteristic speed for DM particles in the dark halo of the Milky Way (that you found in Exercise 1) or the speed of the Sun around the galaxy $v_\odot \simeq 220\,\mathrm{km/s}$ (or you may just use $v_\odot \simeq 10^{-3}c$ or just $v_\odot \simeq 10^{-3}$ in natural units). Also, we will use $\rho = 0.3\,\frac{\mathrm{GeV}}{\mathrm{cm}^3}$ as the local energy density of the DM ("local" meaning at the position of the solar system in our galaxy).

4.2 Find the characteristic flux of dark matter particles in units of number of particles per $(\mathrm{cm}^2\mathrm{s})$ as function of the DM particles mass m.

4.3 How many DM particles (again, given as function of m) are on average in a 1 liter soda bottle?

Exercise 5 WIMPs interact coherently with nuclei

Consider the elastic collision of dark matter (DM) particles of mass m with a target of mass M_T.

5.1 Show that for $m \gtrsim$ GeV the typical momentum tranfer q (momentum imparted to the target, initially at rest, so that the target recoil energy is $E_R = q^2/2M_T$) is such that the interaction is coherent when the target is a nucleus.

Help. The radius of a nucleus is $R_N \simeq 1.25\ \mathrm{fm}\ A^{1/3}$ where A is the mass number A, $M_T \simeq A$ GeV. You may use the limits $m \ll M_T$ and $m \gg M_T$ to establish typical values of q.

5.2 Considering that present direct DM detection experiments have at present energy thresholds not lower than a fraction of keV, show that the maximum energy deposited in an elastic collision by a Light Dark Matter (LDM) particle, defined as DM particles with $1\ \mathrm{keV} \lesssim m \lesssim 100$ MeV is below threshold for detection.

Help: the lightest nuclear mass in use in Direct DM detection is about $M_T \simeq 10$ GeV.

Exercise 6 Relic Abundance

6 a) Assume no asymmetry between particles χ and antiparticles $\bar{\chi}$. Starting from the Boltzmann transport equation in an expanding Universe of the form

$$\frac{dn_\chi}{dt} + 3 H n_\chi = - \langle \sigma_{\chi\bar{\chi} \to l\bar{l}} |\vec{v}| \rangle \left[n_\chi^2 - (n_\chi^{EQ})^2 \right]$$

(l stands for "light particles", and n_χ^{EQ} is the equilibrium number density, $n_\chi^{EQ} = n_{\bar{\chi}}^{EQ}$ and, in general $n_\chi = n_{\bar{\chi}}$)

for the evolution of the particle χ number density derive the equation

$$\frac{x}{Y_\chi^{EQ}} \frac{dY_\chi}{dx} = - \frac{\Gamma_A}{H} \left[\frac{Y_\chi}{Y_\chi^{EQ}} - 1 \right].$$

where $Y_\chi = \frac{n_\chi}{s}$, s is the entropy density ($s = \frac{2\pi^2}{45} g_{*s} T^3$)

$x = m_\chi / T$ and the annihilation rate is $\Gamma_A = \langle \sigma_{\chi\bar{\chi} \to l\bar{l}} |\vec{v}| \rangle n_\chi$

[Hint : use $\dot{n}_\chi + 3 H n_\chi = s \dot{Y}_\chi$ due to the conservation of entropy, $sa^3 = \text{const}$. Here H is the Hubble parameter, a is the scale factor of the Universe]

The last equation shows that when $\frac{\Gamma_A}{H} \ll 1$, Y_χ remains constant, and this means that if s is constant the n_χ is constant too.

[You may consult any astro-particles book e.g. "The Early Universe" by Kolb and Turner to get help in solving this problem]

6 b) Assume now a constant asymmetry $A = Y_\chi - Y_{\bar\chi}$ (where $Y_\chi = n_\chi/s$ and $Y_{\bar\chi} = n_{\bar\chi}/s$), and we assume that n_χ and $n_{\bar\chi}$ can only change due to $\chi\bar\chi$ annihilation

Now we have,

$$\frac{dn_\chi}{dt} + 3Hn_\chi = \frac{dn_{\bar\chi}}{dt} + 3Hn_{\bar\chi} = -\langle\sigma_{\chi\bar\chi}|\vec{v}|\rangle\left(n_\chi n_{\bar\chi} - n_\chi^{EQ} n_{\bar\chi}^{EQ}\right)$$

$\bar\chi$ is the minority component. Until $\bar\chi$ "freeze-out", at $\bar{x}_{fo}$ the density of both components are those of equilibrium

$$\left.\begin{aligned} n_\chi^{EQ} &= g_\chi\left(\frac{m_\chi T}{2\pi}\right)^{3/2} e^{(-m_\chi+\mu_\chi)/T} \\ n_{\bar\chi}^{EQ} &= g_\chi\left(\frac{m_\chi T}{2\pi}\right)^{3/2} e^{(-m_\chi-\mu_\chi)/T} \end{aligned}\right\} \text{(for } \chi \text{ and } \bar\chi \text{ non-relativistic)}$$

where μ_χ is the chemical potential (in equilibrium $\mu_{\bar\chi} = -\mu_\chi$).

For $x > \bar{x}_{f.o.}$ the production term (proportional to the equilibrium number densities) is suppressed, because $Y_{\bar\chi}$ becomes $\ll Y_{\bar\chi}^{EQ}$, and can be neglected. Prove that integrating the $dY_{\bar\chi}/dx$ equation from $x = \bar{x}_{fo}$ to $x = \infty$ (a good approximation for the value at present) we get (assume s is constant)

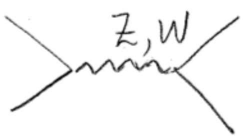
$$Y_{\bar\chi}(x\to\infty) = A\left[e^{\left(A\int_{\bar{x}_{fo}}^{\infty}\frac{\langle\sigma_{\chi\chi}v\rangle s\,dx}{Hx}\right)} - 1\right]^{-1}$$

(you may consult for example Iminniyaz, Drees, Chen 1104.5548 or Gelmini, Huh, Rehagen 1304.3679 to get help)

Then $Y_\chi(x\to\infty) = A + Y_{\bar\chi}(x\to\infty)$.

Exercise 7 Decoupling or freeze-out of active neutrinos

Active neutrinos decouple while they are relativistic.
Just estimate the interaction cross section

$\sigma \simeq \Big|$ 

$\Big|^2 \simeq \dfrac{g_W^4}{M_Z^4}\, T^2$ on dimensional grounds

and $H = \sqrt{\frac{8}{3}\pi G \rho} \sim \sqrt{\dfrac{\rho_{radiation}}{M^2_{Planck}}} \simeq \dfrac{T^2}{M_{Planck}}$

Prove that at large temperatures the reaction rate is $\Gamma > H$ and estimate the freeze-out temperature $T_{f.o.}$ for which $\Gamma(T_{f.o.}) = H(T_{f.o.})$.
(after, for $T < T_{f.o.}$, $\Gamma < H$ and interactions cease)-

Exercise 8

Decoupling or freeze-out of non-relativistic particles

The solution for WIMPs without a particle-antiparticle asymmetry of the Boltzmann equation yields

$$\Omega h^2 \simeq 0.2 \left(\frac{3 \times 10^{-26}\ cm^3/s}{\langle \sigma_A v \rangle} \right)$$

You can check that in natural units

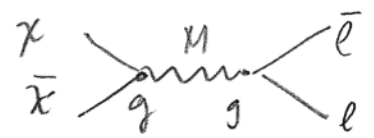
$$0.6 \times 10^{-26}\ cm^3/s \simeq 10^{-37}\ cm^2 \simeq 2 \times 10^{-10}\ GeV^{-2}.$$

Let us consider three examples of the use of this equation

8. a Consider a baryon-symmetric Universe. For strongly interacting particles the freeze-out value of $x = M/T$ is 40 instead of 20 for WIMPs, but still

the order of magnitude of the remaining density of baryons and antibaryons in a baryon symmetric Universe if we use the equation for WIMPs will be fine. Which is this value? Take the nucleon-antinucleon annihilation cross section to be $\langle \sigma_A v \rangle \sim m_\pi^{-2}$ where $m_\pi = 135$ MeV.

8. b. <u>Heavy neutral relics with $m_\chi < M$</u>, M = mediator mass. Assume no particle antiparticle asymmetry. Assuming $M \simeq M_{Z^0}$ and interactions of weak order

$\Big|$  $\Big|^2 \sim \sigma_A$ with $T, m_\ell \ll m_\chi \ll M$

Prove that $\Omega_\chi h^2 \simeq N_a \left(\frac{\text{GeV}}{M}\right)^2$ where N_a is the number of annihilation channels. (This corresponds to the original bound on heavy neutrinos obtained by Lee and Weinberg in 1977)

8. c <u>Heavy neutral relics with $m_\chi > M$</u> (again without a particle-antiparticle asymmetry).
Assume now $T, m_\ell \ll M \ll m_\chi$ and estimate σ_A on dimensional grounds, for a coupling of weak strength. Prove that $\Omega_\chi h^2 \simeq \frac{1}{N_a}\left(\frac{m_\chi}{\text{TeV}}\right)^2$