Dicke model with photon losses

Brief introduction to the Dicke model

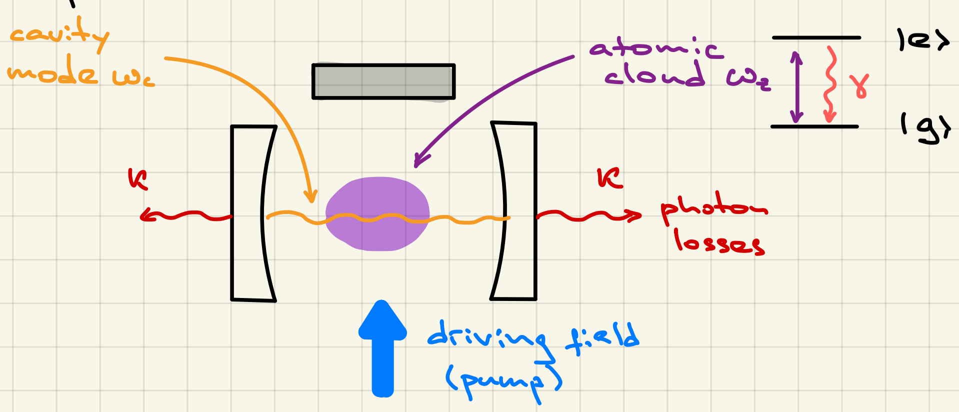
Over the course of lectures thus far you have been introduced to some general concepts of open quantum systems as well as to advanced methods employed to tackle the dynamics of such systems. Today, we'll apply this knowledge to one of the most paradigmatic models in quantum optics and physics of open systems: the Dicke model.

In its simplest form, the Dicke model describes a single bosonic mode (usually a photon in a cavity) coupled to a set of N two-level systems (the atoms). The Dicke Hamiltonian reads

$$\hat{H} = \omega_0 \hat{a}^\dagger \hat{a} + \omega_z \sum_{i=1}^{N} \hat{\sigma}_i^z + \frac{2g}{\sqrt{N}} (\hat{a} + \hat{a}^\dagger) \sum_{i=1}^{N} \hat{\sigma}_i^x . \qquad (1)$$

Here, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators, respectively, satisfying $[\hat{a}, \hat{a}^\dagger] = 1$, and $\hat{\sigma}_i^\alpha$ are spin-1/2 operators: $[\hat{\sigma}_i^\alpha, \hat{\sigma}_j^\beta] = i\delta_{ij}\, \varepsilon^{\alpha\beta\gamma} \hat{\sigma}_i^\gamma$. The $1/\sqrt{N}$ prefactor ensures that the energy is extensive. The simplest way to see it is by "integrating out" the photon field $\hat{a}$, which yields an all-to-all interaction $\sim N^{-1} (\sum_i \hat{\sigma}_i^x)^2 \sim O(N^2)$. Note also that the dependence on the atomic degrees of freedom enters only via the total spin $\hat{S}^\alpha = \sum_i \hat{\sigma}_i^\alpha$. Thus, despite its many-body appearance, the Dicke model describes a "big fat spin" interacting with photons.

In the following, we will focus on the nonequilibrium case of the Dicke model, in which the system is subject to an external coherent driving as well as to dissipative losses. While an actual experimental implementation of the model is far from being trivial, schematically the setup can be visualized as follows:



Potential dissipative processes are modeled using the Lindblad equation formalism:

$$\partial_t \hat{\rho} = -i[\hat{H}, \hat{\rho}] + \sum_i \gamma_i \mathcal{D}[\hat{L}_i], \qquad (2)$$

with $\mathcal{D}[\hat{L}] = 2\hat{L}\hat{\rho}L^\dagger - \{\hat{L}^\dagger \hat{L}, \rho\}$.

Main sources of dissipation:

rate	Lindblad operator	physical process
κ	a	cavity loss
γ	$\sum_j \sigma_j^- = S_-$	collective atomic decay
$\gamma_\downarrow$	σ_j^-	single-atom decay
γ_ϕ	σ_j^z	single-atom dephasing

In this lecture, we will neglect all potential atomic losses and will only take into account photon dissipation

The superradiant transition

The light-matter interaction coupling g is controlled by the external pumping strength. Photons from the external laser field rescatter off the atoms and populate the cavity mode ω_c (and vice versa). At a certain point, as we keep increasing the strength g, the cavity mode becomes macroscopically occupied ("photon condensate"). One says that the Dicke model exhibits the superradiant transition.

As you probably know, (continuous) phase transitions can be related to spontaneous breaking of some symmetry. What is the symmetry associated with the superradiant transition? The Hamiltonian (1) is symmetric under the global $\mathbb{Z}_2$ transformation,

$$\mathbb{Z}_2: \quad \hat{a} \to -\hat{a}, \quad \hat{\sigma}_i^x \to -\hat{\sigma}_i^x. \tag{3}$$

This symmetry reflects the parity-conserving nature of the interaction: $[\hat{H}, \hat{P}] = 0$, $\hat{P} = (-1)^{\hat{N}_{ex}}$, with $\hat{N}_{ex} = \hat{a}^\dagger \hat{a} + \sum_i \hat{\sigma}_i^z$.

Nonequilibrium dynamics of the open Dicke model

We know that depending on the strength of the interaction g, the system may have a very different steady state. What is the critical value of the coupling g_c that separates the two phases? And perhaps even more interestingly, how does the system reach it?

Before jumping to field-theoretic methods let's try a more straightforward approach. Recall that expectation values of observables evolve, according to the Lindblad equation, as

$$\partial_t \langle \hat{O} \rangle = i \langle [\hat{H}, \hat{O}] \rangle + \sum_i \gamma_i \langle \hat{L}_i^\dagger [\hat{O}, \hat{L}_i] + [\hat{L}_i^\dagger, \hat{O}] \hat{L}_i \rangle. \tag{4}$$

In our case, $\gamma_i \to \kappa$, $\hat{L} \to \hat{a}$. After some simple algebra, one finds

$$
\begin{aligned}
\partial_t \langle \hat{a} \rangle &= -(i\omega_c + \kappa)\langle \hat{a} \rangle - 2ig\sqrt{N}\langle \hat{\sigma}^x \rangle, \\
\partial_t \langle \hat{\sigma}^x \rangle &= -\omega_z \langle \sigma^y \rangle, \\
\partial_t \langle \hat{\sigma}^y \rangle &= \omega_z \langle \sigma^x \rangle - \frac{2g}{\sqrt{N}} \langle (\hat{a} + \hat{a}^+)\hat{\sigma}^z \rangle, \\
\partial_t \langle \hat{\sigma}^z \rangle &= \frac{2g}{\sqrt{N}} \langle (\hat{a} + \hat{a}^+)\hat{\sigma}^y \rangle,
\end{aligned}
\tag{5}
$$

with $\langle \hat{\sigma}^\alpha \rangle = \langle \hat{S}^\alpha \rangle / N = s^\alpha$, $\langle \hat{a} \rangle \equiv \alpha$.

Ex: Derive it

We note that the system of equations (5) is not closed: on top of the 1-point functions, it also contains 2-point functions. If we now derived the equations governing the dynamics of the 2-point correlators, they would clearly include 3-point functions, etc. The result is an infinite hierarchy of differential equations, cf. BBGKY. One way to close it, is to truncate the correlation functions at a certain order. For instance, to lowest order, $\langle \hat{a}\hat{\sigma}^\alpha \rangle \simeq \langle \hat{a} \rangle \langle \hat{\sigma}^\alpha \rangle$, which yields the mean-field approximation:

$$
\begin{aligned}
\partial_t \alpha &= -(i\omega_c + \kappa)\alpha - 2ig\sqrt{N} s^x \\
\partial_t s^x &= -\omega_z s^y, \\
\partial_t s^y &= \omega_z s^x - \frac{2g}{\sqrt{N}} (\alpha + \alpha^*) s^z, \\
\partial_t s^z &= \frac{2g}{\sqrt{N}} (\alpha + \alpha^*) s^y.
\end{aligned}
\tag{6}
$$

The corresponding steady-state solution is given by

$$
s^x_* = \pm\sqrt{1/4 - (s^z_*)^2}, \quad s^y_* = 0, \quad s^z_* = -\omega_z \frac{\omega_c^2 + \kappa^2}{8g^2\omega_c}, \quad \alpha_* = -\frac{2g\sqrt{N}}{\omega_c - i\kappa} s^x_*. \tag{7}
$$

Using the condition $|\alpha_*| > 0$ for $g > g_c$, we easily find

$$
g_c^{MF} = \frac{1}{2}\sqrt{\frac{|\omega_z|(\omega^2 + \kappa^2)}{\omega_c}}. \tag{8}
$$

Note that the condition $\langle \hat{a}\hat{\sigma}^\alpha \rangle \simeq \langle \hat{a} \rangle \langle \hat{\sigma}^\alpha \rangle \Longleftrightarrow \langle \hat{a}\hat{\sigma}^\alpha \rangle_c \simeq 0$, so the approximation was in fact neglecting connected two-point correlation functions. Extending upon this idea we might have instead truncated at a higher order. This approximation scheme is known as the cumulant expansion, for obvious reasons.

Though very straightforward, this approach is non-conserving and suffers from secularity problems. Thus, at a certain point, it is bound to fail.

Q: Note, however, that the mean-field approximation is conserving. Could you guess why?

Spin degrees of freedom and the path integral

As has been argued throughout the course, a good candidate for a conserving and self-consistent description of the dynamics of a nonequilibrium quantum system is given by the 2PI formalism To use it, we first need to transform our Dicke Hamiltonian into its appropriate Lagrangian counterpart.

It might feel tempting to work directly with the spin degrees of freedom using the spin coherent-state path integral formulation. Schematically, the latter has the form

$$Z = \int D\vec{n}\, \delta(1-\vec{n}^2)\, e^{-S_E[\vec{n}]} , \qquad (9)$$

where for simplicity we considered a single spin in Eucidean space. Needless to say, the presence of the constraint $\vec{n}^2=1$ doesn't look too appealing, especially having nonequilibrium in mind. Of course, we could instead employ a parametrization that automatically takes care of the constraint ($\rightarrow$ Euler angles), but then the action would contain terms $\sim \cos\theta, \sin\theta$, etc., which is not very convenient for developing approximation schemes.

It is thus suggestive to map the original spin d.o.f. onto some new auxiliary ones. There are many options on the market:

1) Jordan-Wigner:

$$\hat{\sigma}_i^{\mp} = \exp\left(\mp i\pi \textstyle\sum_{j=1}^{i-1} \hat{c}_j^+ \hat{c}_j\right) \hat{c}_i^{(+)} , \quad \hat{\sigma}_i^z = 2\hat{c}_i^+\hat{c}_i - \hat{I} , \qquad (10)$$

with approriate fermionic operators $\{\hat{c}_i^+, \hat{c}_j\} = \delta_{ij}$, $\{\hat{c}_i^+, \hat{c}_j^+\} = \{\hat{c}_i, \hat{c}_j\} = 0$.

This map is only suitable in 1D.

2) Holstein-Primakoff:

$$\hat{S}^z = -N/2 + \hat{b}^+\hat{b} , \quad \hat{S}^+ = \sqrt{N-\hat{b}^+\hat{b}}\,\hat{b}^+ , \quad \hat{S}^- = \hat{b}\sqrt{N-\hat{b}^+\hat{b}} , \qquad (11)$$

with bosonic operators $[\hat{b}, \hat{b}^+] = 1$.

This map is very nonlinear and thus forces one to do a $1/N$ expansion (see below) from the very beginnig. A viable option for semiclassical computations, but we can do "better".

3) Jordan – Schwinger:

$$\hat{G}_i^\alpha = \frac{1}{2}\hat{b}^\dagger_{i,s}\,\tau^\alpha_{ss'}\,\hat{b}_{i,s'}, \qquad s,s' \in \{1,2\} \tag{12}$$

with τ^α being the Pauli matrices and $\hat{b}_1, \hat{b}_2$ being bosonic annihilation operators.

The map is now only bilinear, which is good. The su(2) Casimir element is given by

$$\hat{\vec{G}}^2 = \frac{\hat{M}}{2}\left(\frac{\hat{M}}{2}+1\right), \qquad \hat{M} = \sum_s \hat{b}^\dagger_s \hat{b}_s. \tag{13}$$

Hence, to ensure the constraint, one must have $\hat{M} \equiv 1$ on the operator level (↪ also dynamically). While this sounds doable, the condition suggests to adopt a different type of auxiliary d.o.f., for which the condition will be satisfied by construction.

4) Martin transformation (Majorana fermions):

$$\hat{G}_i^\alpha = -\frac{i}{2}\varepsilon^{\alpha\beta\gamma}\hat{\eta}_i^\alpha\hat{\eta}_i^\beta, \qquad \{\hat{\eta}_i^\alpha, \hat{\eta}_j^\beta\} = \delta_{ij}\delta^{\alpha\beta}, \qquad \alpha,\beta,\gamma \in \{x,y,z\}. \tag{14}$$

Ex: Verify that $[\hat{G}_i^\alpha, \hat{G}_j^\beta] = i\delta_{ij}\,\varepsilon^{\alpha\beta\gamma}\hat{G}_i^\gamma$ and $\hat{\vec{G}}^2 = 3/4$.

2PI approach

Adopting the map (14) the Schwinger-Keldysh action in the presence of photon losses takes the form:

$$S = \int_C dt\left[\frac{i}{2}(a^*\partial_t a - a\partial_t a^*) - \omega_c a^* a + \frac{i}{2}\sum_i(\eta_i^\alpha\partial_t\eta_i^\alpha + 2\omega_z\eta_i^x\eta_i^y) + \right.$$
$$\left. + \frac{2ig}{\sqrt{N}}(a+a^*)\sum_i \eta_i^y\eta_i^z\right] - i\kappa\int dt\,(2a_+a_-^* - a_+^*a_+ - a_-^*a_-). \tag{15}$$

Since the interaction term depends on the photon field only through the combination $\hat{a}+\hat{a}^\dagger$, it is convenient to introduce

$$\hat{a} = \sqrt{\frac{\omega_c}{2}}\,(\hat{\phi} + i\hat{\pi}/\omega_c) \;\hookrightarrow\; \hat{\phi} = \frac{1}{\sqrt{2\omega_c}}(\hat{a}+\hat{a}^\dagger), \quad \hat{\pi} = -i\sqrt{\omega_c/2}\,(\hat{a}-\hat{a}^\dagger). \tag{16}$$

to wit

$$S = \int_C dt\left[\frac{1}{2}(\pi\partial_t\phi - \phi\partial_t\pi) - \frac{1}{2}(\omega_c^2\phi^2 + \pi^2) + \frac{i}{2}\sum(\eta_i^\alpha\partial_t\eta_i^\alpha + 2\omega_z\eta_i^x\eta_i^y) + i\tilde{g}\phi\sum_i\eta_i^y\eta_i^z\right.$$
$$\left. - i\kappa\omega_c\int dt\,\left[\phi_+\phi_- + i\phi_+\pi_-/\omega_c - i\phi_-\pi_+/\omega_c + \pi_+\pi_-/\omega_c^2 - \frac{1}{2}(\phi_+^2 + \pi_+^2/\omega_c^2 + \phi_-^2 + \pi_-^2/\omega_c^2)\right]\right], \tag{17}$$

where $\tilde{g} = \sqrt{8\omega_c/N}\,g$.

The 2PI effective action takes the usual form

Majorana propagator

$$\Gamma = S + \frac{i}{2}\mathrm{Tr}\ln D^{-1} + \frac{i}{2}\mathrm{Tr}\, D_0^{-1} D - \frac{i}{2}\mathrm{Tr}\ln G^{-1} - \frac{i}{2}\mathrm{Tr}\, G_0^{-1} G + \Gamma_2 , \tag{18}$$

photon propagator

with $\Gamma_2 = -i \ln \langle e^{iS_{int}} \rangle_{2PI}$ being the sum of all the 2PI (w.r.t. dressed propagators) connected vacuum diagrams.

The quantum equations of motion take the form

$$\frac{\delta S}{\delta \phi_a(t)} - \frac{i}{2}\frac{\delta \mathrm{Tr}[G_0^{-1}(\phi) G]}{\delta \phi_a(t)} + \frac{\delta \Gamma_2}{\delta \phi_a(t)} = 0 , \tag{19a}$$

$$[(D_0^{-1} - \Sigma)\circ D]_{ab}(t,t') = \delta_{ab}\,\delta(t-t') , \tag{19b}$$

$$[(G_0^{-1} - \Pi)\circ G]^{\alpha\beta}_{ab,ij}(t,t') = \delta_{ab}\,\delta_{ij}\,\delta^{\alpha\beta}\,\delta(t-t') . \tag{19c}$$

One can proceed working either in the original $\pm$-basis or perform the Keldysh rotation,

$$\begin{pmatrix} O_+ \\ O_- \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1 & -1/2 \end{pmatrix} \begin{pmatrix} O_{cl} \\ O_q \end{pmatrix} . \tag{20}$$

In Keldysh basis, the Dyson equations read

$$\begin{aligned}
&\partial_t D^{\phi\phi}_{clq} - D^{\pi\phi}_{clq} + \kappa D^{\phi\phi}_{clq} = 0 , \\
&\partial_t D^{\phi\pi}_{clq} - D^{\pi\pi}_{clq} + \kappa D^{\phi\pi}_{clq} = i\mathbb{1} , \\
&\partial_t D^{\pi\phi}_{clq} + \omega_c^2 D^{\phi\pi}_{clq} + \kappa D^{\pi\phi}_{clq} + i\Sigma^{\phi\phi}_{qcl}\circ D^{\phi\phi}_{clq} = i\mathbb{1} , \\
&\partial_t D^{\pi\pi}_{clq} + \omega_c^2 D^{\phi\pi}_{clq} + \kappa D^{\pi\pi}_{clq} + i\Sigma^{\phi\phi}_{qcl}\circ D^{\phi\pi}_{clq} = 0 ,
\end{aligned} \tag{21a}$$

$$\begin{aligned}
&\partial_t D^{\phi\phi}_{clcl} - D^{\pi\phi}_{clcl} + \kappa D^{\phi\phi}_{clcl} - i\frac{\kappa}{\omega_c} D^{\pi\phi}_{qcl} = 0 , \\
&\partial_t D^{\phi\pi}_{clcl} - D^{\pi\pi}_{clcl} + \kappa D^{\phi\pi}_{clcl} - i\frac{\kappa}{\omega_c} D^{\pi\pi}_{qcl} = 0 , \\
&\partial_t D^{\pi\phi}_{clcl} + \omega_c^2 D^{\phi\phi}_{clcl} + \kappa D^{\pi\phi}_{clcl} + i\kappa\omega_c D^{\phi\phi}_{qcl} + i\Sigma^{\phi\phi}_{qcl}\circ D^{\phi\phi}_{clcl} + i\Sigma^{\phi\phi}_{qq}\circ D^{\phi\phi}_{qcl} = 0 \\
&\partial_t D^{\pi\pi}_{clcl} + \omega_c^2 D^{\phi\pi}_{clcl} + \kappa D^{\pi\pi}_{clcl} + i\kappa\omega_c D^{\phi\pi}_{qcl} + i\Sigma^{\phi\phi}_{qcl}\circ D^{\phi\pi}_{clcl} + i\Sigma^{\phi\phi}_{qq}\circ D^{\phi\pi}_{qcl} = 0 ,
\end{aligned} \tag{21b}$$

+ (21a) with $clq \leftrightarrow qcl$ and $\kappa \to -\kappa$.

Here, $\circ$ denotes temporal convolution, $(f\circ g)(t,t') = \int dt''\, f(t,t'')\, g(t'',t')$, and Σ is the proper self-energy:

$$\Sigma^{\alpha\beta}_{ab}(t,t') = 2i\frac{\delta \Gamma_2}{\delta D^{\alpha\beta}_{ab}(t,t')} . \tag{22}$$

We'll derive one of the equations and leave the rest as an exercise.
In matrix notation, one of the Dyson equations read:

$$\begin{pmatrix} (D_0^{-1})^{\phi\phi}_{qcl} - \Sigma^{\phi\phi}_{qcl} & (D_0^{-1})^{\phi\pi}_{qcl} \\ (D_0^{-1})^{\pi\phi}_{qcl} & (D_0^{-1})^{\pi\pi}_{qcl} \end{pmatrix} \circ \begin{pmatrix} D^{\phi\phi}_{clcl} & D^{\phi\pi}_{clcl} \\ D^{\pi\phi}_{clcl} & D^{\pi\pi}_{clcl} \end{pmatrix} +$$

$$+ \begin{pmatrix} (D_0^{-1})^{\phi\phi}_{qq} - \Sigma^{\phi\phi}_{qq} & 0 \\ 0 & (D_0^{-1})^{\pi\pi}_{qq} \end{pmatrix} \circ \begin{pmatrix} D^{\phi\phi}_{qcl} & D^{\phi\pi}_{qcl} \\ D^{\pi\phi}_{qcl} & D^{\pi\pi}_{qq} \end{pmatrix} = 0. \qquad (23)$$

For instance, the $\phi\phi$-entry then yields

$$(D_0^{-1})^{\phi\phi}_{qcl} \circ D^{\phi\phi}_{clcl} - \Sigma^{\phi\phi}_{qcl} \circ D^{\phi\phi}_{clcl} + (D_0^{-1})^{\phi\pi}_{qcl} \circ D^{\pi\phi}_{clcl} + (D_0^{-1})^{\phi\phi}_{qq} \circ D^{\phi\phi}_{qcl} - \Sigma^{\phi\phi}_{qq} \circ D^{\phi\phi}_{qcl} = 0. \qquad (24)$$

The relevant inverse bare propagators, $D_0^{-1} = -iS^{(2)}$, read

$$(D_0^{-1})^{\phi\phi}_{qcl} = i\omega_c^2\,\delta(t-t'), \quad (D_0^{-1})^{\phi\pi}_{qcl} = i\,\delta(t-t')(\partial_{t'} + \kappa), \quad (D_0^{-1})^{\phi\phi}_{qq} = \kappa\omega_c\,\delta(t-t'). \qquad (25)$$

Ex: Show it.

Putting everything together we end up with

$$\partial_t D^{\pi\phi}_{clcl}(t,t') = -\omega_c^2 D^{\phi\phi}_{clcl}(t,t') - \kappa D^{\pi\phi}_{clcl}(t,t') + i\kappa\omega_c D^{\phi\phi}_{qcl}(t,t') -$$

$$- i\int dt''\, \Sigma^{\phi\phi}_{qcl}(t,t'')\, D^{\phi\phi}_{clcl}(t'',t') - i\int dt''\, \Sigma^{\phi\phi}_{qq}(t,t'')\, D^{\phi\phi}_{qcl}(t'',t'). \qquad (26)$$

Since retarded and advanced propagators are not independent of each other, one sometimes instead with a linear combination thereof,

$$D_{clq}(t,t') = -i\rho(t,t')\,\Theta(t-t'), \qquad D_{qcl}(t,t') = i\rho(t,t')\,\Theta(t'-t), \qquad (27)$$

known as the spectral function $\rho = i(D_{clq} - D_{qcl})$. Likewise,

$$\Sigma_{clq}(t,t') = -i\Sigma^{(0)}(t)\,\delta(t-t') + i\Sigma_\rho(t,t')\,\Theta(t'-t),$$

$$\Sigma_{qcl}(t,t') = -i\Sigma^{(0)}(t)\,\delta(t-t') - i\Sigma_\rho(t,t')\,\Theta(t-t'), \qquad (27)$$

so that

$$\partial_t F^{\pi\phi}(t,t') = -\Omega^2(t)F^{\phi\phi}(t,t') - \kappa F^{\pi\phi}(t,t') - \kappa\omega_c\rho^{\phi\phi}(t,t')\,\Theta(t'-t) -$$

$$- \int_{t_0}^{t} dt''\, \Sigma^{\phi\phi}_\rho(t,t'')\, F^{\phi\phi}(t'',t') + \int_{t_0}^{t'} dt''\, \Sigma^{\phi\phi}_F(t,t'')\, \rho^{\phi\phi}(t'',t'), \qquad (28)$$

with $F \equiv D_{clcl}$, $\Sigma_F \equiv \Sigma_{qq}$, and $\Omega^2(t) \equiv \omega_c^2 + \Sigma^{(0)}_{\phi\phi}(t)$.

Remark: the Keldysh propagator D_{clcl} is sometimes referred to as the statistical function in the literature.

Analogously, the Majorana equations are

$$(\delta^{\alpha\gamma}\partial_t + i m^{\alpha\gamma}) G^{\gamma\beta}_{qq,ij}(t,t') - \Pi^{\alpha\gamma}_{clq,ik} \circ G^{\gamma\beta}_{qcl,kj} = \delta^{\alpha\beta}\delta_{ij}(t-t'),$$
$$(\delta^{\alpha\gamma}\partial_t + i m^{\alpha\gamma}) G^{\gamma\beta}_{clq,ij}(t,t') - \Pi^{\alpha\gamma}_{qcl,ik} \circ G^{\gamma\beta}_{clq,kj} = \delta^{\alpha\beta}\delta_{ij}(t-t'), \quad (29)$$
$$(\delta^{\alpha\gamma}\partial_t + i m^{\alpha\gamma}) G^{\gamma\beta}_{clcl,ij}(t,t') - \Pi^{\alpha\gamma}_{qcl,ik} \circ G^{\gamma\beta}_{clcl,kj} - \Pi^{\alpha\gamma}_{qq,ik} \circ G^{\gamma\beta}_{qcl,kj} = 0,$$

with

$$i m^{\alpha\beta} = \begin{pmatrix} 0 & \omega_z & 0 \\ -\omega_z & 0 & \tilde{g}\phi \\ 0 & -\tilde{g}\phi & 0 \end{pmatrix}, \qquad \Pi^{\alpha\beta}_{ab,ij}(t,t') = -2i\frac{\delta\Gamma_2}{\delta G^{\beta\alpha}_{ba,ji}(t',t)}.$$

Note that action (15) is invariant under the gauge $\mathbb{Z}_2$ symmetry

$$\begin{pmatrix} \eta^x_i(t) \\ \eta^y_i(t) \\ \eta^z_i(t) \end{pmatrix} \longrightarrow \begin{pmatrix} -\eta^x_i(t) \\ -\eta^y_i(t) \\ -\eta^z_i(t) \end{pmatrix},$$

which implies $G^{\alpha\beta}_{ij}(t,t') = \delta_{ij} G^{\alpha\beta}(t,t';i)$, $\Pi^{\alpha\beta}_{ij}(t,t') = \delta_{ij}\Pi^{\alpha\beta}(t,t';i)$. Assuming further homogeneity $G^{\alpha\beta}(t,t';i) = G^{\alpha\beta}(t,t')$, etc.

Finally, the equation yields the MF equation for the photon fields:

$$\partial_t\phi = \pi - \kappa\phi,$$
$$\partial_t\pi = -\omega_c^2\phi - \kappa\pi + i\tilde{g}N \underbrace{G^{yz}_{clcl}(t,t)}_{iS^x(t)}. \quad (30)$$

1/N expansion

To close the set of Dyson (or Kadanoff-Baym) equations, it is left to specify the approximation for the self-energies Σ and Π, or equivalently, for Γ_2. The most common expansion scheme is the familiar perturbative coupling expansion, which relies on the smallness of the coupling g.

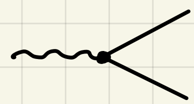
However, the Dicke model offers an alternative, nonperturbative expansion parameter: 1/N. As we have already discussed, the Dicke model actually describe a big spin of length N/2 interacting with photons. Therefore, physically, we expect that at $N\to\infty$ the description becomes fully classical, while quantum fluctuations should be completely suppressed (think of self-averaging).

Formally, this can be argued as follows. The first terms in (18) stem from the classical action and thus scale extensively with the system size N.

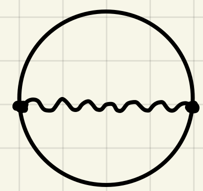
What about Γ_2? While the Kadanoff-Baym equations were derived in the Keldysh basis, it is more convenient to work with Γ_2 on the Schwinger-Keldysh contour (fewer kinds of vertices and propagators ↳ fewer diagrams) and simply decompose Σ at the last step. Introducing the diagrammatic notation,

$$D^{\phi\phi}(t,t') = \langle \mathcal{T}_c \hat{\phi}(t)\hat{\phi}(t')\rangle_c \quad = \quad$$,

$$G^{\alpha\beta}_{ij}(t,t') = \langle \mathcal{T}_c \hat{\eta}_i^{\alpha}(t)\hat{\eta}_j(t')\rangle_c \quad = \quad$$,

$$i\tilde{g}\int_c dt \quad = \quad$$  ,

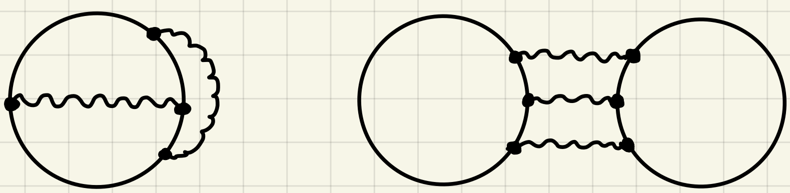
the first contribution takes the form



Each vertex yields $1/\sqrt{N}$ and the trace adds an additional factor of N ↳ $O(N^0)$.

In fact, this is the only next-to-leading order (NLO) diagram in the Dicke model.

Ex: Show that at NNLO, there are two diagrams



Self energies at NLO in $1/N$

The analytic expression for Γ_2^{NLO} reads

$$\Gamma_2^{NLO} = -\frac{1}{2}\tilde{g}^2\sum_{ij}\int_c dt\, \langle \phi(t)\phi(t')\,\eta_i^{y}(t)\,\eta_i^{z}(t)\,\eta_j^{y}(t')\,\eta_j^{z}(t')\rangle_{2PI} =$$

$$= -\frac{1}{2}\tilde{g}^2\sum_{ij}\int_c dt\, D^{\phi\phi}(t,t')\left[G^{yz}_{ij}(t,t')G^{zy}_{ij}(t,t') - G^{yy}_{ij}(t,t')G^{zz}_{ij}(t,t')\right]. \tag{31}$$

Hence, the NLO photon self-energy is

$$\Sigma^{\phi\phi}(t,t') = N\tilde{g}^2\left[G^{yz}(t,t')G^{zy}(t,t') - G^{yy}(t,t')G^{zz}(t,t')\right]. \tag{32}$$

Likewise, e.g.,

$$\Pi^{yy}(t,t') = -\tilde{g}^2 D^{\phi\phi}(t,t')\, G^{zz}(t,t'). \tag{33}$$

Ex: Derive the rest.

If you prefer working with F and ρ, here's a neat trick to decompose Σ. The causal structure of the propagators can be summarized as

$$D(t,t') = F(t,t') - \frac{i}{2}\rho(t,t')\,\mathrm{sgn}_c(t-t'), \tag{34}$$

and likewise for G. Similarly,

$$\Sigma(t,t') = -i\,\Sigma^{(0)}(t)\delta(t-t') + \Sigma_F(t,t') - \frac{i}{2}\Sigma_\rho(t,t')\,\mathrm{sgn}_c(t-t'), \tag{35}$$

and same for Π.

Now it's very simple to decompose the self-energies. For instance,

$$\Sigma_F^{\phi\phi}(t,t') = N\tilde{g}^2\left[F_f^{\gamma 2}(t,t')F_f^{2\gamma}(t,t') - \frac{1}{4}\rho_f^{\gamma 2}(t,t')\rho_f^{2\gamma}(t,t') - F_f^{\gamma\gamma}(t,t')F_f^{22}(t,t') + \frac{1}{4}\rho_f^{\gamma\gamma}(t,t')\rho_f^{22}(t,t')\right], \tag{36a}$$

$$\Sigma_\rho^{\phi\phi}(t,t') = N\tilde{g}^2\left[\rho_f^{\gamma 2}(t,t')F_f^{2\gamma}(t,t') + F_f^{\gamma 2}(t,t')\rho_f^{2\gamma}(t,t') - \rho_f^{\gamma\gamma}(t,t')F_f^{22}(t,t') - F_f^{\gamma\gamma}(t,t')\rho_f^{22}(t,t')\right]. \tag{36b}$$

Some details regarding numerical implementation

It goes without saying that the Kadanoff-Baym equations, being a set of coupled nonlinear partial integro-differential equations, are way too hard to solve analytically, and one has to resort to numerical methods.

Before we proceed, note that the propagators are functions of two time arguments: t and t'. Therefore, to cover the whole time domain, we need to be able to propagate along the t'-direction as well. This can be achieved by considering the "dual" Dyson equations:

$$\left[D\circ(D_0^{-1} - \Sigma)\right]_{ab}(t,t') = \delta_{ab}\delta(t-t'), \tag{37a}$$

$$\left[G\circ(G_0^{-1} - \Pi)\right]^{\alpha\beta}_{ab,ij}(t,t') = \delta_{ab}\delta_{ij}\delta^{\alpha\beta}\delta(t-t'). \tag{37b}$$

On the other hand, using the symmetry properties

$$\begin{aligned} &F^{\alpha\beta}(t,t') = F^{\beta\alpha}(t,t'), \quad \rho^{\alpha\beta}(t,t') = -\rho^{\beta\alpha}(t',t), \quad &&\text{(bosons)} \\ &F_f^{\alpha\beta}(t,t') = -F_f^{\beta\alpha}(t,t'), \quad \rho_f^{\alpha\beta}(t,t') = \rho_f^{\beta\alpha}(t',t), \quad &&\text{(fermions)} \end{aligned} \tag{38}$$

we can propagate equations along the t axis and then reflect the result using (38).

The only missing ingredient in the above scheme is propagation along the diagonal. Introducing the notation,

$$d_t f(t,t) \equiv (\partial_t + \partial_{t'}) f(t,t')\Big|_{t=t'}, \tag{39}$$

one finds, for instance,

$$d_t F^{\pi\phi}(t,t) = F^{\pi\pi}(t,t) - \Omega^2(t) F^{\phi\phi}(t,t) - 2\kappa F^{\pi\phi}(t,t) - \int_{t_0}^{t} dt'' \left[\Sigma_\rho^{\phi\phi}(t,t'') F^{\phi\phi}(t'',t) - \Sigma_F^{\phi\phi}(t,t'') \rho^{\phi\phi}(t'',t) \right] \tag{40}$$

$$\overset{\shortparallel}{\partial_t F^{\pi\phi}(t,t')\big|_{t=t'}} + \partial_{t'} F^{\pi\phi}(t,t')\big|_{t=t'} = \partial_t \left[F^{\pi\phi}(t,t') + F^{\phi\pi}(t,t') \right]\big|_{t=t'}$$

and

"quantum ½"

$$d_t F^{\phi\phi}(t,t) = 2 \left[F^{\phi\pi}(t,t) - \kappa \left(F^{\phi\phi}(t,t) - \frac{1}{2\omega_k} \right) \right]. \tag{41}$$

The spectral components, on the other hand, remain unchanged on the diagonal:

$$\rho^{\pi\phi}(t,t) = -\rho^{\phi\pi}(t,t) = -1, \qquad \rho_f^{\alpha\beta}(t,t) = i\delta^{\alpha\beta}, \tag{42}$$

and follow from the (anti) commutation relations for bosons (fermions).

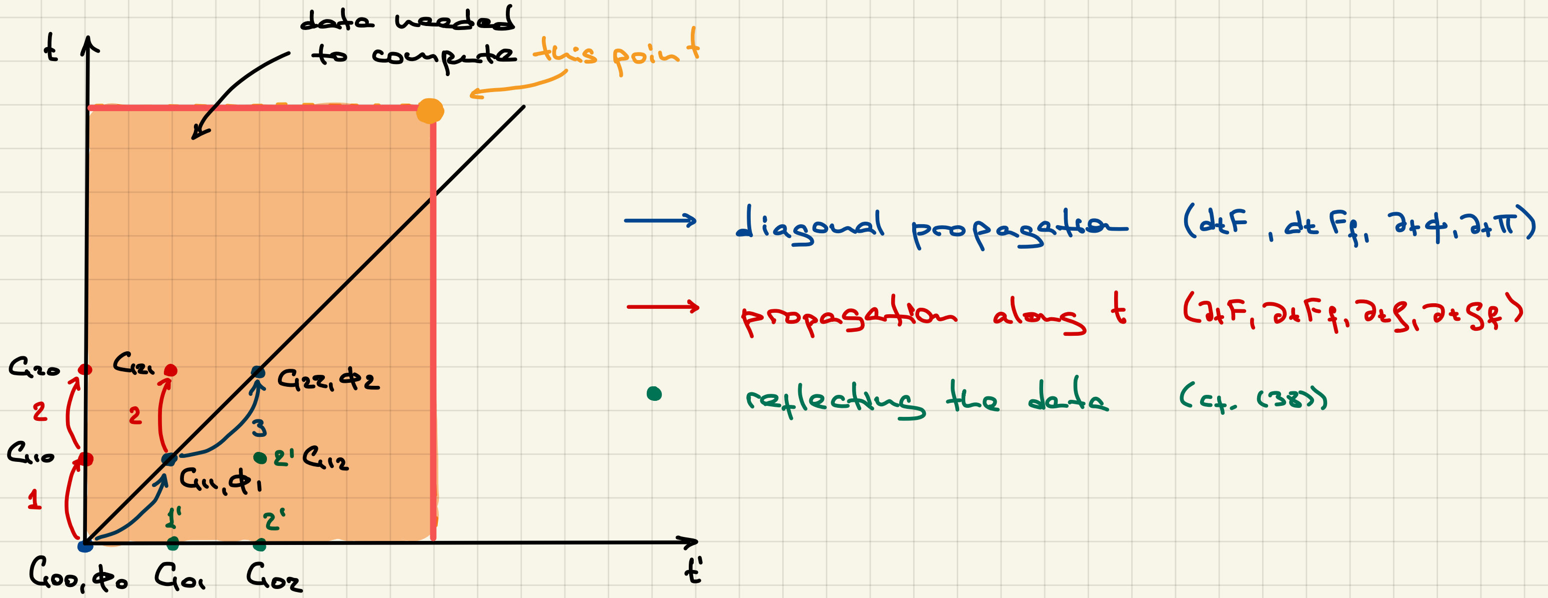
Remark: One may show that

$$F^{\alpha\beta}(t,t') = \frac{1}{2} \langle \{ \hat\phi^\alpha(t), \hat\phi^\beta(t') \} \rangle - \langle \hat\phi^\alpha(t) \rangle \langle \hat\phi^\beta(t') \rangle, \quad \rho^{\alpha\beta}(t,t') = i \langle [\hat\phi^\alpha(t), \hat\phi^\beta(t')] \rangle, \quad \text{(bosons)}$$

$$F_f^{\alpha\beta}(t,t') = \frac{1}{2} \langle [\hat\eta^\alpha(t), \hat\eta^\beta(t')] \rangle, \quad \rho_f^{\alpha\beta}(t,t') = i \langle \{ \hat\eta^\alpha(t), \hat\eta^\beta(t') \} \rangle. \quad \text{(fermions)}$$

and $d_t \rho_{(f)}^{\alpha\beta} = 0$.

To summarize, one can sketch the following scheme:



What is the numerical cost of solving the 2PI equations? Well, suppose we want to propagate an equation along t at fixed (t,t'). Due to numerical integration on the RHS this requires $O(t)$ operations. Going through each t' from 0 to t makes it $O(t^2)$. Finally, there are N_t steps like this, so the computational cost is $\sum_{t=0}^{N_t} O(t^2) = O(N_t^3)$. So the Kadanoff-Baym complexity scales cubically with the number of timesteps. Obviously, the memory costs, however, scales quadratically with the number of timesteps.

What schemes can one use? In order not to lose the self-cosistency of 2PI, it is very advisable to use implicit method. In practice, one normally adopts the predictor - corrector approach for that and iterates until the desired convergence is reached. Of course, one can use any sensible solver combinations for this. Suppose, however, we compute the memory integrals on the RHS using the simple trapezoidal rule. The error is then $O(\Delta t^2)$ making using fancy higher-order schemes essentially meaningless. The most commonly used scheme in this case is the so-called implicit Heun's method:

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0 \quad \Leftrightarrow \quad y_{i+1}^{(0)} = y_i + \Delta t \cdot f(t_i, y_i), \qquad \text{(predictor)}$$

$$y_{i+1}^{(n+1)} = y_i + \frac{\Delta t}{2}\left[f(t_i, y_i) + f(t_{i+1}, y_{i+1}^{(n)})\right]. \qquad \text{(corrector)}$$

More sophisticated approaches may involve having an adaptive grid, using higher-order schemes, etc. In addition, one may consider approximations on the level of the Kadanoff-Baym equations themselves. The most prominent ones are (a) employing the so-called generalized Kadanoff-Baym ansatz and (b) memory truncation ($\int_0^t dt \sim \int_{t-t_{trunc}}^t$). A good pedagogical read on the subject with some references is, e.g., arxiv: 2110.04793.

To get some feeling, a typical numerical run for the Dicke model with $N_t = 250$ timesteps takes about ~1 minute on a single thread.

Remark: In the class presentation, the numerical tolerances were set too low, which affected the performance without much accuracy gain. The lesson is: play with your numbers to find the right balance!