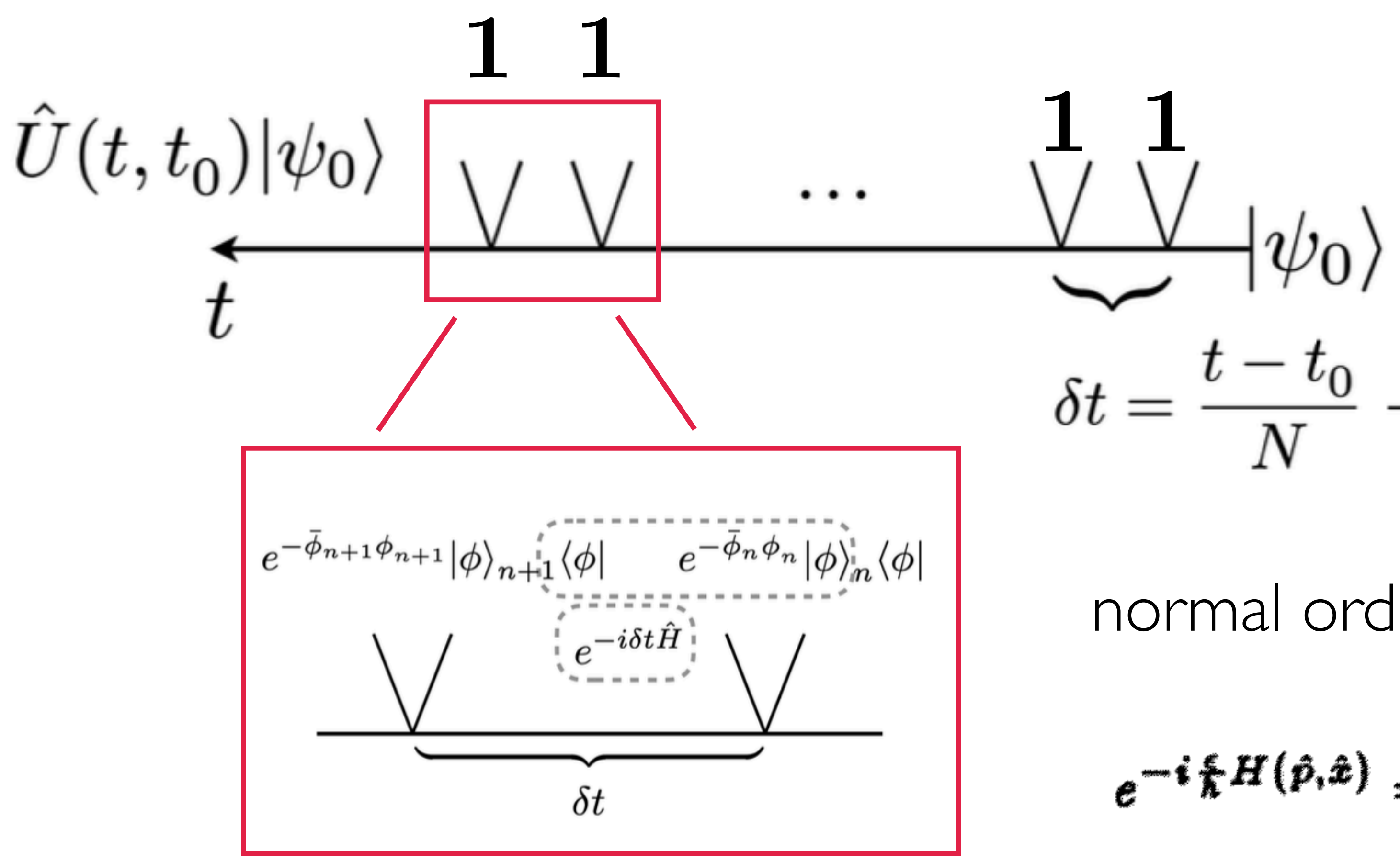
Recap of path integrals: unitary dynamics

→ unitary evolution (recap from your reading assignment of Negele/Orland)

$$i\partial_t \left|\psi\right\rangle(t) = \hat{H}\left|\psi\right\rangle(t) \quad \Longrightarrow \quad \left|\psi\right\rangle(t) = \hat{U}(t,t_0)\left|\psi_0\right\rangle, \quad \hat{U}(t,t_0) = e^{-i\hat{H}(t-t_0)}$$



$$\delta t = \frac{t-t_0}{N} \longrightarrow 0$$

$$\hat{U}(t,t_0) = e^{-i\hat{H}(t-t_0)} = \left(e^{-i\hat{H}\delta t}\right)^N = \lim_{N\to\infty}\left(\hat{1} - i\hat{H}\delta t\right)^N$$

normal ordering is <u>crucial</u>: it allows to convert functions of operators into regular functions (that's why we use coherent states!)

$$e^{-i\frac{\epsilon}{\hbar}H(\hat{p},\hat{x})} = :e^{-i\frac{\epsilon}{\hbar}H(\hat{p},\hat{x})}: - \left(\frac{\epsilon}{\hbar}\right)^2 \sum_{n=0}^{\infty} \frac{(-i\frac{\epsilon}{\hbar})^n}{(n+2)!}\left(H(\hat{p},\hat{x})^{n+2} - :[H(\hat{p},\hat{x})]^{n+2}:\right)$$

error is order $(\delta t)^2$

for N infinitesimal time steps δt

↓

$$\langle\phi_N|\hat{U}(t,t_0)|\psi(t_0)\rangle \approx \int \prod_{n=0}^{N-1} d[\bar\phi_n,\phi_n]\, e^{iS}\,\langle\phi_0|\psi(t_0)\rangle,$$

$$S = \sum_{n=0}^{N-1}\delta t\left(-i\frac{\bar\phi_{n+1}-\bar\phi_n}{\delta t}\phi_n - H(\bar\phi_{n+1},\phi_n)\right).$$

bosonic coherent states (brief summary)

$$a|\phi\rangle = \phi|\phi\rangle$$

$$\langle\phi_1|\phi_2\rangle = e^{\phi_1^*\phi_2}$$

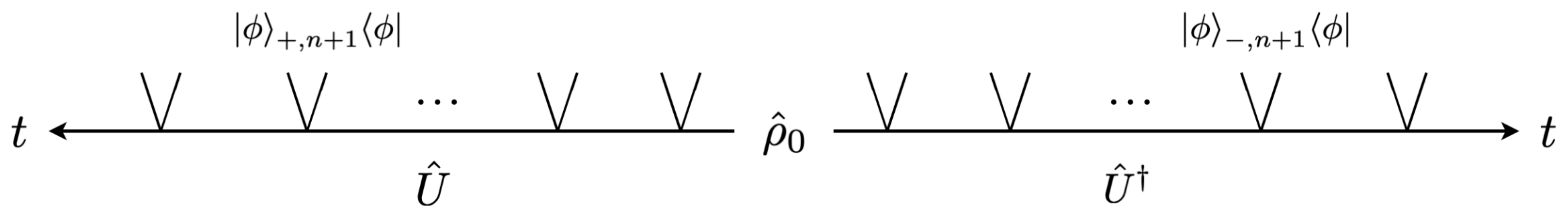
$$\mathbf{1} = \int d\phi d\phi^* e^{-|\phi|^2}|\phi\rangle\langle\phi|$$

Master equation and path integrals

Buchhold/Sieberer/Diehl, arXiv1512
Buchhold/Sieberer/Marino/Diehl, arXiv2312

→ open quantum systems

$$\partial_t \hat{\rho}(t) = -i[\hat{H}, \hat{\rho}(t)] \quad \Longrightarrow \quad \hat{\rho}(t) = \hat{U}(t, t_0)\hat{\rho}_0 \hat{U}^\dagger(t, t_0)$$



the goal is to find a representation of $Z(t) = \mathrm{Tr}(\rho(t))$

since later on we will equip $Z(t)$ with 'sources' to have a generating functional:

$Z[J] = \mathrm{Tr}[\rho(t) e^{i\int_{t,\mathbf{x}} J\psi(x)}]$ such that we can evaluate expectation values of n-pt functions, taking functional derivatives wrt J and then $J \to 0$

→ for now we focus only on the dynamics of the density matrix: $\rho(t)$

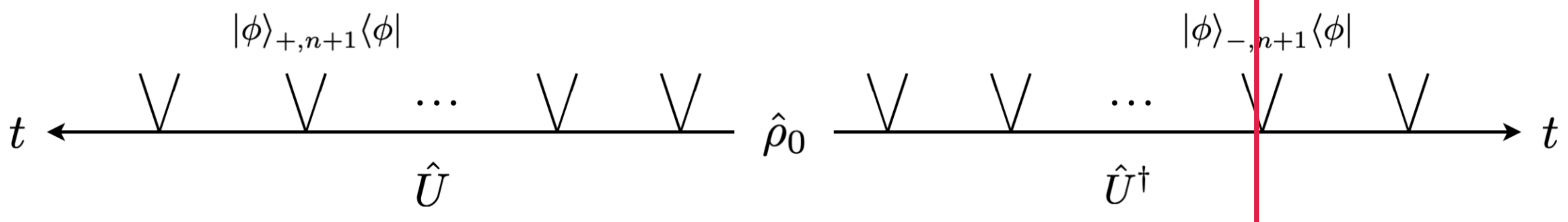
Master equation and path integrals

$$\rho(t) = e^{(t-t_0)\mathcal{L}}\rho(t_0) \equiv \lim_{N\to\infty} \left(\mathbb{1} + \delta_t \mathcal{L}\right)^N \rho(t_0)$$ infinitesimal evolution of the Liouvillian

$$\rho_{n+1} = e^{\delta_t \mathcal{L}}\rho_n = (\mathbb{1} + \delta_t \mathcal{L})\,\rho_n + O(\delta_t^2).$$

Let's now decompose the density matrix at time $n+1$ using coherent states insertions

$$\rho_{n+1} = \int \frac{d\psi_{+,n+1} d\psi^*_{+,n+1}}{\pi} \frac{d\psi_{-,n+1} d\psi^*_{-,n+1}}{\pi} e^{-\psi^*_{+,n+1}\psi_{+,n+1} - \psi^*_{-,n+1}\psi_{-,n+1}} \langle \psi_{+,n+1} | \rho_{n+1} | \psi_{-,n+1} \rangle |\psi_{+,n+1}\rangle\langle\psi_{-,n+1}|$$



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$$\rho_n = \int \frac{d\psi_{+,n} d\psi^*_{+,n}}{\pi} \frac{d\psi_{-,n} d\psi^*_{-,n}}{\pi} e^{-\psi^*_{+,n}\psi_{+,n} - \psi^*_{-,n}\psi_{-,n}} \langle \psi_{+,n} | \rho_n | \psi_{-,n} \rangle |\psi_{+,n}\rangle\langle\psi_{-,n}|$$

Master equation and path integrals

$$\rho(t) = e^{(t-t_0)\mathcal{L}}\rho(t_0) \equiv \lim_{N\to\infty} (\mathbb{1} + \delta_t\mathcal{L})^N \rho(t_0)$$

infinitesimal evolution of the Liouvillian

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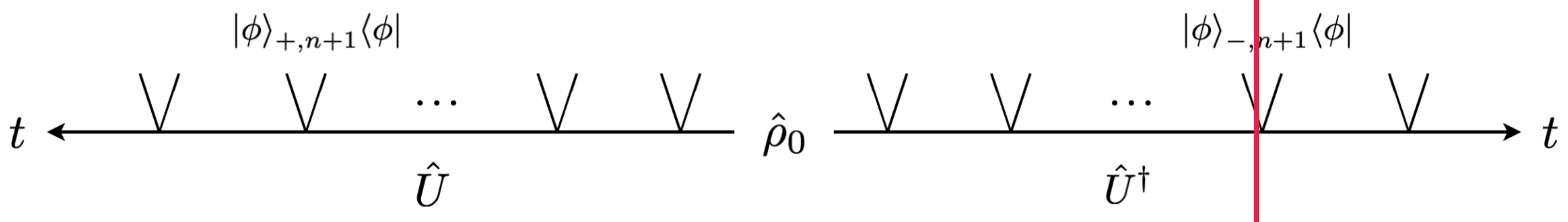

$$\rho_{n+1} = e^{\delta_t \mathcal{L}}\rho_n = (\mathbb{1} + \delta_t\mathcal{L})\rho_n + O(\delta_t^2).$$

1) the actual job is getting the action of $\mathscr{L}$ on ρ_n

2) we need two set of coh. states ($\pm$)

to convert the dens. matrix at time n into a number

$$\rho_n = \int \frac{d\psi_{+,n}d\psi^*_{+,n}}{\pi}\frac{d\psi_{-,n}d\psi^*_{-,n}}{\pi} e^{-\psi^*_{+,n}\psi_{+,n}-\psi^*_{-,n}\psi_{-,n}} \langle\psi_{+,n}|\rho_n|\psi_{-,n}\rangle |\psi_{+,n}\rangle\langle\psi_{-,n}|$$

Master equation and path integrals

$$\langle \psi_{+,n+1} | \mathcal{L}(|\psi_{+,n}\rangle \langle \psi_{-,n}|) | \psi_{-,n+1} \rangle$$

Master equation and path integrals

$$\langle \psi_{+,n+1} | \mathcal{L}(|\psi_{+,n}\rangle \langle \psi_{-,n}|) | \psi_{-,n+1}\rangle$$

$$\langle \psi_{+,n+1} | \mathcal{L}(|\psi_{+,n}\rangle \langle \psi_{-,n}|) | \psi_{-,n+1}\rangle = -i \left(\langle \psi_{+,n+1} | H | \psi_{+,n}\rangle \langle \psi_{-,n} | \psi_{-,n+1}\rangle - \langle \psi_{+,n+1} | \psi_{+,n}\rangle \langle \psi_{-,n} | H | \psi_{-,n+1}\rangle \right) + \ldots$$

Master equation and path integrals

$$\langle\psi_{+,n+1}|\mathcal{L}(|\psi_{+,n}\rangle\langle\psi_{-,n}|)|\psi_{-,n+1}\rangle$$

$$\langle\psi_{+,n+1}|\mathcal{L}(|\psi_{+,n}\rangle\langle\psi_{-,n}|)|\psi_{-,n+1}\rangle = -i\left(\langle\psi_{+,n+1}|H|\psi_{+,n}\rangle\langle\psi_{-,n}|\psi_{-,n+1}\rangle - \langle\psi_{+,n+1}|\psi_{+,n}\rangle\langle\psi_{-,n}|H|\psi_{-,n+1}\rangle\right) + \ldots$$

$$\langle\psi_{+,n+1}|\mathcal{L}(|\psi_{+,n}\rangle\langle\psi_{-,n}|)|\psi_{-,n+1}\rangle = -i\left(\langle\psi_{+,n+1}|H|\psi_{+,n}\rangle\langle\psi_{-,n}|\psi_{-,n+1}\rangle - \langle\psi_{+,n+1}|\psi_{+,n}\rangle\langle\psi_{-,n}|H|\psi_{-,n+1}\rangle\right)$$
$$+\sum_\alpha \gamma_\alpha\left[\langle\psi_{+,n+1}|L_\alpha|\psi_{+,n}\rangle\langle\psi_{-,n}|L_\alpha^\dagger|\psi_{-,n+1}\rangle - \frac{1}{2}\left(\langle\psi_{+,n+1}|L_\alpha^\dagger L_\alpha|\psi_{+,n}\rangle\langle\psi_{-,n}|\psi_{-,n+1}\rangle + \langle\psi_{+,n+1}|\psi_{+,n}\rangle\langle\psi_{-,n}|L_\alpha^\dagger L_\alpha|\psi_{-,n+1}\rangle\right)\right]$$

assume several dissipative channels (α)

Master equation and path integrals

$$\langle \psi_{+,n+1} | \mathcal{L}(|\psi_{+,n}\rangle \langle \psi_{-,n}|) | \psi_{-,n+1}\rangle$$

$$\langle \psi_{+,n+1} | \mathcal{L}(|\psi_{+,n}\rangle \langle \psi_{-,n}|) | \psi_{-,n+1}\rangle = -i \left(\langle \psi_{+,n+1} | H | \psi_{+,n}\rangle \langle \psi_{-,n} | \psi_{-,n+1}\rangle - \langle \psi_{+,n+1} | \psi_{+,n}\rangle \langle \psi_{-,n} | H | \psi_{-,n+1}\rangle \right)$$

$$\langle \psi_{+,n+1} | \mathcal{L}(|\psi_{+,n}\rangle \langle \psi_{-,n}|) | \psi_{-,n+1}\rangle = -i \left(\langle \psi_{+,n+1} | H | \psi_{+,n}\rangle \langle \psi_{-,n} | \psi_{-,n+1}\rangle - \langle \psi_{+,n+1} | \psi_{+,n}\rangle \langle \psi_{-,n} | H | \psi_{-,n+1}\rangle \right)$$
$$+ \sum_\alpha \gamma_\alpha \left[\langle \psi_{+,n+1} | L_\alpha | \psi_{+,n}\rangle \langle \psi_{-,n} | L_\alpha^\dagger | \psi_{-,n+1}\rangle - \frac{1}{2} \left(\langle \psi_{+,n+1} | L_\alpha^\dagger L_\alpha | \psi_{+,n}\rangle \langle \psi_{-,n} | \psi_{-,n+1}\rangle + \langle \psi_{+,n+1} | \psi_{+,n}\rangle \langle \psi_{-,n} | L_\alpha^\dagger L_\alpha | \psi_{-,n+1}\rangle \right) \right]$$

case of several dissipative channels (α)

$\rightarrow \quad \mathcal{L}(\psi^*_{+,n+1}, \psi_{+,n}, \psi^*_{-,n+1}, \psi_{-,n})$ function of the coherent state eigenvalues

.. and we have a residual $\langle \psi_{+,n+1} | \psi_{+,n}\rangle$ and $\langle \psi_{-,n} | \psi_{-,n+1}\rangle$

e.g. if $L = a \rightarrow L|\psi\rangle = \psi|\psi\rangle$

Master equation and path integrals

Remember we were starting from here

$$\rho_{n+1} = \int \frac{d\psi_{+,n+1} d\psi^*_{+,n+1}}{\pi} \frac{d\psi_{-,n+1} d\psi^*_{-,n+1}}{\pi} e^{-\psi^*_{+,n+1}\psi_{+,n+1} - \psi^*_{-,n+1}\psi_{-,n+1}} \langle \psi_{+,n+1} | \rho_{n+1} | \psi_{-,n+1} \rangle \, | \psi_{+,n+1} \rangle \langle \psi_{-,n+1} |$$

$$\rho_{n+1} = e^{\delta_t \mathcal{L}} \rho_n = (\mathbb{1} + \delta_t \mathcal{L}) \rho_n + O(\delta_t^2).$$

And now we can re-exponentiate the outcome of the previous calculation:

$$\langle \psi_{+,n+1} | \rho_{n+1} | \psi_{-,n+1} \rangle = \int \frac{d\psi_{+,n} d\psi^*_{+,n}}{\pi} \frac{d\psi_{-,n} d\psi^*_{-,n}}{\pi} e^{i\delta_t \left(-\psi_{+,n} i\partial_t \psi^*_{+,n} - \psi^*_{-,n} i\partial_t \psi_{-,n} - i\mathcal{L}(\psi^*_{+,n+1}, \psi_{+,n}, \psi^*_{-,n+1}, \psi_{-,n}) \right)} \langle \psi_{+,n} | \rho_n | \psi_{-,n} \rangle + O(\delta_t^2)$$

??

We have three leftovers:

- $\langle \psi_{+,n+1} | \psi_{+,n} \rangle$ and $\langle \psi_{-,n} | \psi_{-,n+1} \rangle$ from the previous matrix element calculation
- $e^{-\psi^*_{+,n}\psi_{+,n} - \psi^*_{-,n}\psi_{-,n}}$ from ρ_n
- $e^{-\psi^*_{+,n+1}\psi_{+,n+1} - \psi^*_{-,n+1}\psi_{-,n+1}}$ from ρ_{n+1}

$$\rho_n = \int \frac{d\psi_{+,n} d\psi^*_{+,n}}{\pi} \frac{d\psi_{-,n} d\psi^*_{-,n}}{\pi} e^{-\psi^*_{+,n}\psi_{+,n} - \psi^*_{-,n}\psi_{-,n}} \langle \psi_{+,n} | \rho_n | \psi_{-,n} \rangle \, | \psi_{+,n} \rangle \langle \psi_{-,n} |$$

Master equation and path integrals

Remember we were starting from here

what happens to this?

$$\rho_{n+1} = \int \frac{d\psi_{+,n+1} d\psi^*_{+,n+1}}{\pi} \frac{d\psi_{-,n+1} d\psi^*_{-,n+1}}{\pi} e^{-\psi^*_{+,n+1}\psi_{+,n+1} - \psi^*_{-,n+1}\psi_{-,n+1}} \langle \psi_{+,n+1} | \rho_{n+1} | \psi_{-,n+1} \rangle | \psi_{+,n+1} \rangle \langle \psi_{-,n+1} |$$

what happens to this?

$$\rho_{n+1} = e^{\delta_t \mathcal{L}} \rho_n = (\mathbb{1} + \delta_t \mathcal{L}) \rho_n + O(\delta_t^2).$$

And now we can re-exponentiate the outcome of the previous calculation:

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??

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- $\langle \psi_{+,n+1} | \psi_{+,n} \rangle$ and $\langle \psi_{-,n} | \psi_{-,n+1} \rangle$ from the previous matrix element calculation
- $e^{-\psi^*_{+,n}\psi_{+,n} - \psi^*_{-,n}\psi_{-,n}}$
- $e^{-\psi^*_{+,n+1}\psi_{+,n+1} - \psi^*_{-,n+1}\psi_{-,n+1}}$ from ρ_{n+1}

$\longrightarrow$

- $\langle \psi_{+,n+1} | \psi_{+,n} \rangle = e^{\psi^*_{+,n+1}\psi_{+,n}}$ and $\langle \psi_{-,n} | \psi_{-,n+1} \rangle = e^{\psi^*_{-,n}\psi_{-,n+1}}$
- $e^{-\psi^*_{+,n}\psi_{+,n} - \psi^*_{-,n}\psi_{-,n}}$

$\longrightarrow$

$$\longrightarrow \quad e^{\psi^*_{+,n+1}\psi_{+,n}} e^{-\psi^*_{+,n}\psi_{+,n}} = e^{(\psi^*_{+,n+1} - \psi^*_{+,n})\psi_{+,n}} = e^{-i^2 \delta_t \frac{(\psi^*_{+,n+1} - \psi^*_{+,n})}{\delta_t} \psi_{+,n}} = e^{i\delta_t(-i\psi_{+,n}\partial_t \psi^*_{+,n})}$$

Master equation and path integrals

By iteration of Eq. (23), the density matrix can be evolved from $\rho(t_0)$ at t_0 to $\rho(t_f)$ at $t_f = t_N$. This leads in the limit $N \to \infty$ (and hence $\delta_t \to 0$) to

$$Z_{t_f,t_0} = \operatorname{tr}\rho(t_f) = \operatorname{tr} e^{(t_f - t_0)\mathcal{L}}\rho(t_0) = \int \mathscr{D}[\psi_+, \psi_+^*, \psi_-, \psi_-^*]\, e^{iS}\, \langle \psi_+(t_0)|\rho(t_0)|\psi_-(t_0)\rangle\,, \tag{25}$$

ignored for open quantum systems, where initial conditions can be neglected for $t_0 \to -\infty, t \to \infty$

where the integration measure is given by

$$\mathscr{D}[\psi_+, \psi_+^*, \psi_-, \psi_-^*] = \lim_{N\to\infty} \prod_{n=0}^{N} \frac{d\psi_{+,n} d\psi_{+,n}^*}{\pi} \frac{d\psi_{-,n} d\psi_{-,n}^*}{\pi}\,, \tag{26}$$

and the Keldysh action reads

$$S = \int_{t_0}^{t_f} dt\, (\psi_+^* i\partial_t \psi_+ - \psi_-^* i\partial_t \psi_- - i\mathcal{L}(\psi_+^*, \psi_+, \psi_-^*, \psi_-))\,. \tag{27}$$

just continuous version of what we just did

$$\mathcal{L}(\psi_+^*, \psi_+, \psi_-^*, \psi_-) = -i\,(H_+ - H_-) + \sum_\alpha \gamma_\alpha \left[L_{\alpha,+} L_{\alpha,-}^* - \frac{1}{2}\left(L_{\alpha,+}^* L_{\alpha,+} + L_{\alpha,-}^* L_{\alpha,-} \right) \right], \tag{28}$$

where $H_\pm = H(\psi_\pm^*, \psi_\pm)$ contains fields on the $\pm$ contour only, and the same is true for $L_{\alpha,\pm}$. We clearly recognize the Lindblad superoperator structure of Eq. (16): operators acting on the density matrix from the left (right) reside on the forward, + (backward, -) contour. This gives a simple and direct translation ta-

Keldysh basis

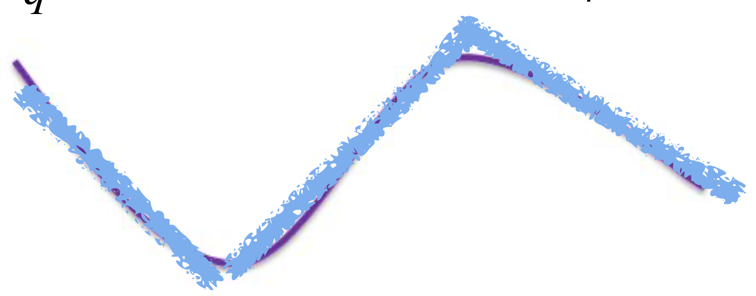
^ L. V. Keldysh (1965). "Diagram technique for nonequilibrium processes". *Soviet Physics JETP*. **20**: 1018–1026.

$$\phi_c = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-), \quad \phi_q = \frac{1}{\sqrt{2}}(\psi_+ - \psi_-)$$

useful for physical interpretation and avoid redundancies in the action

$\partial_t \phi_c =$ (classical eq. of motion)

ϕ_q describe flucts. on top of classical eq. of motion



if $T \to 0$ these flucts. are purely quantum

in open q. systems, this is a mix of classical and quantum

want to know more? ask me in pvt!

Let's try to figure this out with one example:

$$H = \omega_0 a^\dagger a, \quad L = \sqrt{2\kappa}\, a$$

$$S = \int_t \{a_+^* (i\partial_t - \omega_0)\, a_+ - a_-^* (i\partial_t - \omega_0)\, a_- \\ -i\kappa\, [2a_+ a_-^* - (a_+^* a_+ + a_-^* a_-)]\}$$

$$\mathcal{L}(\psi_+^*, \psi_+, \psi_-^*, \psi_-) = -i\,(H_+ - H_-) \\ + \sum_\alpha \gamma_\alpha \left[L_{\alpha,+} L_{\alpha,-}^* - \frac{1}{2}\left(L_{\alpha,+}^* L_{\alpha,+} + L_{\alpha,-}^* L_{\alpha,-}\right)\right], \quad (28)$$

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$$\phi_c = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-)\,, \quad \phi_q = \frac{1}{\sqrt{2}}(\psi_+ - \psi_-)$$

→ conservation probability (cf. Sieberer et al. arXiv1512, page 17)

$$S = \int_\omega \left(a_c^*(\omega), a_q^*(\omega)\right) \begin{pmatrix} 0 & P^A(\omega) \\ P^R(\omega) & P^K \end{pmatrix} \begin{pmatrix} a_c(\omega) \\ a_q(\omega) \end{pmatrix}$$

$$P^R(\omega) = P^A(\omega)^* = \omega - \omega_0 + i\kappa, \quad P^K = 2i\kappa$$

What's the physical content of this action?

imagine to compute expectation value of some observable O

$$\langle O \rangle = \int Da_{cl} Da_{cl}^* \int Da_q Da_q^* \; O \; e^{iS[a_{cl}, a_{cl}^*, a_q, a_q^*]}$$

auxiliary variable

notice that the 'Keldysh' term can be written as $e^{-\int dt\,\kappa\, a_q^2(t)} = \int D\xi e^{-\int dt\left[\frac{1}{\kappa}\xi(t)^2 - 2i\xi(t)a_q(t)\right]}$ using Gaussian integration formula

which results in $\langle O \rangle = \int D\xi e^{-\int dt \frac{1}{\kappa}\xi(t)^2} \int Da_{cl} \ldots \int Da_q \ldots \; O \; e^{-2i\int dt a_q(\kappa\partial_t a_{cl} + \omega_0 a_{cl} - \xi(t))}$

Keldysh basis

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$$S = \int_t \{a_+^* (i\partial_t - \omega_0) a_+ - a_-^* (i\partial_t - \omega_0) a_- \\ -i\kappa [2a_+a_-^* - (a_+^* a_+ + a_-^* a_-)]\}$$

$$\phi_c = \frac{1}{\sqrt{2}}(\psi_+ + \psi_-), \quad \phi_q = \frac{1}{\sqrt{2}}(\psi_+ - \psi_-)$$

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$$P^R(\omega) = P^A(\omega)^* = \omega - \omega_0 + i\kappa, \quad P^K = 2i\kappa$$

What's the physical content of this action?

- the loss rate in the *retarded* sector (cl-q) is a damping term;
- the constant term in the *Keldysh* sector (q-q) has the meaning of noise variance

imagine to compute expectation value of some observable O

$$\langle O \rangle = \int Da_{cl} Da_{cl}^* \int Da_q Da_q^* \ O \ e^{iS[a_{cl}, a_{cl}^*, a_q, a_q^*]}$$

auxiliary variable

notice that the 'Keldysh' term can be written as $e^{-\int dt\ \kappa\ a_q^2(t)} = \int D\xi e^{-\int dt \left[\frac{1}{\kappa}\xi(t)^2 - 2i\xi(t) a_q(t)\right]}$ using Gaussian integration formula

Langevin equation

which results in $\langle O \rangle = \int D\xi e^{-\int dt \frac{1}{\kappa}\xi(t)^2} \int Da_{cl}.. \int Da_q.. \ O \ e^{-2i\int dt a_q(\kappa\partial_t a_{cl} + \omega_0 a_{cl} - \xi(t))} = \int D\xi e^{-\int dt \frac{1}{\kappa}\xi(t)^2} \int Da_{cl}.. \int Da_q.. \ O \ \delta(\kappa\partial_t a_{cl} + \omega_0 a_{cl} - \xi(t))$

Keldysh basis

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$$S = \int_t \{a_+^* \left(i\partial_t - \omega_0\right) a_+ - a_-^* \left(i\partial_t - \omega_0\right) a_- \\ -i\kappa \left[2a_+ a_-^* - \left(a_+^* a_+ + a_-^* a_-\right)\right]\}$$

$$\phi_c = \frac{1}{\sqrt{2}}\left(\psi_+ + \psi_-\right), \quad \phi_q = \frac{1}{\sqrt{2}}\left(\psi_+ - \psi_-\right)$$

→ conservation probability (cf. Sieberer et al. arXiv1512, page 17)

$$S = \int_\omega \left(a_c^*(\omega), a_q^*(\omega)\right) \begin{pmatrix} 0 & P^A(\omega) \\ P^R(\omega) & P^K \end{pmatrix} \begin{pmatrix} a_c(\omega) \\ a_q(\omega) \end{pmatrix}$$

$$P^R(\omega) = P^A(\omega)^* = \omega - \omega_0 + i\kappa, \quad P^K = 2i\kappa$$

Exercise 2

Write the Keldysh action for the anharmonic oscillator $H = \omega_0 a^\dagger a + \lambda(a^\dagger a)^2$ (careful about normal ordering!)

Write the Keldysh action for the anharmonic oscillator $H = \omega_0 a^\dagger a + \lambda(a^\dagger a)^2$ with two body loss $L = \sqrt{\gamma} a^2$

and interpret the result distinguishing the effect of γ in the retarded and Keldysh sectors

Computing non-equilibrium Green's functions

- compute them: introduce sources (cf. Stat Mech)

$$Z = \mathrm{Tr}(1 \cdot \rho) = \langle 1 \rangle$$

$$Z[j_+, j_-] = \langle e^{\mathrm{i}\int (j_+ \phi_+^* + j_- \phi_-^* + c.c.)} \rangle \qquad Z[0,0] = \langle 1 \rangle = 1$$

normalization

- example

$$\langle \mathcal{T}_C[\hat{\phi}^\dagger(t)\hat{\phi}(t')]\rangle = \frac{\delta^2 Z[j_+, j_-]}{\delta j_+(t)\delta j_+^*(t')}\Big|_{j=0}$$

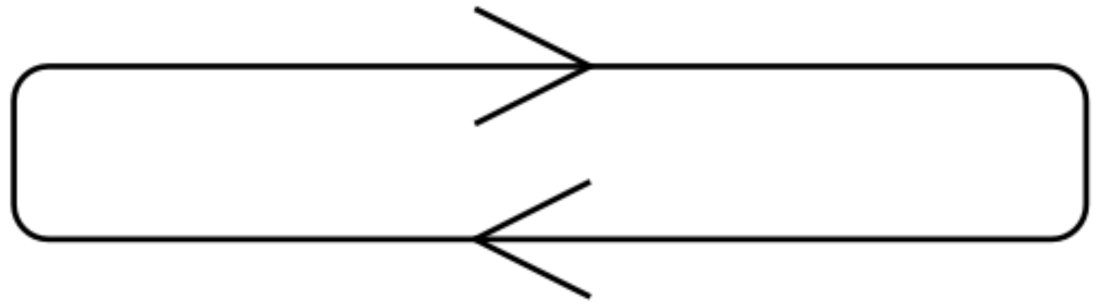
NB: Functional integrals always compute time-ordered correlation functions

But …

- what does physically mean the index ±?
- how can I connect operator exp .value with path integral?
- what's the relation between corr. functions in ± basis and c/q basis?
- how these correlation functions connect with conventional ones? (e.g. retarded Green's function)

Computing non-equilibrium Green's functions

- Along the contour, we imagine that an initial state $\hat{\rho}(t=0)$ is evolved from $t=0$ to the final time $\hat{\rho}(t=t_\mathrm{f})$ for some $t_\mathrm{f}>0$. Then the trace is performed. Pictorially this looks like



- We now switch to a Heisenberg picture for operators

$$\hat{\phi}(x,t) \equiv e^{i\hat{H}t} \underbrace{\hat{\phi}(x)}_{\text{Schrödinger operator}} e^{-i\hat{H}t}, \qquad \hat{\phi}^\dagger(x',t') \equiv e^{i\hat{H}t'}\hat{\phi}^\dagger(x')e^{-i\hat{H}t'}.$$

and $0<t,t'<t_\mathrm{f}$.

- **Definition:** a fixed pair of times $0<t,t'<t_\mathrm{f}$, we can define two independent Green's functions for an initial state $\hat{\rho}$:

$$\text{"G-greater"}: \quad G^>(x,t,x',t') \equiv -i\langle\hat{\phi}(x,t)\hat{\phi}^\dagger(x',t')\rangle = -i\,\mathrm{tr}(\hat{\phi}(x,t)\hat{\phi}^\dagger(x',t')\hat{\rho}),$$

$$\text{"G-lesser"}: \quad G^<(x,t,x',t') \equiv -i\langle\hat{\phi}^\dagger(x',t')\hat{\phi}(x,t)\rangle = -i\,\mathrm{tr}(\hat{\phi}^\dagger(x',t')\hat{\phi}(x,t)\hat{\rho}).$$

- **Note:** It is convention for $G^{<,>}(\underbrace{x,t}_{\text{annihilation}}, \underbrace{x',t'}_{\text{creation}})$ that the arguments on the left correspond to annihilation and those on the right to creation operators. $G^{<,>}$ exchange **the order** of creation/annihilation **not their arguments**.

Computing non-equilibrium Green's functions

Definition: a fixed pair of times $0 < t, t' < t_\mathrm{f}$, we can define two independent Green's functions for an initial state $\hat{\rho}$:

$$\text{"G-greater"}: \quad G^>(x,t,x',t') \equiv -i\langle \hat{\phi}(x,t)\hat{\phi}^\dagger(x',t')\rangle = -i\,\mathrm{tr}(\hat{\phi}(x,t)\hat{\phi}^\dagger(x',t')\hat{\rho}),$$

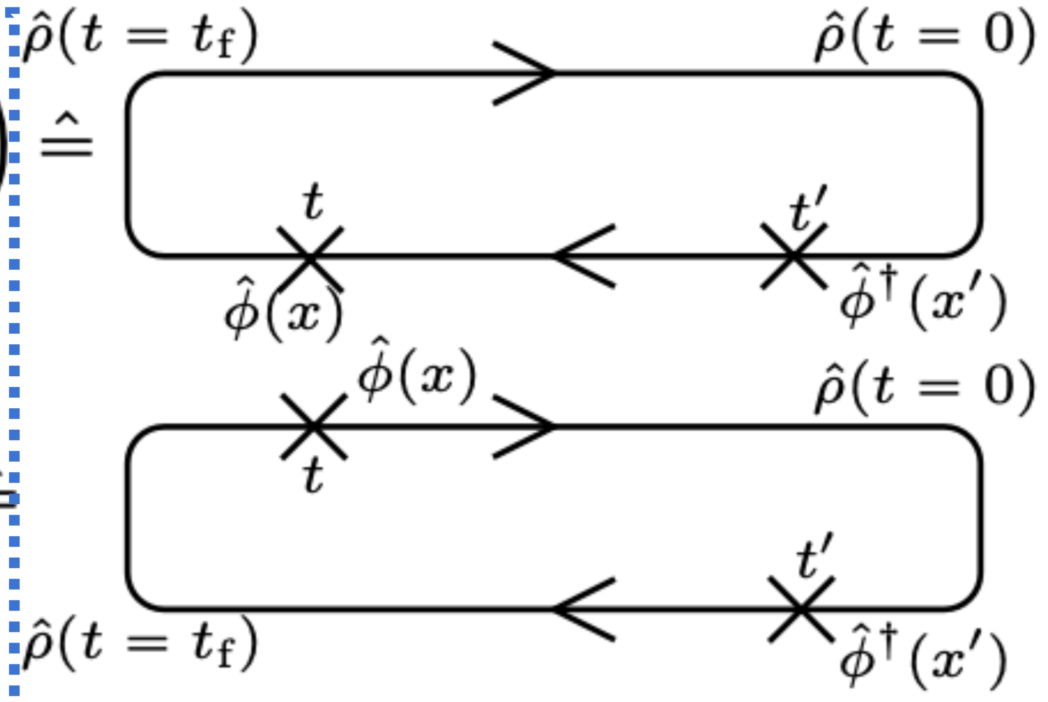
$$\text{"G-lesser"}: \quad G^<(x,t,x',t') \equiv -i\langle \hat{\phi}^\dagger(x',t')\hat{\phi}(x,t)\rangle = -i\,\mathrm{tr}(\hat{\phi}^\dagger(x',t')\hat{\phi}(x,t)\hat{\rho}).$$

- For the path integral, we need to distinguish two cases:

  (i) $\underline{t > t'}$: We can write this as

$$iG^>(x,t,x',t') = \mathrm{tr}\left(\underbrace{e^{-i\hat{H}(t_\mathrm{f}-t)}\hat{\phi}(x)e^{-i\hat{H}(t-t')}\;\hat{\phi}(x')^\dagger e^{-i\hat{H}t'}}_{\text{forward}}\underbrace{\hat{\rho}_0 e^{i\hat{H}t_\mathrm{f}}}_{\text{backward}}\right) \hat{=}$$

$$= \mathrm{tr}\left(\underbrace{e^{-i\hat{H}(t_\mathrm{f}-t')}\hat{\phi}^\dagger(x')e^{-i\hat{H}t'}}_{\text{forward}}\underbrace{\hat{\rho}_0 e^{i\hat{H}t}\hat{\phi}(x)e^{i\hat{H}(t_\mathrm{f}-t)}}_{\text{backward}}\right) \hat{=}$$



$$\equiv \langle \phi_+(x,t)\phi_+^*(x',t')\rangle$$

$$\equiv \langle \phi_-(x,t)\phi_+^*(x',t')\rangle$$

operator averages

path integral averages

$$iG^>(x,t,x',t') = \begin{cases} \int \mathcal{D}[\phi,\phi^*]\phi_+(x,t)\phi_+^*(x',t')e^{iS} \equiv \langle \phi_+(x,t)\phi_+^*(x',t')\rangle \\ \int \mathcal{D}[\phi,\phi^*]\phi_-(x,t)\phi_+^*(x',t')e^{iS} \equiv \langle \phi_-(x,t)\phi_+^*(x',t')\rangle \end{cases}$$

Computing non-equilibrium Green's functions

- **Definition:** For a fixed pair of quantum states x, x' and a fixed pair of times $0 < t, t' < t_\mathrm{f}$, we can define two independent Green's functions for an initial state $\hat{\rho}$:

$$\text{"G-greater"}: \quad G^>(x,t,x',t') \equiv -i\langle\hat{\phi}(x,t)\hat{\phi}^\dagger(x',t')\rangle = -i\,\mathrm{tr}(\hat{\phi}(x,t)\hat{\phi}^\dagger(x',t')\hat{\rho}),$$
$$\text{"G-lesser"}: \quad G^<(x,t,x',t') \equiv -i\langle\hat{\phi}^\dagger(x',t')\hat{\phi}(x,t)\rangle = -i\,\mathrm{tr}(\hat{\phi}^\dagger(x',t')\hat{\phi}(x,t)\hat{\rho}).$$

The we find with the same logic as above for $t < t'$

$$iG^>(x,t,x',t') = \begin{cases} \int \mathcal{D}[\phi,\phi^*]\phi_-(x,t)\phi_+^*(x',t')e^{iS} \equiv \langle\phi_-(x,t)\phi_+^*(x',t')\rangle \\ \int \mathcal{D}[\phi,\phi^*]\phi_-(x,t)\phi_-^*(x',t')e^{iS} \equiv \langle\phi_-(x,t)\phi_-^*(x',t')\rangle \end{cases}$$

- **In summary**

$$iG^>(x,t,x',t') = \underbrace{\langle\hat{\phi}(x,t)\hat{\phi}^\dagger(x',t')\rangle}_{\text{operator average}} = \begin{cases} \underbrace{\langle\phi_-(x,t)\phi_+^*(x',t')\rangle}_{\text{path integral average}} & \text{for } t,t' \text{ arbitrary} \\ \langle\phi_+(x,t)\phi_+^*(x',t')\rangle & \text{for } t > t' \\ \langle\phi_-(x,t)\phi_-^*(x',t')\rangle & \text{for } t < t'. \end{cases}$$

- The same considerations can be repeated for $G^<(x,t,x',t')$

Computing non-equilibrium Green's functions

- **In summary**

$$iG^>(x,t,x',t') = \underbrace{\langle \hat\phi(x,t)\hat\phi^\dagger(x',t')\rangle}_{\text{operator average}} = \begin{cases} \underbrace{\langle \phi_-(x,t)\phi_+^*(x',t')\rangle}_{\text{path integral average}} & \text{for } t,t' \text{ arbitrary} \\ \langle \phi_+(x,t)\phi_+^*(x',t')\rangle & \text{for } t > t' \\ \langle \phi_-(x,t)\phi_-^*(x',t')\rangle & \text{for } t < t'. \end{cases}$$

- The same considerations can be repeated for $G^<(x,t,x',t')$. This yields

Path integral contour Green's functions:
On the $\pm$-contour, we can define **four** possible two-point correlations.
no time ordering:

$$\langle \phi_-(x,t)\phi_+^*(x',t')\rangle = \langle \hat\phi(x,t)\hat\phi^\dagger(x',t')\rangle = iG^>(x,t,x',t')$$
$$\langle \phi_+(x,t)\phi_-^*(x',t')\rangle = \langle \hat\phi^\dagger(x',t')\hat\phi(x,t)\rangle = iG^<(x,t,x',t')$$

time-ordered:

$$\langle \phi_+(x,t)\phi_+^*(x',t')\rangle = \Theta(t-t')\langle \hat\phi(x,t)\hat\phi^\dagger(x',t')\rangle + \Theta(t'-t)\langle \hat\phi^\dagger(x',t')\hat\phi(x,t)\rangle$$

anti-time-ordered:

$$\underbrace{\langle \phi_-(x,t)\phi_-^*(x',t')\rangle}_{\substack{\text{path integral averages} \\ \langle\ldots\rangle=\int \mathcal{D}[\phi,\phi^*]\ldots e^{iS}}} = \underbrace{\Theta(t'-t)\langle \hat\phi(x,t)\hat\phi^\dagger(x',t')\rangle + \Theta(t-t')\langle \hat\phi^\dagger(x',t')\hat\phi(x,t)\rangle}_{\text{operator averages } \langle\ldots\rangle=\mathrm{tr}(\ldots\hat\rho)}$$

Computing non-equilibrium Green's functions

Relation to cl/q Green's functions

$$G^R(t,t') = -i\langle a_c(t)a_q^*(t')\rangle$$
$$= -\frac{i}{2}\langle (a_+(t) + a_-(t))\,(a_+^*(t') - a_-^*(t'))\rangle$$

$t > t'$

$\langle a(t)a^\dagger(t')\rangle + \langle a(t)a^\dagger(t')\rangle - \langle a^\dagger(t')a(t)\rangle - \langle a^\dagger(t')a(t)\rangle = 2\langle [a(t), a^\dagger(t')]\rangle$

$t' > t$

$\langle a^\dagger(t')a(t)\rangle + \langle a(t)a^\dagger(t')\rangle - \langle a^\dagger(t')a(t)\rangle - \langle a(t)a^\dagger(t')\rangle$

Exercise

Repeat the same for G^K

$$G^K(t,t') = -i\langle\{a(t), a^\dagger(t')\}\rangle$$

Reminder:

Path integral contour Green's functions:
On the $\pm$-contour, we can define **four** possible two-point correlations.
no time ordering:

$$\langle\phi_-(x,t)\phi_+^*(x',t')\rangle = \langle\hat\phi(x,t)\hat\phi^\dagger(x',t')\rangle = iG^>(x,t,x',t')$$
$$\langle\phi_+(x,t)\phi_-^*(x',t')\rangle = \langle\hat\phi^\dagger(x',t')\hat\phi(x,t)\rangle = iG^<(x,t,x',t')$$

time-ordered: $\langle T\hat\phi(x,t)\hat\phi^\dagger(x',t')\rangle$

$$\langle\phi_+(x,t)\phi_+^*(x',t')\rangle = \Theta(t-t')\langle\hat\phi(x,t)\hat\phi^\dagger(x',t')\rangle + \Theta(t'-t)\langle\hat\phi^\dagger(x',t')\hat\phi(x,t)\rangle$$

anti-time-ordered: $\langle\tilde T\hat\phi(x,t)\hat\phi^\dagger(x',t')\rangle$

$$\underbrace{\langle\phi_-(x,t)\phi_-^*(x',t')\rangle}_{\substack{\text{path integral averages}\\ \langle\ldots\rangle=\int\mathcal{D}[\phi,\phi^*]\ldots e^{iS}}} = \underbrace{\Theta(t'-t)\langle\hat\phi(x,t)\hat\phi^\dagger(x',t')\rangle + \Theta(t-t')\langle\hat\phi^\dagger(x',t')\hat\phi(x,t)\rangle}_{\text{operator averages } \langle\ldots\rangle=\mathrm{tr}(\ldots\hat\rho)}$$

More on R and K Green's functions

- master equation for decaying cavity:

$$\partial_t \rho = -i[\omega_0 \hat{a}^\dagger \hat{a}, \rho] + \kappa(2\hat{a}\rho\hat{a}^\dagger - \{\hat{a}^\dagger \hat{a}, \rho\})$$

- action:

$$S = \int dt (a_{cl}^*, a_q^*) \begin{pmatrix} 0 & i\partial_t - \omega_0 - i\kappa \\ i\partial_t - \omega_0 + i\kappa & 2i\kappa \end{pmatrix} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

time domain

$$a_\nu(t)$$

$$= \int \frac{d\omega}{2\pi} (a_{cl}^*, a_q^*) \begin{pmatrix} 0 & \omega - \omega_0 - i\kappa \\ \underbrace{\omega - \omega_0 + i\kappa} & 2i\kappa \end{pmatrix} \begin{pmatrix} a_{cl} \\ a_q \end{pmatrix}$$

frequency domain

$$a_\nu(\omega)$$

More on R and K Green's functions

From retarded Green function → spectral density $A(\omega) = -2\,\mathrm{Im}\,G^R(\omega).$

which satisfies normalization $\int_\omega A(\omega) = \langle [a, a^\dagger] \rangle = 1$ and contains information on spectral properties of the system

example of the lossy bosonic mode

$$A(\omega) = \frac{2\kappa}{(\omega - \omega_0)^2 + \kappa^2},$$

i.e., it is a Lorentzian, which is centered at the cavity frequency ω_0 and has a half-width at half-maximum given by κ. Note that for $\kappa \to 0$, the photon number states become exact eigenstates and the spectral density reduces to a δ-function peaked at ω_0,

like in equilibrium (cf. Mahan, Negele/Orland, Altland/Simons)

$G^K(t,t') = -i\langle \{a(t), a^\dagger(t')\} \rangle$ $\quad t \to t' \quad$ $iG^K(t,t) = 2\langle a^\dagger(t) a(t) \rangle + 1$

cfr G^R, commutator carries no info on mode occup. ($[a, a^\dagger] = 1$)

example of the lossy bosonic mode

$$\langle a^\dagger a \rangle = \frac{1}{2}\left(i \int_\omega G^K(\omega) - 1\right) = 0.$$

system is empty in the steady state (loss, with no pump)

Relation to equilibrium

At equilibrium there is no need of 2 Green's functions; it is sufficient to specify the response properties (i.e. spectrum), since the occupation is universally given by a Gibbs state (canonical, grandcanonical, etc)

$$\langle a^\dagger a \rangle = n(\omega_0) = \frac{1}{e^{\beta\omega_0} - 1}$$

There is a conceptual way to formalize it, know as *fluctuation-dissipation theorem*, which strongly ties G^R to G^K at equilibrium

$$G^K(\omega) = (2n(\omega) + 1)\left(G^R(\omega) - G^A(\omega)\right)$$

Exercise (pg. 20 of Sieberer et al arXiv1512)

For a bosonic mode at finite temperature, compute $G^{R/K}$ and prove that they satisfy the fluctuation-dissipation theorem:

$$G^R(\omega) = G^A(\omega)^* = \frac{1}{\omega - \omega_0 + i\delta},$$
$$G^K(\omega) = -i2\pi\delta(\omega - \omega_0)\,(2n(\omega) + 1)$$