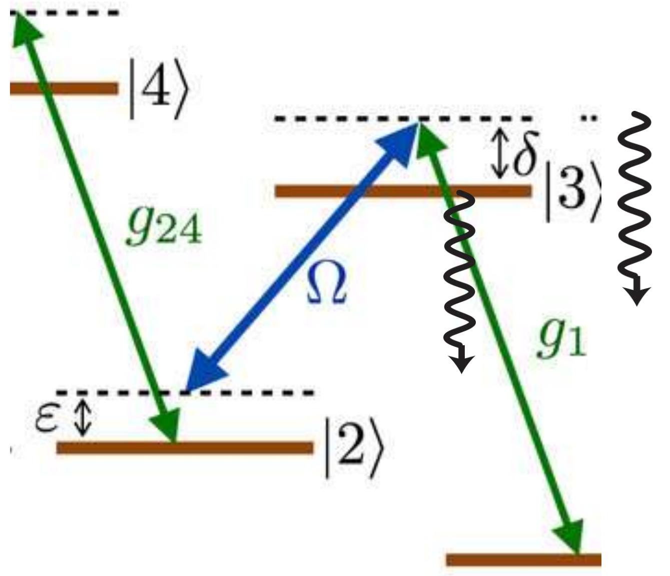
Motivation & purpose

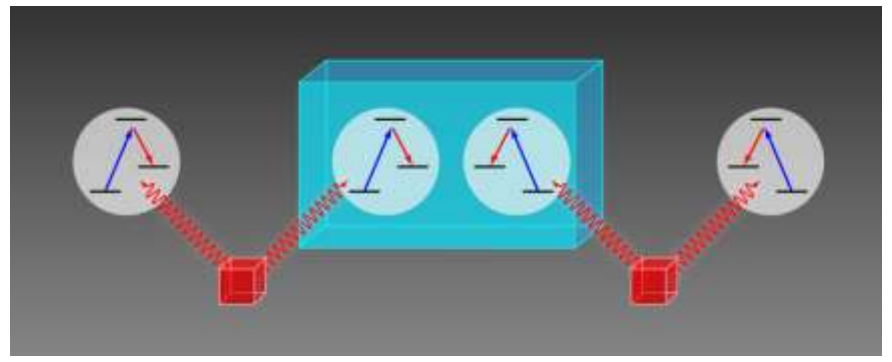
Quantum optics (the course)

Drive

pump of energy/particles



Few atoms & photons



losses

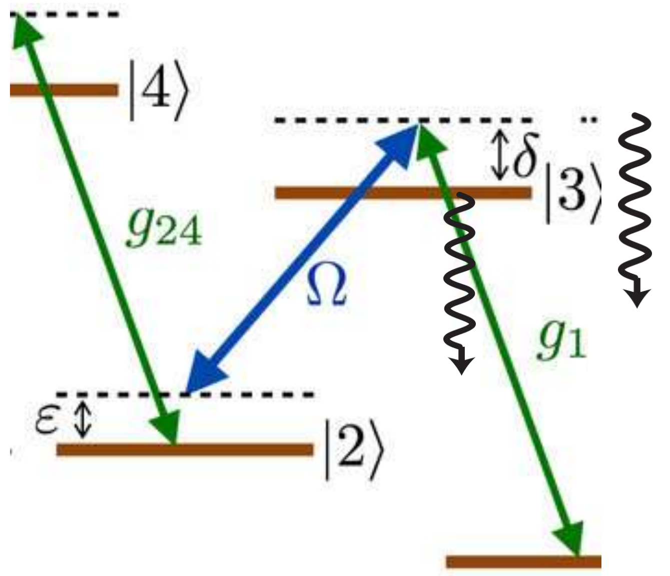
Bath

dissipation
(decoherence/dephasing)

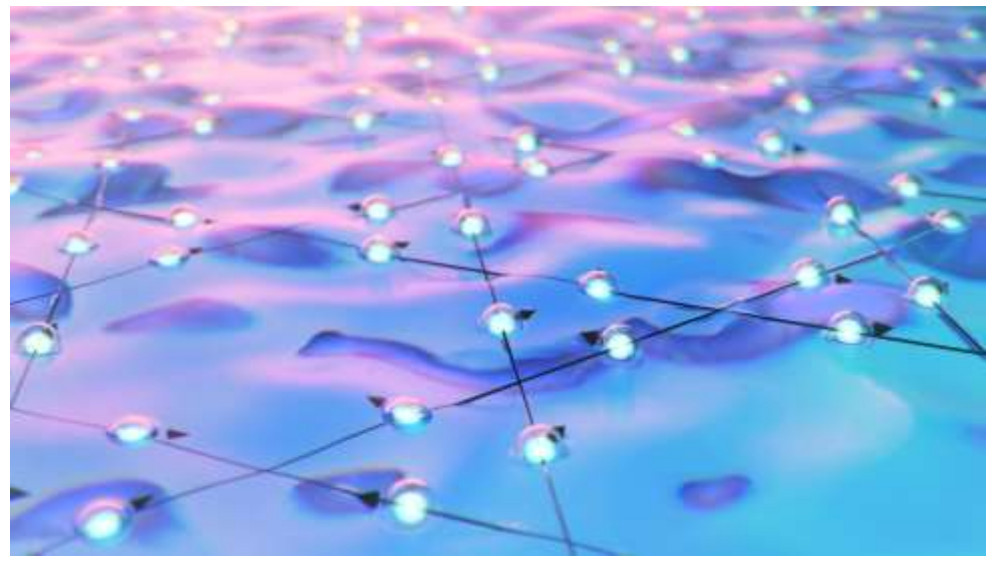
$N \to \infty$ limit of quantum optics (research subject)

Drive

pump of energy/particles



Interacting quantum many-body system

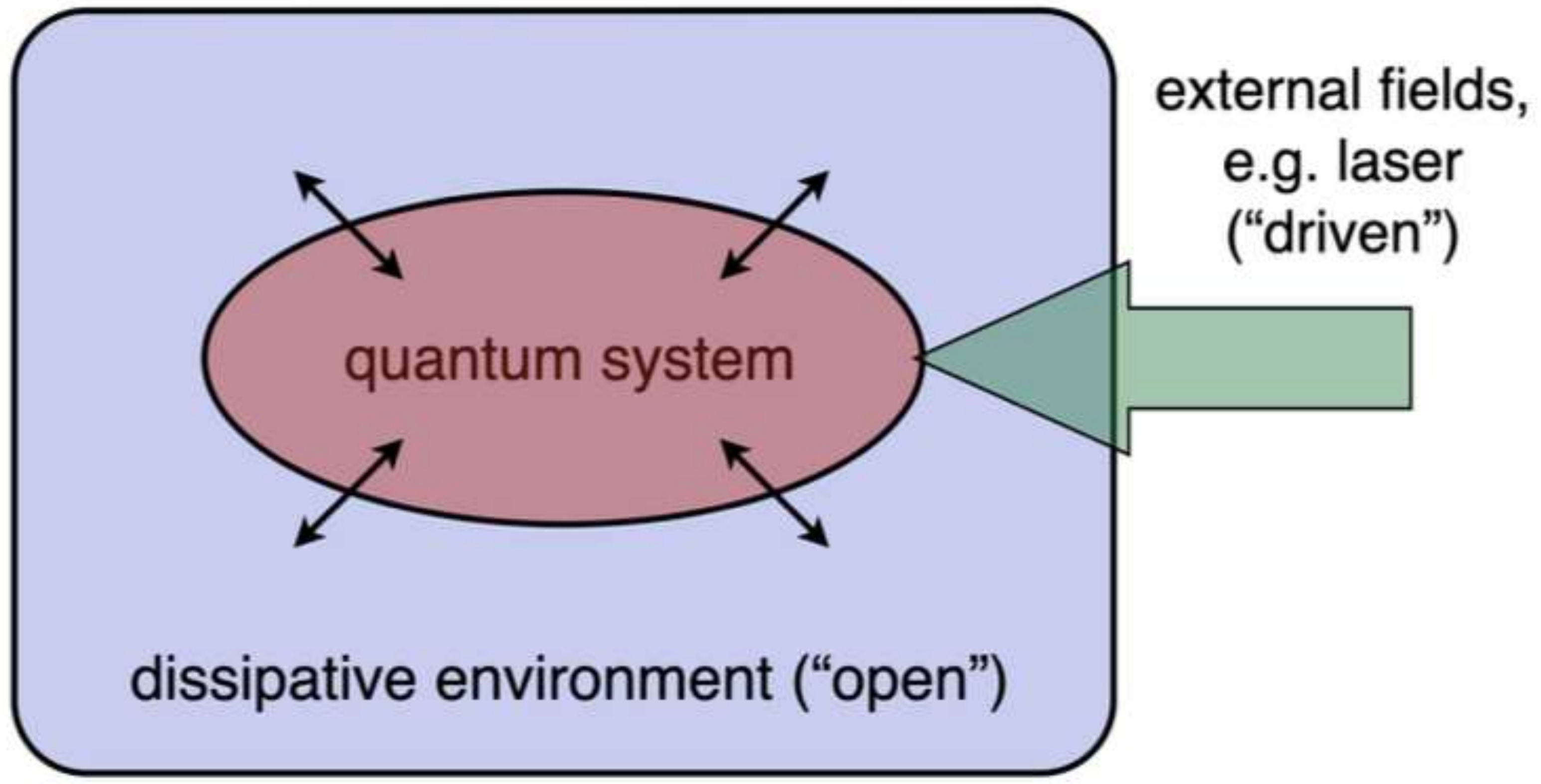


losses

Bath

dissipation
(decoherence/dephasing)

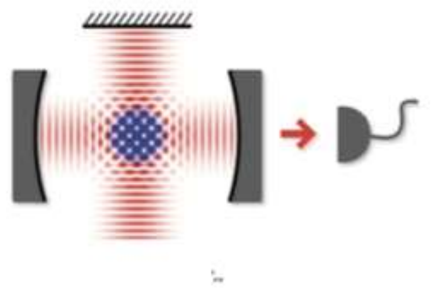
Motivation & purpose



→ <u>in this course:</u>

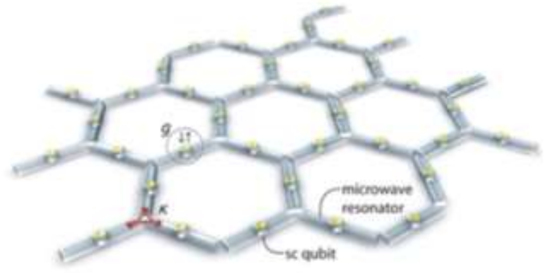
single body problems

Atoms



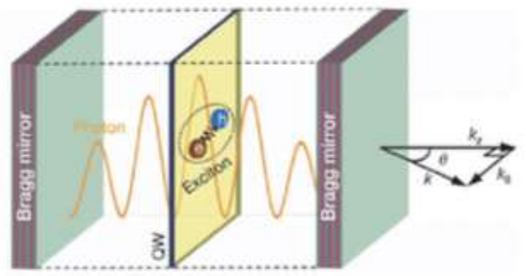
Bose-Einstein condensate in a cavity
Baumann et al., Nature 2010

Light

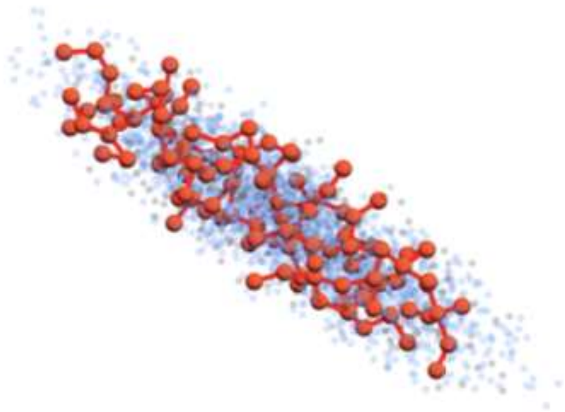


Microcavity arrays
Houck, Türeci, Koch, Nat. Phys. 2012

Solids

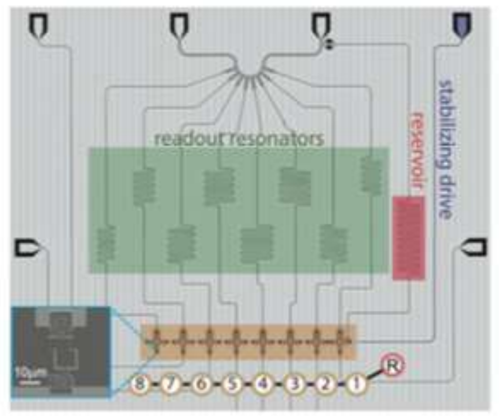


Exciton-polariton condensates
Kasprzak et al., Nature 2006



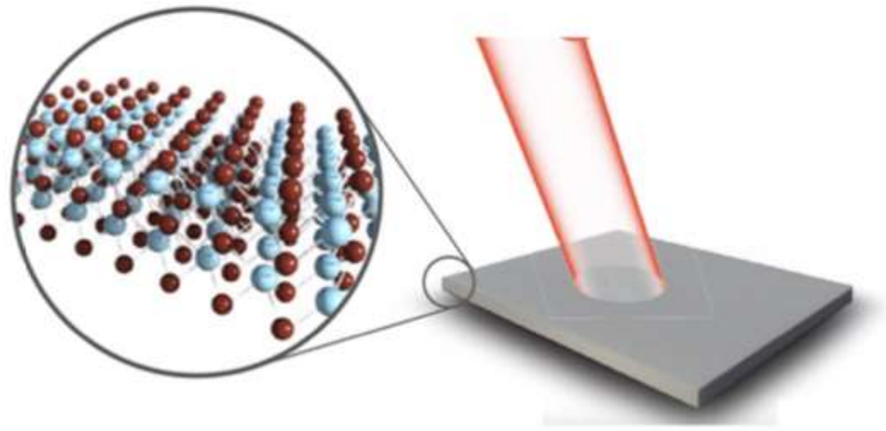
driven-dissipative Rydberg gases

S. Helmrich et al., Nature (2020)



Mott insulator of light

R. Ma et al., Nature (2019)



Ultrafast matter

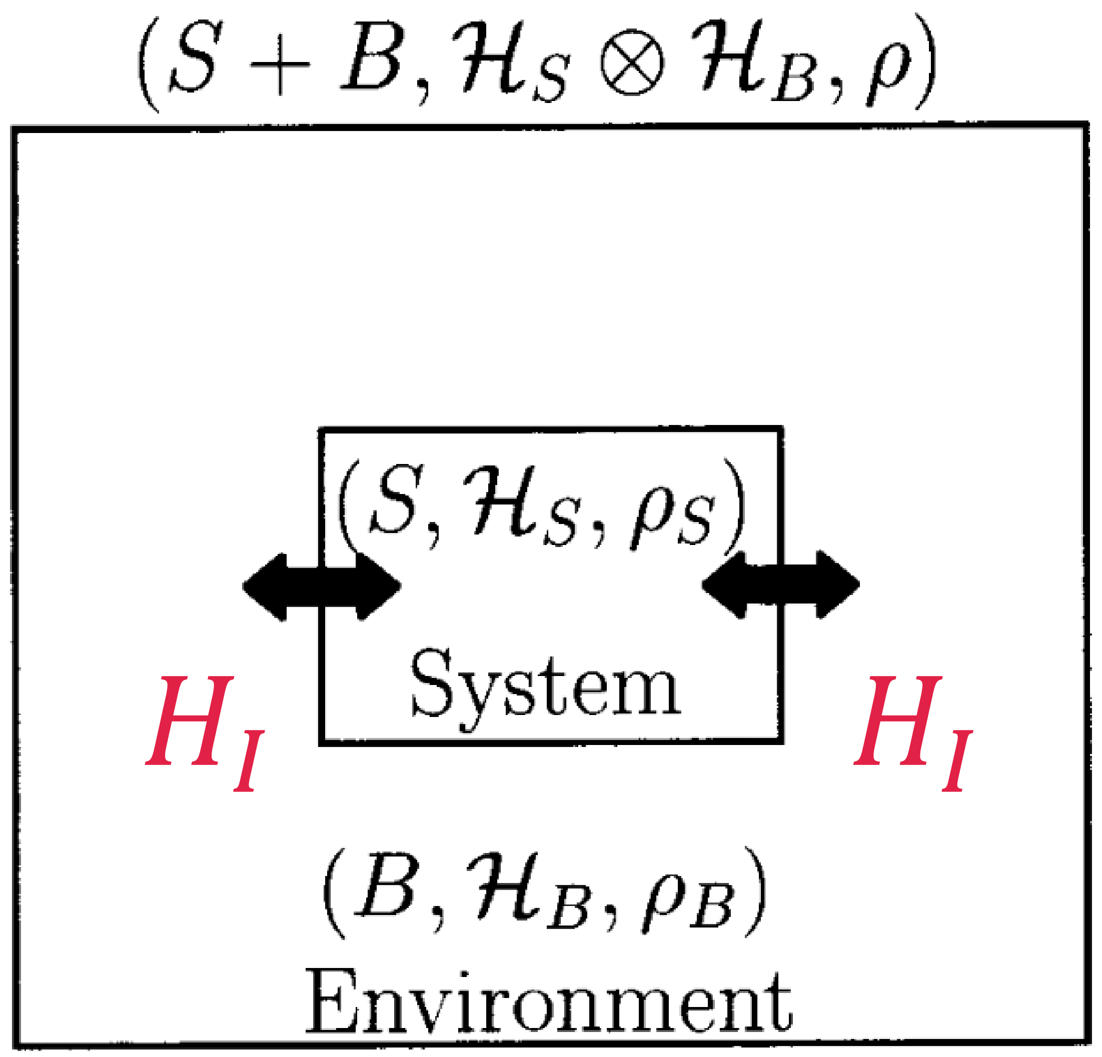
A. de la Torre et al., Rev. Mod. Phys. (2021)

→ <u>in research:</u>

many-body systems

Derivation of quantum master equation

$$H = H_S + H_B + H_I$$



Breuer/Petruccione (Sec. 3.1.2)

→ Starting point is 'interaction picture' dynamics for total density matrix $\rho(t)$

$\rho(t)$ is a state, it makes sense it evolves under H_I

$$\frac{d}{dt}\rho(t) = -i[H_I(t), \rho(t)]$$ or equivalently $$\rho(t) = \rho(0) - i\int_0^t ds[H_I(s), \rho(s)]$$

plug it back!

$$\frac{d}{dt}\rho_S(t) = -\int_0^t ds \mathrm{tr}_B [H_I(t), [H_I(s), \rho(s)]]$$

and do one extra step: take *partial trace* over bath (*B*)

$\rho_S = Tr_B(\rho(t))$

BUT on RHS we have still total density matrix dependence $\rho(s)$

FIRST APPROX: *Born approximation* $\rho(s) \approx \rho_S(t) \otimes \rho_B$

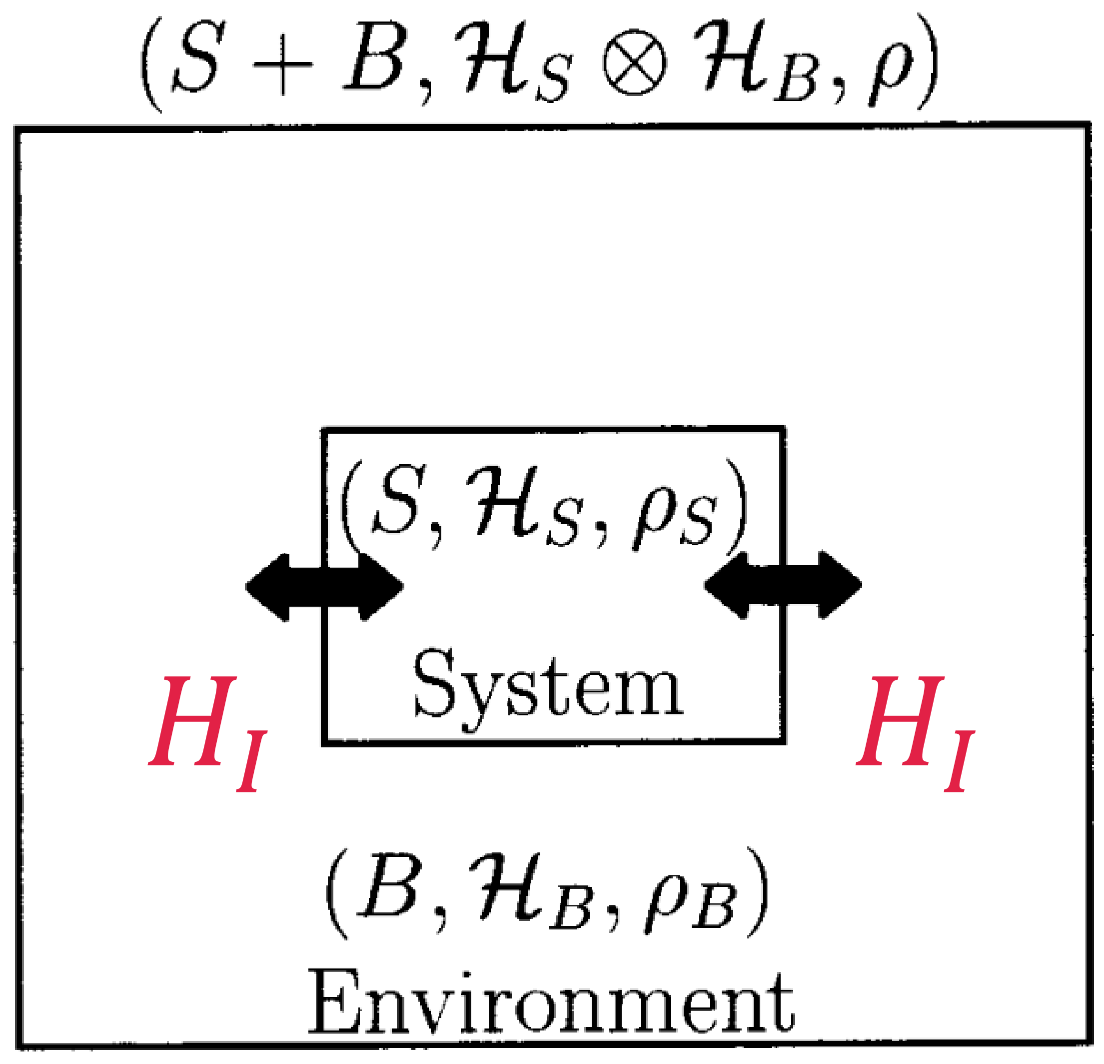
weak system-bath coupling (H_I)

$$\rightarrow \frac{d}{dt}\rho_S(t) = -\int_0^t ds \mathrm{tr}_B [H_I(t), [H_I(s), \rho_S(s) \otimes \rho_B]]$$

source: Breuer/Petruccione 's book

Derivation of quantum master equation

$$H = H_S + H_B + H_I$$



$$\rightarrow \frac{d}{dt}\rho_S(t) = -\int_0^t ds\,\mathrm{tr}_B\,[H_I(t),[H_I(s),\rho_S(s)\otimes\rho_B]]$$

$\rho_S(s) \rightarrow \rho_S(t)$

Redfield equation
(for certain physical systems one has to stop here!)

SECOND APPROX: *Markov approximation* $\rho(s) \approx \rho_S(t) \otimes \rho_B$

→ now perform the simple change of variables $s \rightarrow t - s$

$$\frac{d}{dt}\rho_S(t) = -\int_0 ds\,\mathrm{tr}_B\,[H_I(t),[H_I(t-s),\rho_S(t)\otimes\rho_B]]$$

→ ASSUMPTION on timescales separation:

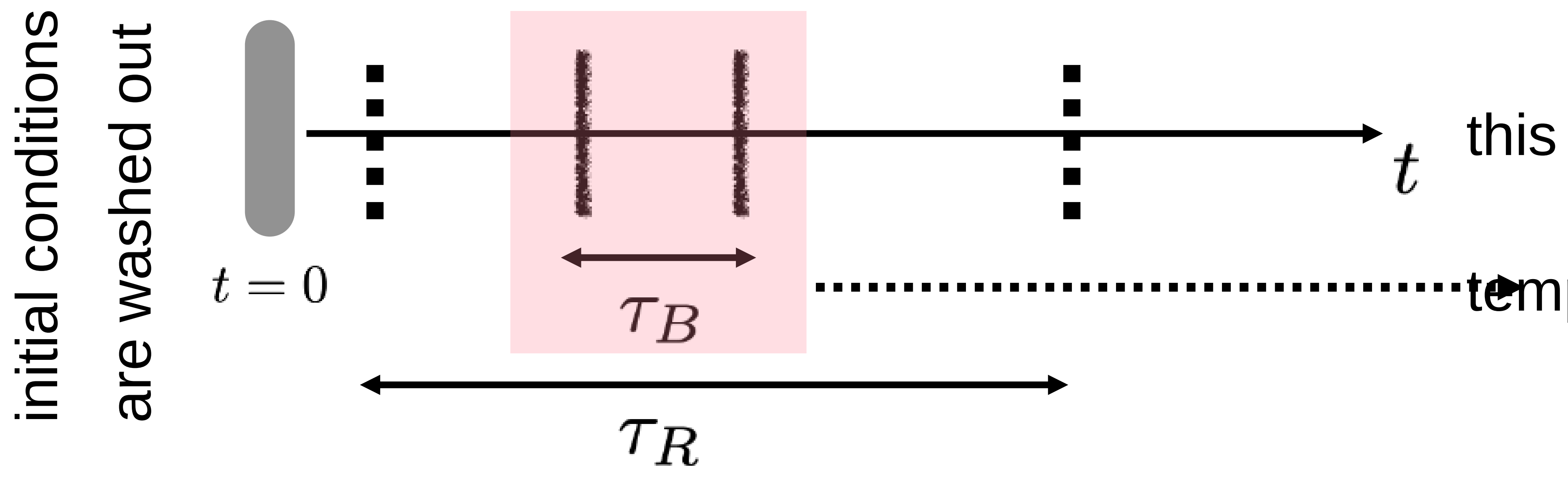
- we want the integrand to decay sufficiently fast for $s \gg \tau_R$ (τ_R is the time scale over which corr. functions of the bath decay)

memory effects with it ($\rho_S(s) \approx \rho_S(t)$)

assuming here infinitely large reservoir, otherwise there are recurrences!

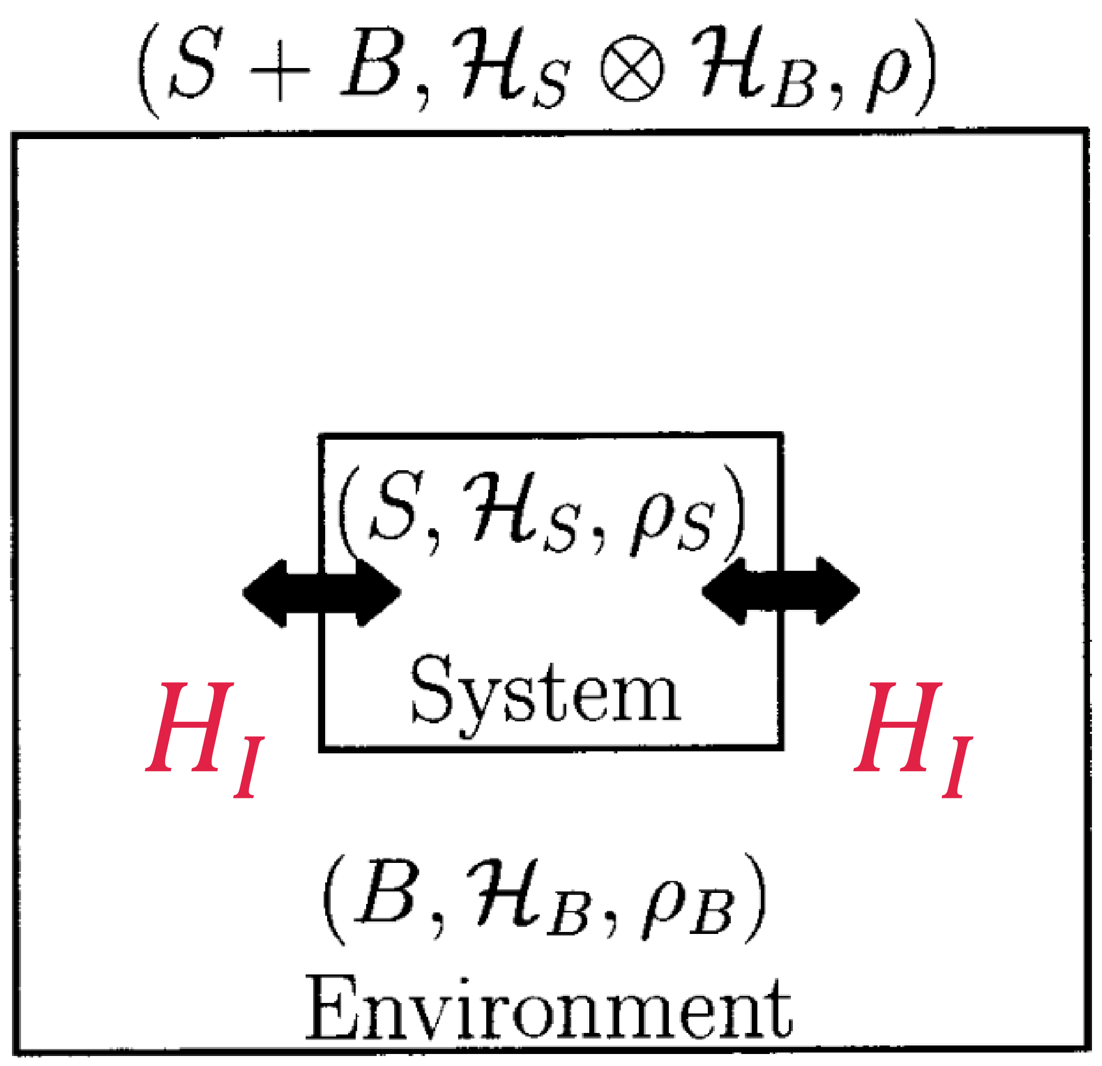
Derivation of quantum master equation

$$\frac{d}{dt}\rho_S(t) = -\int_0^\infty ds \, \mathrm{tr}_B \left[H_I(t), [H_I(t-s), \rho_S(t) \otimes \rho_B]\right]$$



this equation is valid over <u>coarse-grained</u> time scales

temporal effects in the window $\sim \tau_B$ are not resolved



→ our final goal is to convert the integro-differential equation for ρ_s into a differential one with fully local time-dependent terms

$$H_I = \sum_\alpha A_\alpha \otimes B_\alpha \; +h.c.$$

operators of the bath

operators of the system

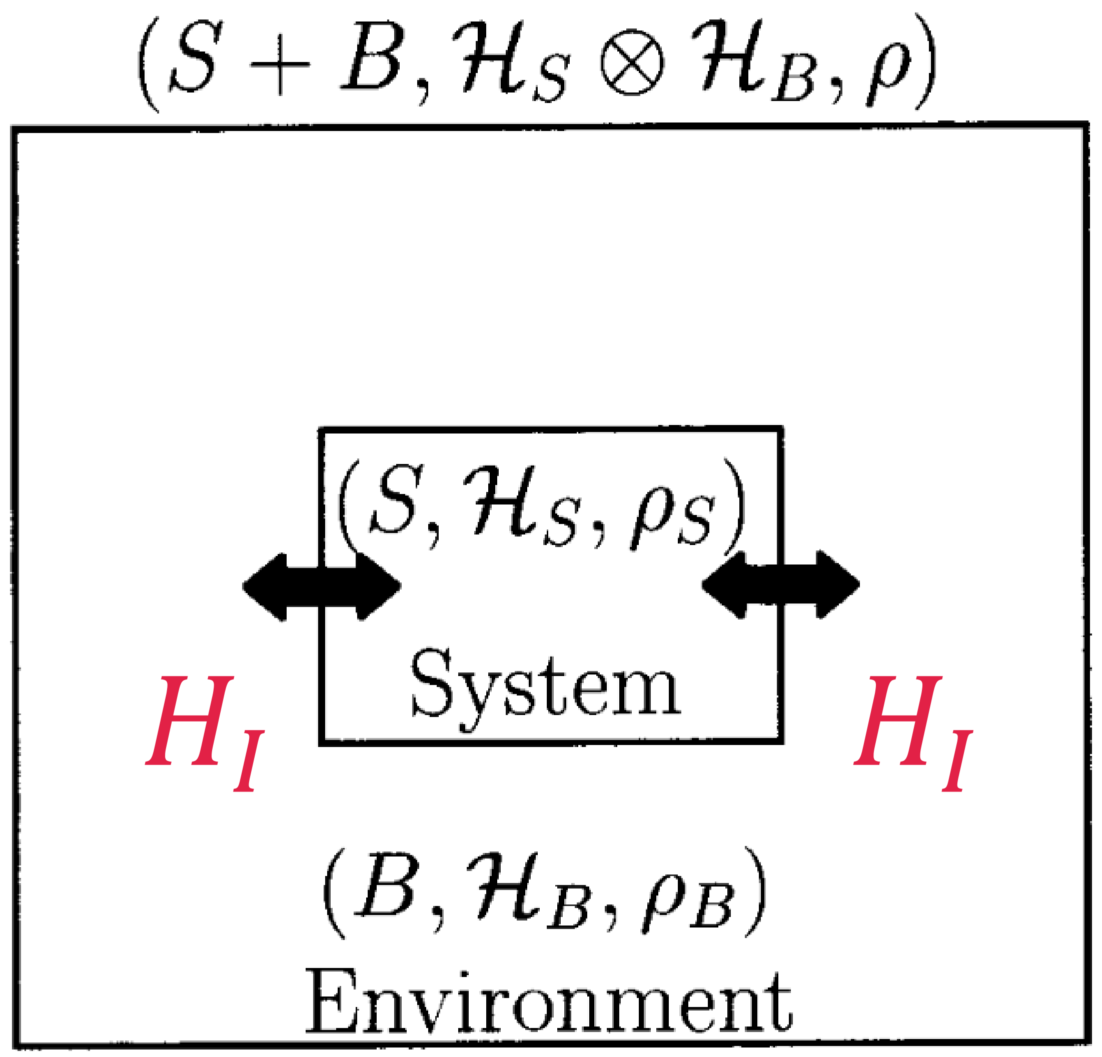
for instance, α can be lattice indices

→ A_α and B_α evolve with their respective free hamiltonians

(reminder: we are in interaction picture). | e.g. $e^{iH_S t} A_\alpha(\omega) e^{-iH_S t} = e^{-i\omega t} A_\alpha(\omega)$

$$H_I(t) = \sum_{\alpha,\omega} e^{-i\omega t} A_\alpha(\omega) \otimes B_\alpha(t) \; \text{+h.c.}$$

Derivation of quantum master equation



$$\frac{d}{dt}\rho_S(t) = -\int_0^\infty ds \mathrm{tr}_B \left[H_I(t), [H_I(t-s), \rho_S(t) \otimes \rho_B]\right]$$

$$H_I(t) = \sum_{\alpha,\omega} e^{-i\omega t} A_\alpha(\omega) \otimes B_\alpha(t) \text{ +h.c.}$$

$$\frac{d}{dt}\rho_S(t) = \int_0^\infty ds \mathrm{tr}_B \left\{H_I(t-s)\rho_S(t)\rho_B H_I(t) - H_I(t)H_I(t-s)\rho_S(t)\rho_B\right\} + \text{h.c.}$$

$$= \sum_{\omega,\omega'}\sum_{\alpha,\beta} e^{i(\omega'-\omega)t}\Gamma_{\alpha\beta}(\omega)\left(A_\beta(\omega)\rho_S(t)A_\alpha^\dagger(\omega') - A_\alpha^\dagger(\omega')A_\beta(\omega)\rho_S(t)\right)$$
$$+\text{h.c.}$$

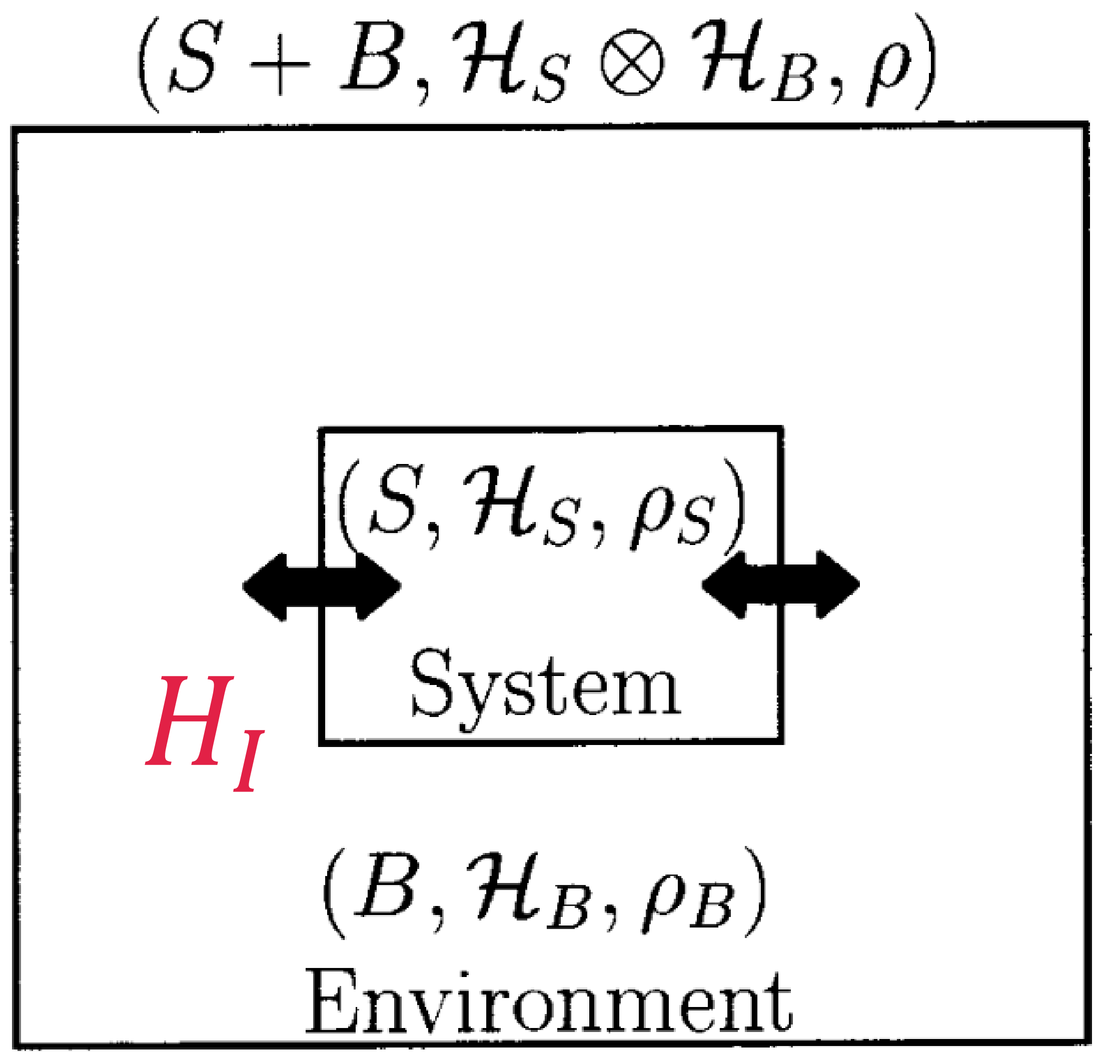
$$\Gamma_{\alpha\beta}(\omega) \equiv \int_0^\infty ds e^{i\omega s}\langle B_\alpha^\dagger(t)B_\beta(t-s)\rangle$$

$$\int_0^\infty ds \mathrm{tr}_B \; H_I(t)H_I(t-s)\rho_S(t)\rho_B \rightarrow A^\dagger(t)B^\dagger(t)A(t-s)B(t-s)\rho_S(t)\rho_B + .. = A_\alpha^\dagger(\omega')e^{i\omega' t}A_\beta(\omega)e^{-i\omega(t-s)}B_\alpha^\dagger(t)B_\beta(t-s)\rho_S(t)\rho_B$$

we assume the reservoir is in a steady state ($[H_B, \rho_B] = 0$) $\rightarrow$ its correlation functions are time translational invariant

$\rightarrow \Gamma_{\alpha\beta}(\omega)$ will be t-independent and the integral can be performed freely over *s*

Derivation of quantum master equation



$$\frac{d}{dt}\rho_S(t) = -\int_0^\infty ds \mathrm{tr}_B \left[H_I(t), [H_I(t-s), \rho_S(t) \otimes \rho_B]\right]$$

$$H_I(t) = \sum_{\alpha,\omega} e^{-i\omega t} A_\alpha(\omega) \otimes B_\alpha(t)$$

$$\frac{d}{dt}\rho_S(t) = \int_0^\infty ds \mathrm{tr}_B \left\{H_I(t-s)\rho_S(t)\rho_B H_I(t) - H_I(t)H_I(t-s)\rho_S(t)\rho_B\right\} + \mathrm{h.c.}$$

$$= \sum_{\omega,\omega'}\sum_{\alpha,\beta} e^{i(\omega'-\omega)t}\Gamma_{\alpha\beta}(\omega)\left(A_\beta(\omega)\rho_S(t)A_\alpha^\dagger(\omega') - A_\alpha^\dagger(\omega')A_\beta(\omega)\rho_S(t)\right)$$
$$+\mathrm{h.c.}$$

$$\Gamma_{\alpha\beta}(\omega) \equiv \int_0^\infty ds e^{i\omega s}\langle B_\alpha^\dagger(t)B_\beta(t-s)\rangle$$

we have assumed that the reservoir is in a steady state ($[H_B, \rho_B] = 0$) $\rightarrow$ the correlation functions are time translational invariant
$\rightarrow \Gamma_{\alpha\beta}(\omega)$ will be t-independent and the integral can be performed freely over *s*

$e^{i(\omega-\omega')t}$ $\longrightarrow$ $t \sim |\omega - \omega'|^{-1} \sim \tau_S$ $\quad$ $\tau_R \gg \tau_S \rightarrow$ the oscillatory terms goes fast and acts as a $\delta_{\omega,\omega'}$

Derivation of quantum master equation

$$\frac{d}{dt}\rho_S(t) = -\int_0^\infty ds\,\mathrm{tr}_B\,[H_I(t),[H_I(t-s),\rho_S(t)\otimes\rho_B]]$$

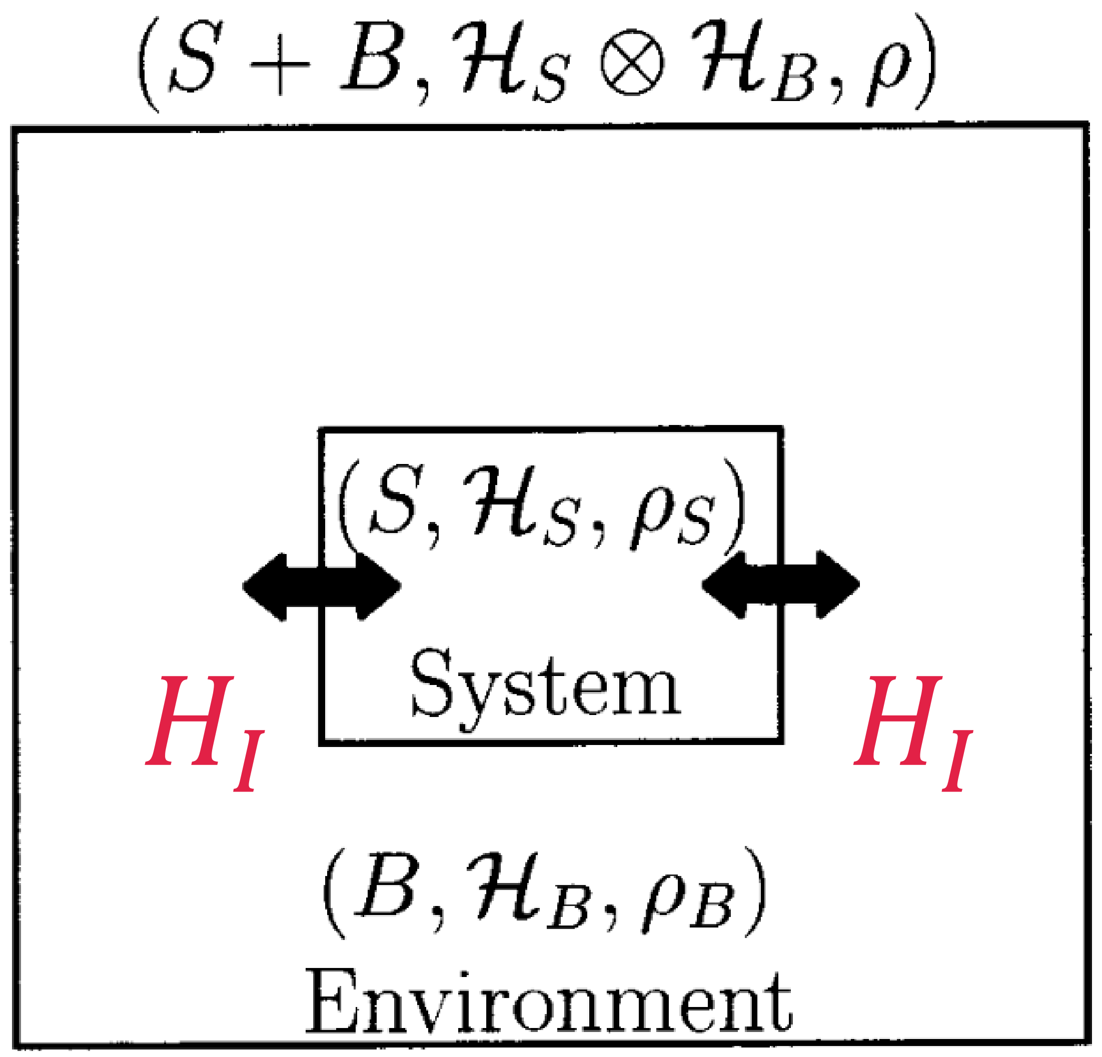
$$H_I(t) = \sum_{\alpha,\omega} e^{-i\omega t} A_\alpha(\omega)\otimes B_\alpha(t)$$

$$\frac{d}{dt}\rho_S(t) = \int_0^\infty ds\,\mathrm{tr}_B\,\{H_I(t-s)\rho_S(t)\rho_B H_I(t) - H_I(t)H_I(t-s)\rho_S(t)\rho_B\} + \mathrm{h.c.}$$

$$= \sum_{\omega,\omega'}\sum_{\alpha,\beta} e^{i(\omega'-\omega)t}\Gamma_{\alpha\beta}(\omega)\left(A_\beta(\omega)\rho_S(t)A_\alpha^\dagger(\omega') - A_\alpha^\dagger(\omega')A_\beta(\omega)\rho_S(t)\right)$$

$$+\mathrm{h.c.}$$

$$\Gamma_{\alpha\beta}(\omega) \equiv \int_0^\infty ds\, e^{i\omega s}\langle B_\alpha^\dagger(t)B_\beta(t-s)\rangle$$



$e^{i(\omega-\omega')t} \longrightarrow t \sim |\omega-\omega'|^{-1} \sim \tau_S$ $\quad\tau_R \gg \tau_S \rightarrow$ the oscillatory terms goes fast and acts as a $\delta_{\omega,\omega'}$

$$\int_0^\infty ds\,\mathrm{tr}_B\; H_I(t)H_I(t-s)\rho_S(t) \rightarrow A^\dagger B(t)B^\dagger(t)A(t-s)B(t-s)\rho_S(t)\rho_B + .. = A_\alpha^\dagger(\omega')e^{i\omega' t}A_\beta(\omega)e^{-i\omega(t-s)}B_\alpha^\dagger(t)B_\beta(t-s)\rho_S(t)\rho_B$$

$A(t)A(t-s) \sim e^{-i(\omega+\omega')t} \rightarrow 0$ fast oscillations (rotating-wave approximation/RWA)

Derivation of quantum master equation

$$\frac{d}{dt}\rho_S(t) = -i\,[H_{LS}, \rho_S(t)] + \mathcal{D}(\rho_S(t))$$

some dressing of the hamiltonian (called conventionally 'Lamb shift')

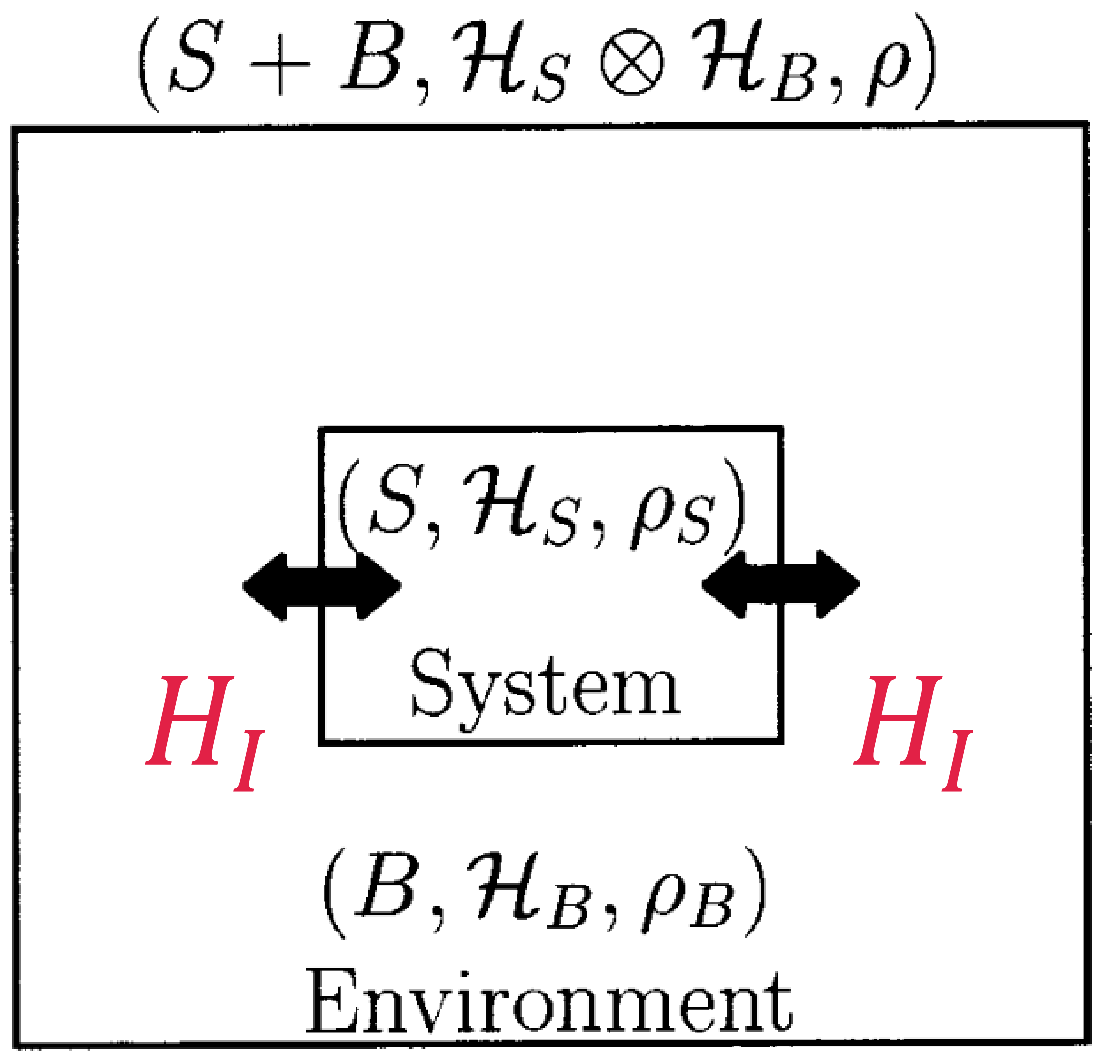
$$\mathcal{D}(\rho_S) = \sum_{\omega}\sum_{\alpha,\beta}\gamma_{\alpha\beta}(\omega)\left(A_\beta(\omega)\rho_S A_\alpha^\dagger(\omega) - \frac{1}{2}\left\{A_\alpha^\dagger(\omega)A_\beta(\omega), \rho_S\right\}\right)$$

$$\gamma_{\alpha\beta}(\omega) = \Gamma_{\alpha\beta}(\omega) + \Gamma^*_{\beta\alpha}(\omega) = \int_{-\infty}^{+\infty} ds\, e^{i\omega s}\langle B_\alpha^\dagger(s)B_\beta(0)\rangle$$

diagonalize and get the standard form of *Lindblad* quantum master equation

$$= -i\,[H, \rho_S] + \sum_{k=1}^{N^2-1}\gamma_k\left(A_k\rho_S A_k^\dagger - \frac{1}{2}A_k^\dagger A_k\rho_S - \frac{1}{2}\rho_S A_k^\dagger A_k\right)$$

the generator of dynamics for ρ_S is called *Liouvillian*



→ alternative derivation in the book by Gardiner & Zoller

→ **concrete derivation for an atom coupled to EM field (Breuer/Petruccione Sec. 3.4)**

→ **quantum Brownian motion and Caldeira-Leggett model (Breuer/Petruccione Sec. 3.6)**