Basic Particle Physics

Lagrangians and Feynman Rules

Lecture 2 - Felipe Ortega Gama

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- Comments, questions or suggestions: fgortegagama@berkeley.edu

DoF	Classical Mechanics	Quantum
Finite	$\vec{x}, \vec{p}$	b $\hat{x}, \hat{p}, \lvert\psi\rangle$
Infinite	a $\phi(x)$	c $\hat{\phi}(x), \hat{\Pi}(x), \lvert\psi\rangle$

a Hamiltonian formulation (equivalent E-L)

→ Conjugate momenta: $\pi^a = \boxed{\dfrac{\partial \mathcal{L}}{\partial(\partial_t \phi^a)}}$

$$\mathcal{H} = \pi^a \cdot \partial_t \phi^a - \mathcal{L} \Big|_{\pi^a}$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \quad , \quad \Pi = \partial_t \phi$$

$$\mathcal{H} = \Pi \cdot \underbrace{\partial_t \phi}_{\Pi} - \mathcal{L} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2} m^2 \phi^2$$

$$\mathcal{L} = T - V \quad ; \quad \mathcal{H} = T + V$$

Interactions $\rightarrow$ in V "potential" part of the theory

$$\underbrace{\mathcal{H}_{int}} = -\mathcal{L}_{int}$$

$$\partial_t \phi_a = \{\phi_a, H\} \qquad \{f, g\} = \frac{\partial f}{\partial \phi_a}\frac{\partial g}{\partial \Pi_a} - \frac{\partial g}{\partial \phi_a}\frac{\partial f}{\partial \Pi_a}$$

b QM → Harmonic Oscillator $(\hbar = 1)$

$\vec{x}, \vec{p} \rightarrow$ promote to operators

Quantize them

$$[x_a, p_b] = i\delta_{ab}$$

$$[x_a, x_b] = 0$$

$$[p_a, p_b] = 0$$

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{x}^2$$

$$\hat{x} = \frac{1}{\sqrt{2\omega}}(a^+ + a), \quad \hat{p} = -i\sqrt{\frac{\omega}{2}}(a - a^+)$$

$$\hat{H} = \omega(\underbrace{a^+a}_{\text{one op}} + \frac{1}{2})$$

$$H|n\rangle = \underbrace{E_n}|n\rangle$$

$$[H, a^\dagger] = \omega a^\dagger \quad \Rightarrow \quad [H, a] = -\omega a$$

$$H a |n\rangle = (E_n - \omega) \underbrace{a|n\rangle} \qquad a \text{ is lowering the energy of } |n\rangle$$

$$H a^\dagger |n\rangle = (E_n + \omega) a^\dagger |n\rangle \qquad a^\dagger \text{ is raising the energy}$$

ground state

$$a|0\rangle = 0$$

$$H|0\rangle = \frac{\omega}{2}|0\rangle \quad ; \quad |n\rangle = (a^\dagger)^n |0\rangle$$

$$H|n\rangle = \underbrace{\left(n + \tfrac{1}{2}\right)\omega}|n\rangle$$

$$\hat{N} = \underbrace{a^\dagger a}$$ → number operator ; $$\hat{H} = \omega\left(\hat{N} + \tfrac{1}{2}\right)$$

$$\hat{N}|n\rangle = \underbrace{n}\,|n\rangle$$

Quantum Field theory $(\phi_a(x), \underline{\pi_a(x)})$

promote them to operators

$$[\phi_a(\vec{x}), \pi_b(\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

$$[\phi_a(\vec{x}), \phi_b(\vec{y})] = 0, \quad [\pi_a(\vec{x}), \pi_b(\vec{y})] = 0$$

Equal time commutation relations.

$$\underbrace{i\frac{d}{dt}|\Psi\rangle = H|\Psi\rangle} \quad \text{where} \quad \underline{H} = \int d^3x\, \underline{\mathcal{H}}$$

$$\underline{\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2} \quad \left(\text{Hamiltonian density}\right)$$

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{\dagger}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\text{FT} : V = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left[(\vec{p})^2 \tilde{\phi}^2 + m^2 \tilde{\phi}^2 \right] e^{ipx}$$

$$\omega_p^2 = (\vec{p})^2 + m^2$$

$$\Pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^{\dagger}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\left[a_{\vec{p}} , a_{\vec{q}}^{\dagger} \right] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) , \quad [a_{\vec{p}}, a_{\vec{q}}] = 0$$

$$[a_{\vec{p}}^{\dagger}, a_{\vec{q}}^{\dagger}] = 0$$

$$H = \int \frac{d^3p}{(2\pi)^3} \Big(\underbrace{\omega_{\vec p}\, a_{\vec p}^\dagger a_{\vec p}} + \underbrace{\frac{1}{2}\omega_{\vec p}\, (2\pi)^3 \delta^{(3)}(\cancel{\vec x - \vec x})}\ {}^{0} \Big)$$

$$N_{\vec p} = a_{\vec p}^\dagger a_{\vec p}$$

$$\underline{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2}\omega_{\vec p}\, (2\pi)^3 \delta^{(3)}(0)}$$

$$\underline{:} H \underline{:} = \int d^3x\, \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

Normal ordering

$\pi \sim a - a^\dagger$ $\qquad \phi \sim a + a^\dagger$

put <u>all "a" to the right</u>

$$:H: = \int \frac{d^3p}{(2\pi)^3} \big(\omega_p\, \underbrace{a_{\vec p}^\dagger a_{\vec p}} \big)$$

$$[H, a_p^\dagger] = \omega_p a_p^\dagger \quad -- \quad [H, a_p] = -\omega_p a_p^\dagger$$

$$|p\rangle = \sqrt{2\omega_p}\, a_p^\dagger |0\rangle \quad \text{s.t.} \quad \langle p|q\rangle = \underbrace{2\omega_p (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})}_{\text{Lorentz Inv}}$$

$$H|p\rangle = \omega_p |p\rangle,$$ "physical momentum"

$$H = \frac{1}{2}(\omega x - ip)(\omega x + ip)$$ Classical

↓ quantize

$$H = \frac{1}{2} a^\dagger a$$

"physical momentum"

$$\text{Continuous Symmetries} \xrightarrow{\text{Noether theorem}} \text{Conserved current}$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

$x \to x' = x + a$] momentum

$\vec{x} \to R\vec{x}$] angular momentum

$t, x \to$ Lorentz Transf] CM velocity

$$\underbrace{:\vec{P}:}_{\substack{\text{physical momentum} \\ \text{conserved charge} \\ \text{of } x \to x+a \text{ sym}}} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\, \underbrace{\vec{p}}\; \underbrace{a^\dagger_{\vec{p}}\, a_{\vec{p}}}$$

$$|p\rangle = \sqrt{2\omega_p}\, a_p^\dagger |0\rangle \quad \begin{matrix} \nearrow \text{Energy} = \sqrt{|\vec{p}|^2 + m^2} \\ \searrow \text{momentum} = \vec{p} \end{matrix} \Bigg\} \Rightarrow$$

Relativistic Energy-Mom dispersion

$$|p_1 p_2\rangle = \sqrt{2\omega_{p_1}}\sqrt{2\omega_{p_2}}\, a_{p_1}^\dagger a_{p_2}^\dagger |0\rangle$$

$$\downarrow$$

$$E = \underbrace{(\omega_{p_1} + \omega_{p_2})}, \quad \vec{P} = \underbrace{\vec{p}_1 + \vec{p}_2}$$

Fock space : $\underbrace{|0\rangle,\ |p\rangle,\ |p_1\rangle \otimes |p_2\rangle,\ |p_1 p_2 p_3\rangle \cdots}$

$|p_1 p_2\rangle$

$$\mathbb{1} = |0\rangle\langle 0| + \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} |p\rangle\langle p| + \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{1}{2\omega_{p_1}} \frac{1}{2\omega_{p_2}} |p_1 p_2\rangle\langle p_1 p_2| + \cdots$$

$$N = \int \frac{d^3p}{(2\pi)^3} \underbrace{N_{\vec{p}}}_{a_p^\dagger a_p}$$

Complex scalar field $\psi = \phi_1 + i\phi_2$, $\psi^\dagger$

$$\mathcal{L} = \partial_\mu \underbrace{\psi \, \partial_\mu \psi^\dagger}_{\text{real}} - m^2 \underbrace{\psi \psi^\dagger}_{\text{real}}$$

$$\psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(b_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + c_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\psi^\dagger = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(b_p^\dagger e^{-i\vec{p}\cdot\vec{x}} + c_p^\dagger e^{+i\vec{p}\cdot\vec{x}} \right)$$

$$H = \int \frac{d^3p}{(2\pi)^3} \left(\omega_p \left(\underbrace{b_p^\dagger b_p} + \underbrace{c_p^\dagger c_p} \right) \right)$$

→ Symmetry $\quad \psi \to \underline{e^{i\alpha}\psi} \quad$ → Noether's theorem → Q

$$Q = \int \frac{d^3p}{(2\pi)^3} \left(\underbrace{c_p^\dagger c_p} - \underbrace{b_p^\dagger b_p} \right) = \underbrace{N_c - N_b}$$

$$A^\alpha \to \underbrace{R^\alpha{}_\beta} A^\beta$$

→ ϕ → scalars $\quad J=0$

spinors $\quad J=\frac{1}{2}$

vectors $\quad J=1$

$x \to Rx$

Schrödinger vs Heisenberg picture

$$i\frac{d|\psi\rangle}{dt} = H|\psi\rangle$$

$$\Downarrow$$

$$|\psi_n(t)\rangle = e^{-iE_nt}|\psi_n(0)\rangle$$

$$\langle x\rangle = \langle\psi_n(t)|\hat{x}|\psi_n(t)\rangle$$

$$\hat{O}_H = \underbrace{e^{iHt}\,\hat{O}_S\,e^{-i\hat{H}t}}$$

$$\frac{dO_H}{dt} = i[H, O_H]$$

* similiraty to the Poisson bracket

$$\langle x\rangle = \langle\psi_n|\hat{x}(t)|\psi_n\rangle$$

$$= \underbrace{\langle\psi_n|e^{iHt}}\,\hat{x}\,\underbrace{e^{-iHt}|\psi_n\rangle}$$

$$\underbrace{\frac{d\phi}{dt} = i[H, \phi] = \Pi(x)}\ ; \quad \underbrace{\frac{d\Pi(x)}{dt} = \nabla^2\phi - m^2\phi^2}$$

$$\partial_\mu \partial^\mu \hat{\phi} + m^2 \hat{\phi} = 0 \rightarrow \text{operators in QFT}$$

$$a_p(t) = \underline{e^{iHt} a_p e^{-iHt} \underbrace{=} e^{-i\omega_{\vec{p}} t} a_p}$$

$$\underbrace{a_p^+(t) = e^{i\omega_{\vec{p}} t}}$$

$$p^\mu x_\mu = \omega_{\vec{p}} t - \vec{p}\cdot\vec{x}$$

$$\underbrace{\phi(\vec{x}, t)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{-ip\vec{x}} + a_p^+ e^{ipx} \right)$$

vacuum expectation value of ϕ

$\langle 0 | \phi(\vec{x}, t) | 0 \rangle = 0$; Higgs $\neq 0$ vev

Higgs $\to$ mechanism

$\phi \sim a + a^{+}$

$\langle 0 | \phi(x) \phi(y) | 0 \rangle$

$\mathbb{1} = \underbrace{|0\rangle\langle 0|} + \int \frac{d^3p}{(2\pi)^3} \frac{|p\rangle\langle p|}{2\omega_p} + \dots$

$= \langle 0 | \phi(x) \, \mathbb{1} \, \phi(y) | 0 \rangle$

$= \langle 0 | \int \frac{d^3p}{(2\pi)^3} \frac{\phi(x) | p \rangle \langle p | \phi(y) | 0 \rangle}{2\omega_p}$

$\phi(x) = e^{i\hat{P}\cdot x} \phi(0) e^{-i\hat{P}\cdot x}$ $\quad \hat{P} = \begin{pmatrix} :H: \\ :\vec{P}: \end{pmatrix}$

$e^{i\hat{P}\cdot x} |p\rangle = e^{ip\cdot x} |p\rangle \quad p = \begin{pmatrix} \omega_p \\ \vec{p} \end{pmatrix}$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip(x-y)}}{2\omega_p} \left| \cancelto{1}{\langle 0 | \phi(0) | p \rangle} \right|^2$$

$$\phi \sim a + a^\dagger$$

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip(x-y)}}{2\omega_p}$$

Time ordering

$$T(\phi(x)\phi(y)) = \begin{cases} x^0 > y^0 & \phi(x)\phi(y) \\ y^0 > x^0 & \phi(y)\phi(x) \end{cases}$$

$\langle 0 | T(\phi(x)\phi(y)) | 0 \rangle =$ Feynman propagator.